Quelques contributions à l'étude qualitative et quantitative de modèles de la physique et de la biologie

Soutenance d'habilitation à diriger les recherches

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What I like to (try to) do ...

- → Study qualitatively and quantitatively models from physics and biology.
- → Accent is put on nonlocal models, which have become more and more important and remain very difficult to study.
- → Try to design and/or expand flexible methods for this kind of problems.

Of course, this a very general (and too big of a) deal. In the past few years, I have worked on ...

1. **Progagation phenomena in nonlocal models from ecology: Bramson corrections, accelerations.**
2. **Trend to equilibrium and scaling limits in kinetic theory.**

Propagation in nonlocal reaction-diffusion models

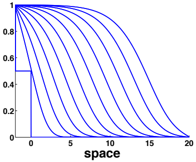
Propagation in nonlocal reaction-diffusion models

1 - Bramson delay in the nonlocal Fisher-KPP equation

The logarithmic delay for Fisher-KPP

The most historical reaction-diffusion equation for population dynamics is ...

$$u_t = u_{xx} + u\,(1-u)\,,$$
$$u(0,x) = u_0$$



... because it is known to exhibit front propagation since Fisher, KPP, Aronson-Weinberger, Fife-McLeod...

- With probabilistic techniques, Bramson ('78, '83) showed that if u_0 is compactly supported, the front of u is located at

$$X(t) = 2t - \frac{3}{2}\ln t + s_0,$$

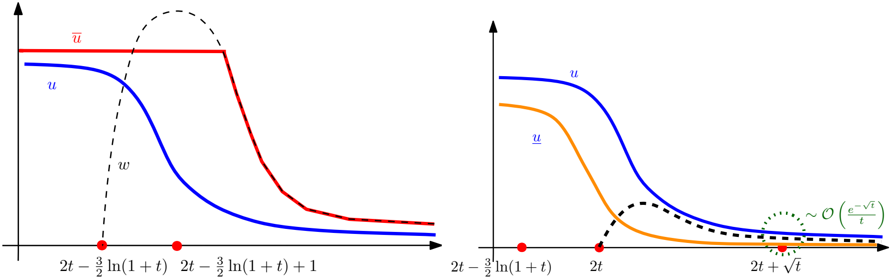
  where s_0 is a shift depending only on u_0.

- See $2t$ as the position of a traveling wave, and $\frac{3}{2}\ln t$ as the delay due to the fact that the initial condition u_0 is compactly supported, so that the solution lags behind the traveling wave.

The Hamel Nolen Roquejoffre Ryzhik strategy

These proofs have been simplified in recent years by Roberts (probabilistic approach) and Hamel-Nolen-Ryzhik-Roquejoffre ('12, '13).

Main strategy : Linearised problem with a Dirichlet condition at well chosen spots to create sub- and super- solutions.



Needs: comparison principles, precise quantitative estimates.

A model with nonlocal competition

Consider a situation for which the competition (*e.g.* for resources) is nonlocal, with a kernel Φ.

$$u_t = u_{xx} + u\left(1 - \Phi \star u\right),$$
$$u(0,x) = u_0$$

where

$$\int_{\mathbb{R}} \Phi(x)dx = 1, \quad \text{and} \quad \Phi(x) = \Phi(-x) \text{ for all } x \in \mathbb{R},$$

such that

$$A_\Phi^{-1}(1+|x|)^{-r} \leq \Phi(x) \leq A_\Phi(1+|x|)^{-r},$$

for all $x \in \mathbb{R}$, with some positive constants $r \in (1,\infty)$ and $A_\phi > 0$.

Known: Propagation in a weak sense at speed 2 (Hamel-Ryzhik), uniform in time L^∞ bound, non easy behaviour at the back (wave trains ...), steady states and travelling waves (sometimes!).

No comparison and maximum principles available.

References. Hamel-Ryzhik ('14), Berestycki-Nadin-Perthame-Ryzhik ('09).

Logarithmic delay for the non-local Fisher-KPP equation

Question: Is the Bramson correction also true here and if yes, how to prove it?

Theorem (B., Henderson, Ryzhik ('17))

Take u_0 compactly supported and $\Phi \asymp |\cdot|^{-r}$.

- *If $r > 3$, then the solution u propagates with the $-\frac{3}{2}\ln(t)$ logarithmic delay.*
- *If $r = 3$, then the solution u propagates with a larger logarithmic delay between $-S_\phi \ln(t)$ and $-s_\phi \ln(t)$.*
- *If $r \in (1,3)$, then the delay is algebraic between $c_\phi t^{\frac{3-r}{1+r}}$ and $C_\phi t^{\frac{3-r}{1+r}}$.*

Elements of proof:

- A **local-in-time Harnack inequality** of the form

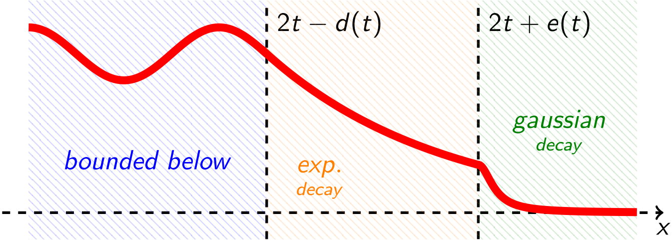
$$u(T, x+y) \leq C\|u\|_{L^\infty([t,T])\times\mathbb{R}}^{1-\frac{1}{p}} u(T,x)^{\frac{1}{p}} e^{\alpha t + \frac{\beta y^2}{t}}.$$

  to estimate

$$\Phi \star u \lesssim \ln\left(\frac{M}{u}\right)^{1-r}.$$

- **Adapt the H-N-R-R strategy** to the local equation with a Gompertz type nonlinearity.

Heuristics for the exponent $\frac{3-r}{1+r}$



$$\phi \star u(t,x) \gtrsim (e(t)+d(t))^{1-r} \quad \text{for } x \in (2t-d(t), 2t+e(t))$$

$$(e(t)+d(t))^{1-r} \gtrsim d'(t) \qquad e(t)^2 \geq 4td(t)$$

1. Since $e(t) = o(t)$, we get $\lim_{t\to+\infty} \frac{d(t)}{e(t)} = 0$. This gives, for t large,

$$d'(t) \lesssim e(t)^{1-r} \lesssim t^{\frac{1-r}{2}} d(t)^{\frac{1-r}{2}} \implies d(t) \lesssim t^{\frac{3-r}{1+r}}.$$

2. We deduce also $e(t) \gtrsim t^{\frac{2}{1+r}}$.

Conclusions and related topics

- New results for nonlocal KPP ☺ !
- Keep in mind the **local-in-time Harnack inequality** for later ... ♡
- This lead to study more precisely travelling waves and delay for ...

$$u_t = u_{xx} + u\left(1 - A\left(\ln\left(\frac{\nu}{u}\right)\right)^{1-r}\right),$$
$$u(0,x) = u_0$$

Theorem (B., Henderson ('21))

- *If $r > 3$, then the delay is $\frac{3}{2}\log t$.*
- *If $r = 3$, then the delay is $\left(1 + \frac{1}{2}\sqrt{1+4A}\right)\log t$.*
- *If $r \in (1,3)$, then the delay is $\Theta_r A^{\frac{2}{1+r}} t^{\frac{3-r}{1+r}}$, with $\Theta_r = \psi(0)$, where ψ solves*

$$\psi' = \frac{y}{1+r} - \sqrt{\frac{y^2}{(1+r)^2} + Ay^{1-r} - \frac{3-r}{1+r}\psi},$$
$$\psi\left((1+r)^{\frac{2}{1+r}}\right) = (1+A)\,\frac{(1+r)^{\frac{3-r}{1+r}}}{3-r}.$$

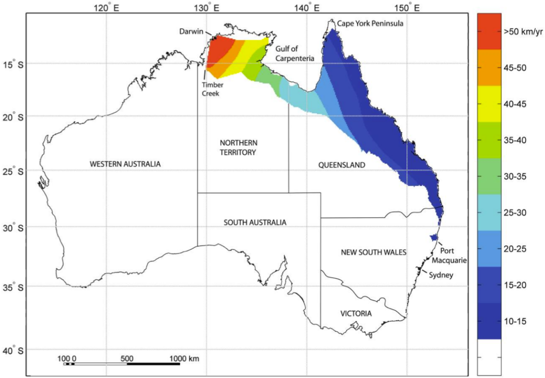
Propagation in nonlocal reaction-diffusion models

2 - Cane toads equation and related topics

In short...

- We will discuss **non-local reaction-diffusion-mutation models** inspired by evolution in cane toads populations in Australia.
- Important feature : the propagation is actively influenced by a microscopic structure of the population : **the leg-length/motility**.
- **Aim:** We seek a precise description of the propagation, in particular, estimates of finite or infinite speeds of propagation.

Evolution of dispersal in cane toads populations (*e.g.*)



- Speed increased by 5.
- At the edge, faster toads in majority.
- **Spatial sorting** : Dynamic selection of traits along the invasion.

We need models with both space and dispersion variables.

Reference. M. Urban *et al* (2008).

Data of acceleration

Data from Urban *et al.* (Am. Nat. 2008): 1.63 ± 0.13.

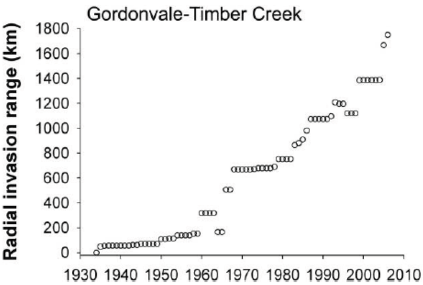


Figure: Position of the front with years - Section Gordonvale-Timber Creek, for which spatial sorting is presumably the main effect.

Reference. M. Urban *et al* (2008).

Modelling the cane toads invasion

$t \in \mathbb{R}^+$: time, $x \in \mathbb{R}$: space variable, $\theta \in \Theta$: dispersal ability.

mutations, reproduction rate.

$$\begin{cases} n_t = \theta n_{xx} + r\alpha n_{\theta\theta} + r\, n\,(1-\rho)\,, & (t,x,\theta) \in \mathbb{R}^+ \times \mathbb{R} \times \Theta, \\ \rho(t,x) = \int_\Theta n(t,x,\theta')\, d\theta'\,, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \end{cases}$$

Neumann boundary conditions in $\Theta := \left[\underline{\theta}, \overline{\theta}\right] \subset (0,\infty]$.

Crucial difference with standard Fisher - KPP:

No full maximum/comparison principles available

References. Desvillettes *et al.* ('04), Champagnat *et al.* ('07), Bénichou *et al.* ('12)

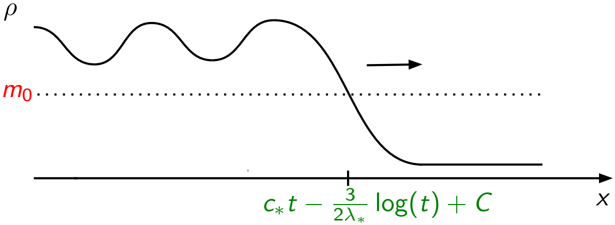
Propagation for the Cauchy problem with bounded traits

Theorem (B., Henderson, Ryzhik ('16))

Assume that Θ **is bounded** *and* $n_0 \sim \mathbb{1}_{[-\infty,0]\times\Theta}$. *Then there exists* m_0 *such that for all* $\varepsilon \in (0, m_0)$, *there is a positive constant* C_ε *such that*

$$\liminf_{t\to\infty} \inf_{x \le c_* t - \frac{3}{2\lambda_*}\log(t) - C_\varepsilon} n(t,x) \ge m_0 - \varepsilon,$$

$$\limsup_{t\to\infty} \sup_{x \ge c_* t - \frac{3}{2\lambda_*}\log(t) + C_\varepsilon} n(t,x) \le \varepsilon.$$



See also. B. - Calvez ('14) (travelling waves).

Acceleration result: local case

From now on **Θ in unbounded**: $\Theta = [\underline{\theta}, \infty)$. Consider first the local cane toads counterpart:

$$n_t = \theta n_{xx} + r\alpha n_{\theta\theta} + rn(1 - n)$$

Theorem (B., Henderson, Ryzhik ('15))

Let n the unique solution of the LOCAL cane toads equation. Fix any constant $m \in (0,1)$.

$$\lim_{t\to\infty} \frac{\max\{x \in \mathbb{R} : \exists \theta \in \Theta, n(t,x,\theta) = m\}}{t^{3/2}} = \frac{4}{3} r\alpha^{1/4}.$$

Proof hinges on: linearized equation and comparison principle (*NOT* available for non-local problem)

See also. Berestycki *et al.* ('15) : same result with probability techniques.

The trajectories

The only natural scaling to make in the linearised cane toads equation is

$$(t, x, \theta) \mapsto \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{3/2}}, \frac{\theta}{\varepsilon}\right) \qquad \Longrightarrow \qquad \varepsilon w_t^\varepsilon = \varepsilon^2 \theta w_{xx}^\varepsilon + \varepsilon^2 w_{\theta\theta}^\varepsilon + w^\varepsilon.$$

Hopf-Cole transformation $w^\varepsilon = \exp\left(-\frac{\varphi^\varepsilon}{\varepsilon}\right)$, so that

$$\varphi_t^\varepsilon + \theta|\varphi_x^\varepsilon|^2 + |\varphi_\theta^\varepsilon|^2 + 1 = \varepsilon\theta\varphi_{xx}^\varepsilon + \varepsilon\varphi_{\theta\theta}^\varepsilon,$$

and obtain, in the formal limit as $\varepsilon \to 0$, the Hamilton-Jacobi equation

$$\varphi_t + \theta|\varphi_x|^2 + |\varphi_\theta|^2 + 1 = 0.$$

We obtain...

- ▷ **an explicit formula** for φ,

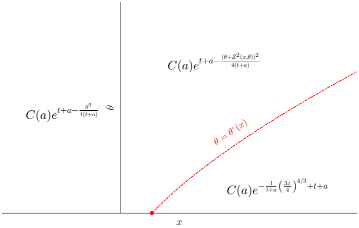
$$\varphi(t, x, \theta) = \frac{1}{4t}\left(\theta + Z(x,\theta)^2\right)^2 - t, \qquad \text{where } Z^3 + 3\theta Z + 3x = 0.$$

- ▷ **explicit Lagrangian trajectories**.

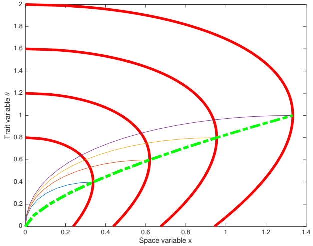
Reference. B. et. al. 2012.

Upper and lower bounds

Upper bound: explicit super-solution (*somewhat miraculously*)



Lower bound: the moving ball technique.



- ▷ Take an optimal Lagrangian trajectory given by the HJ equation,
- ▷ Slide a suitable bump over this trajectory,
- ▷ Use "time-dependent" Dirichlet principal eigenelements to maintain enough mass in the bump.

Acceleration result for the nonlocal equation

Theorem (B., Henderson, Ryzhik ('15))

Let u the unique solution of the cane toads equation. Fix any constant $m \in (0,1)$.

$$\frac{8}{3\sqrt{3\sqrt{3}}} r\alpha^{1/4} \le \limsup_{t\to\infty} \frac{\max\{x \in \mathbb{R} : \rho(t,x) = m\}}{t^{3/2}} \le \frac{4}{3} r\alpha^{1/4}.$$

Not sharp in *two* ways:

- Only lim sup : comes from *proof by contradiction* argument
- $\frac{4}{3} - \frac{8}{3\sqrt{3\sqrt{3}}} \sim .16$: can (should?) *not* follow previous optimal trajectories !

Improved later by Calvez *et al.* refining the trajectories to take into account the nonlinearity: the nonlocal nonlinearity slows down the propagation!

See also. Berestycki *et al.* ('15) : propagation in a modified cane toads model with a windowed non-linearity.

A model with a mortality trade-off

We take into account a penalization of very large traits on the reproduction rate (via a mortality trade-off).

$t \in \mathbb{R}^+$: time, $x \in \mathbb{R}$: space variable, $\theta \in \Theta$: dispersal ability.

$$\begin{cases} n_t = \theta n_{xx} + r\alpha n_{\theta\theta} + rn\left(1 - m(\theta) - \rho\right), & (t,x,\theta) \in \mathbb{R}^+ \times \mathbb{R} \times \Theta, \\ \rho(t,x) = \int_\Theta n(t,x,\theta')\, d\theta', & (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \end{cases}$$

with Neumann boundary conditions in $\theta \in \Theta := [\underline{\theta}, +\infty) \subset \mathbb{R}_+^*$.

$$m(\underline{\theta}) = 0, \qquad m \text{ is increasing}, \qquad \lim_{\theta \to +\infty} m(\theta) = +\infty.$$

See also. Chan *et al* ('15)

The spreading result

Denote by γ_∞ the principal Neumann eigenvalue of $\alpha Q_{\theta\theta} + (1-m)Q$.

Theorem (B., Chan, Henderson, Kim ('17))

- $\gamma_\infty \leq 0 \quad \Longrightarrow \quad$ Extinction
- $\gamma_\infty > 0 \quad \Longrightarrow \quad$ Propagation
  - $\lim_{\theta\to+\infty} \frac{m(\theta)}{\theta} > 0 \quad \Longrightarrow \quad$ ***Finite speed of propagation***, *(explicit speed, travelling waves, linear spreading).*
  - $\frac{m(\theta)}{\theta} \underset{\theta\to+\infty}{\searrow} 0 \quad \Longrightarrow \quad$ ***Acceleration***. *There exist $\underline{a}$, $\overline{a}$ such that,*

$$\liminf_{t\to\infty} \inf_{x \leq \underline{a}\eta(t)^{3/2}} \rho(t,x) > 0 \qquad \textit{and} \qquad \lim_{t\to\infty} \sup_{x\geq \overline{a}\eta(t)^{3/2}} n(t,x,\cdot) = 0.$$

where $\int_0^{\eta(t)} \sqrt{m(s)}\, ds = t$.

Numerics with $m \sim \theta^p$, $p = \frac{1}{3}$, $p = \frac{2}{3}$.

Theorem: $p < 1 \Longrightarrow$ **Acceleration regime**: we have $\eta(t) \sim C t^{\frac{2}{2+p}}$, and thus the Cauchy problem spreads at $t^{\frac{3}{2+p}}$.

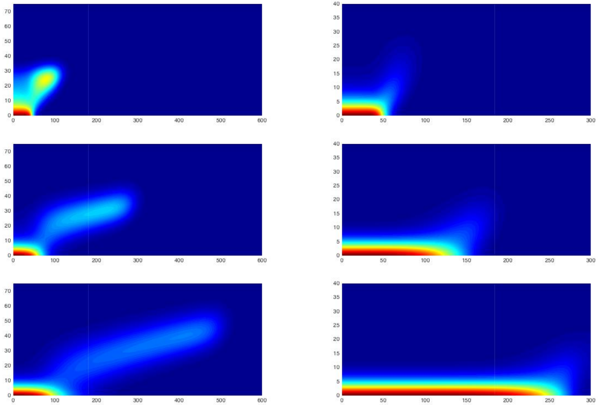


Figure: Cauchy problem at times (from top to bottom) $t = 10$, $t = 30$, $t = 50$. Left column: $p = \frac{1}{3}$. Right column: $p = \frac{2}{3}$. Propagation at a super-linear rate.

Numerics with $m \sim \theta^p$, $p = 1$, $p = \frac{4}{3}$

Theorem: $p \geq 1 \Longrightarrow$ **Linear regime** : travelling wave solutions and the Cauchy problem spreads **linearly**.

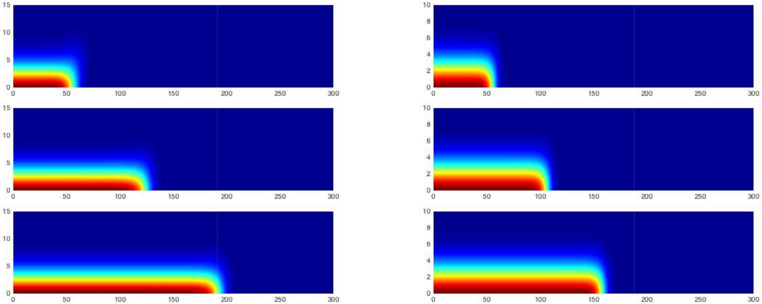


Figure: Cauchy problem at times $t = 10$, $t = 30$, $t = 50$. Left column: $p = 1$. Right column: $p = \frac{4}{3}$. Propagation at a linear rate.

Upper and lower bounds

Upper bound: **Li-Yau estimates**. The Lagrangian

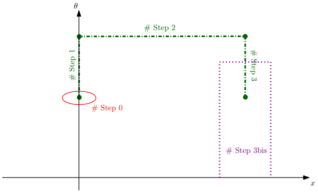
$$\zeta = \inf_{Z(0)=(y,\eta), Z(t)=(x,\theta), Z\in C^{0,1}([0,t])} \left\{ \int_0^t \left(\frac{|\dot{Z}_1|^2}{4Z_2} + \frac{|\dot{Z}_2|^2}{4} + m(Z_2) \right) ds \right\}.$$

allows to estimate from above and gives for $\theta \leq \eta(t)$,

$$u \lesssim \exp\left\{ Ct - \frac{\zeta}{2} \right\}.$$

The Hamilton-Jacobi solution is not necessarily a supersolution!

Lower bound: the moving ball technique.



Main difference with standard toads:

Explicit trajectories not available,

taking a reasonable path is fine.

Reference. P. Li, S.T. Yau, (1986).

Perspectives and related topics

- ▷ We got quantitative estimates on acceleration phenomena appearing in biology. More information on the (flattening) profiles?
- ▷ Related but different, we got with Coville and Legendre quantitative acceleration results (rates and flattening) on

$$u_t = P.V.\left(\int_{\mathbb{R}}[u(t,y) - u(t,x)]J(x-y)\,dy\right) + u^\beta(1-u),$$

where J is a jump operator with **a fat-tailed measure J**: $J(z) \approx \frac{1}{|z|^{\mathbf{1+2}s}}$

Theorem (B., Coville, Legendre ('21))

Assume that

$$\beta < 1 + \frac{1}{2s-1}.$$

Then for any $\lambda \in (0,1)$, the level line $x_\lambda(t)$ accelerates with the following rate,

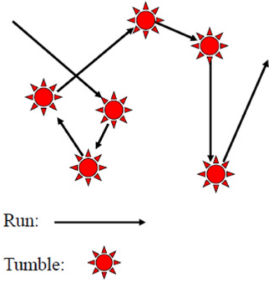
$$x_\lambda(t) \asymp_\lambda t^{\frac{\beta}{\mathbf{2}s(\beta-\mathbf{1})}}.$$

Scaling limits and hypocoercivity in kinetic theory

In short...

- We will talk about kinetic theory. More precisely, long time behaviour of linear kinetic equations coming originally from physics but that have now plenty of applications.
- Schematically, imagine that we observe a density of particules that "run" with some velocity, and "reorientate" due to internal processes or interaction with the environment.
- The specificity of the models we look at is that particules ...
  1. will **not be confined** in space ☺,
  2. will **change** their velocity **to a high velocity with a "large" probability**.

An example: the run and tumble process of E. Coli.



Persistent motion, with two phases, alternately:

→ straight run (deterministic),

$$\dot{X} = V,$$

→ change of velocity (random).

Every τ (Poissonian time), random choice of V following a density M.

The main equation of this process is a kinetic equation.

$$\underbrace{\partial_t f + v \cdot \nabla_x f}_{\text{Free transport}} = \frac{1}{\tau} \underbrace{\int_{\mathbb{R}^d} [f(v')M(v) - f(v)M(v')] \, dv'}_{\text{Reorientation}}$$

The models at hand

Density of particules $f(t,x,v)$: time $t \in \mathbb{R}^+$, position $x \in \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$.

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L} f \,,$$
$$f(0,\cdot,\cdot) = f^{\mathrm{in}}$$

Three types of *reorientation operators*:

- the generalized *Fokker-Planck*

$$\mathcal{L} f = \nabla_v \cdot \left(M \, \nabla_v \left(M^{-1} f \right) \right) ,$$

- the *linear Boltzmann* operator, or *scattering* collision operator

$$\mathcal{L} f = \int_{\mathbb{R}^d} b(\cdot, v') \left[f(v') \, M(\cdot) - f(\cdot) \, M(v') \right] dv' \,,$$

- the *fractional Fokker-Planck* operator of exponent $s \in (0,2)$.

$$\mathcal{L} f = \Delta_v^{\frac{s}{2}} f + \nabla_v \cdot (E \, f) \,,$$

All satisfy

$$\boxed{\mathrm{Ker}\, \mathcal{L} = \mathrm{Span}(M)}$$

Two crucial parameters...

First one: M decays **algebraically** with an exponent α.

$$\forall\, v \in \mathbb{R}^d, \quad M(v) = c_\alpha \lfloor v \rceil^{-d-\alpha} \quad \text{where} \quad \lfloor v \rceil = (1+|v|^2)^{\frac{1}{2}}\,.$$

Second one: The operator $\mathcal{L}$ **looses a weight** β.

- Weighted coercivity inequality

$$-Re\,\langle \mathcal{L}\varphi, \varphi\rangle_{L^2(M^{-1})} \geq \lambda \left\| \varphi - \left(\int_{\mathbb{R}^d} \varphi(v') \lfloor v' \rceil^{-\beta} dv' \right) M \right\|^2_{L^2(\lfloor v \rceil^{-\beta} M^{-1})}.$$

- Write formally as $\mathrm{B}[f] - \nu(v)\, f$ and define $-\beta$ as the exponent at infinity of the function ν.

Fokker-Planck	Scattering	Levy-Fokker-Planck
$\beta_{\mathrm{FP}} = 2$	$\int \mathrm{b}(v, v')\, M(v')\, dv' \sim \lvert v \rvert^{-\beta_{\mathrm{SC}}}$	$\beta_{\mathrm{LFP}} = s - \alpha$

Scaling limits and hypocoercivity in kinetic theory

1- Scaling limits

Fractional limit, of not fractional limit, that is the question!

Claim: Take an initial data $f^{\text{in}} \in L^2(M^{-1}dxdv)$ and rescale space and time:

$$\theta(\varepsilon)\partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon = \mathcal{L} f_\varepsilon.$$

Formally,

$$\mathrm{f}_\varepsilon \longrightarrow \mathrm{M}\rho.$$

What we want to do:

- $\rightarrow$ Find the relevant time change θ such that ρ satisfies a proper limit equation.
- $\rightarrow$ Study quantitatively the macroscopic limit $\varepsilon \to 0$. Provide rates of convergence and explicit constants.
- $\rightarrow$ Have a unified approach (as much as possible...) working for a wide type of $\mathcal{L}$'s.

Existing works ...

- Take a Gaussian M and a scattering operator. Then

$$\boxed{\theta(\varepsilon) \sim \varepsilon^2} \qquad (\text{and} \quad x \sim \varepsilon^{-1})$$

  and leads to a diffusion equation for ρ:

$$\partial_t \rho = \nabla_x \cdot (A \nabla_x \rho) .$$

  Larsen–Keller'74, Degond–Goudon–Poupaud'00.

- Roughly, **if M is a power law and has enough moments the limit is still diffusive**. **If not**, the diffusion matrix A is infinite!
- More precisely Mellet, Mischler and Mouhot considered

$$\mathcal{L}f(v) = \int_{\mathbb{R}^d} [f(v')M(v) - f(v)M(v')] \lfloor v \rceil^{-\beta} \lfloor v' \rceil^{-\beta} dv dv'$$

  with $\beta > 0$ and $\alpha \in (0, 2+\beta)$. Then

$$\boxed{\theta(\varepsilon) = \varepsilon^{\zeta := \frac{\alpha+\beta}{1+\beta}}} \qquad \text{and} \qquad \partial_t \rho = \kappa \Delta_x^{\frac{\zeta}{2}} \rho.$$

- Result reproved with different methods : Mellet'10 (moment method), Ben Abdallah–Mellet–Puel'11 (modified Hilbert expansion).

Fokker-Planck, around Puel *et al.* and Fournier *et al.*

New activity from mid-2010s on around M. Puel & collaborators and around Fournier and Tardif with probabilistic methods. Take

$$\mathcal{L}f(v) := \nabla_v \cdot \left(M \nabla_v \left(\frac{f}{M} \right) \right)$$

- Case $\alpha > 4$ in Nasreddine–Puel'15 (standard diffusion),
- Critical case $\alpha = 4$ in Cattiaux–Nasreddine–Puel'19 (standard diffusion with time scaling $\varepsilon^2 |\ln \varepsilon|$) by probabilistic method
- Case $\alpha \in (0,4)$ in dimension $d = 1$ (fractional diffusion) in Lebeau–Puel'19 by PDE method and the study of a spectral problem reminiscent of Ellis–Pinsky'75 seminal work,
- Case $\alpha \in (0,4)$ in dimension $d = 1$ in Fournier-Tardif '19 and then dimension $d \geq 2$ treated in Fournier-Tardif '20 (fractional diffusion).

$$\boxed{\theta(\varepsilon) = \varepsilon^{\frac{\alpha+2}{3}}} \qquad (\text{and} \quad x \sim \varepsilon^{-1})$$

Scaling function and diffusion coefficient

Define the *diffusion exponent*

$$\zeta = \zeta(\alpha, \beta) := \begin{cases} 2 & \text{when } \alpha \in [2+\beta, +\infty] \\ \dfrac{\alpha+\beta}{1+\beta} & \text{when } \alpha \in [0, 2+\beta), \end{cases}$$

and the *scaling function*

$$\theta(\varepsilon) := \begin{cases} \varepsilon^{\zeta} & \text{when } \alpha \in (-\beta, +\infty] \setminus \{0, 2+\beta\}, \\ \varepsilon^{\zeta} |\ln \varepsilon| & \text{when } \alpha = 2+\beta, \\ \dfrac{\varepsilon^{\zeta}}{|\ln \varepsilon|} & \text{when } \alpha = 0, \end{cases}$$

Note that the **threshold** $\alpha = 2 + \beta$ between standard and fractional diffusion corresponds to whether or not $\lfloor \cdot \rceil^{-\beta} M$ **has finite variance**.

The abstract result

Theorem (B., Mouhot '20)

Take a weak solution $f \in L^\infty_t([0,+\infty); L^2_{x,v}(M^{-1}))$ with initially, say,

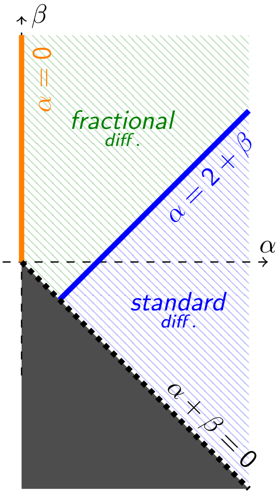
$$\left\| \frac{f_\varepsilon}{M}(0,\cdot,\cdot) - r_\varepsilon(0,\cdot) \right\|_{-\beta} \lesssim \theta(\varepsilon)^{\frac{1}{2}} \left\| \frac{f_\varepsilon}{M}(0,\cdot,\cdot) \right\|,$$

$$\lim_{\varepsilon \to 0} r_\varepsilon(0,\cdot) := r(0,\cdot) \text{ in } H^{-\zeta}(\mathbb{R}^d).$$

Under some assumptions presented later on, on $[0, T]$,

$$\lim_{\varepsilon \to 0} \frac{f_\varepsilon}{M} = r, \text{ where } r \text{ solves } \partial_t r = \kappa \Delta_x^{\frac{\zeta}{2}} r,$$

with explicit topologies, rates of convergence and coefficients.



Prepare a smart attack... : construction of a fluid mode

With a structural hypothesis on $\mathcal{L}$, for $\eta \in (0, \eta_0)$ there is a unique *eigenpair*,

Eigenvector $\phi_\eta \in L^2_v(\lfloor \cdot \rceil^{-\beta} M)$ Eigenvalue $\mu(\eta) \in (0, r_0)$

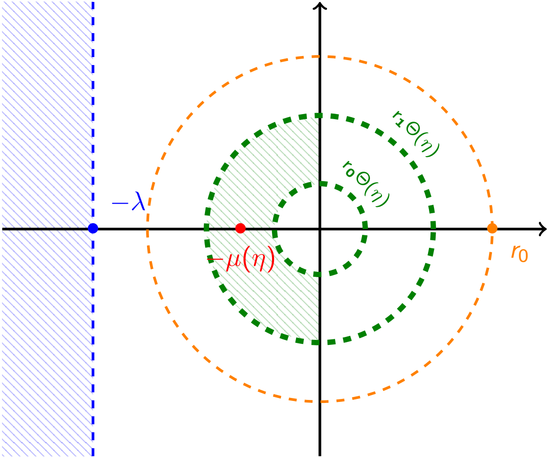
solving, with $Lh := M^{-1}\mathcal{L}(Mh)$,

$$-L^* \phi_\eta - i\eta(v \cdot \sigma)\phi_\eta = \mu(\eta) \lfloor v \rceil^{-\beta} \phi_\eta \quad \text{with} \quad \int_{\mathbb{R}^d} \phi_\eta(v) \lfloor v \rceil^{-\beta} M(v)\, dv = 1.$$

Moreover, the branch $(\phi_\eta, \mu(\eta))$ connects to $(1, 0)$ as $\eta \to 0$, with

$$\|\phi_\eta - 1\|_{-\beta} \lesssim \mu(\eta)^{\frac{1}{2}}.$$

... that you can explain on a small picture!



The main computation...

Following a strategy *a la* Ellis-Pinsky and Lebeau-Puel,

Denote $h_\varepsilon := \frac{f_\varepsilon}{M} \in L^2_{x,v}(M)$. Fourier-transform in x to get on $\hat{h}_\varepsilon(t,\xi,v)$

$$\theta(\varepsilon)\partial_t \hat{h}_\varepsilon = L\hat{h}_\varepsilon + i\varepsilon(v\cdot\xi)\hat{h}_\varepsilon.$$

Denote $\xi =: |\xi|\sigma$ and $\eta := \varepsilon|\xi|$. Test (in v) against $M\phi_\eta$,

$$\begin{aligned}\theta(\varepsilon)\frac{d}{dt}\left\langle \hat{h}_\varepsilon, \phi_\eta \right\rangle &= \left\langle L\hat{h}_\varepsilon + i\varepsilon(v\cdot\xi)\hat{h}_\varepsilon, \phi_\eta \right\rangle = \left\langle \hat{h}_\varepsilon, L^*\left(\phi_\eta\right) + i\varepsilon(v\cdot\xi)\phi_\eta \right\rangle \\ &= -\mu(\eta)\left\langle \hat{h}_\varepsilon, \lfloor v\rceil^{-\beta}\phi_\eta \right\rangle.\end{aligned}$$

Then,

$$\left\langle \hat{h}_\varepsilon, \lfloor v\rceil^{-\beta} \right\rangle = \hat{r}_\varepsilon + small, \qquad \left\langle \hat{h}_\varepsilon, \phi_\eta \right\rangle = \langle 1, \phi_\eta\rangle\, \hat{r}_\varepsilon + small$$

and thus, roughly, $\partial_t \hat{r} \approx -\lim_{\varepsilon\to 0}\left(\frac{\mu(\varepsilon\xi)}{\theta(\varepsilon)}\right)\hat{r}$.

Scaling of μ determines θ and then the limit equation!

Heuristics about the scaling ...

Start from

$$\mu(\eta) = \eta \int_{\mathbb{R}^d} (v \cdot \sigma) \, \mathrm{Im}(\phi_\eta) M(v) \, dv.$$

- Either you can phone your dearest friend Lebesgue ...

$$\mu(\eta) \sim \eta^2 \int_{\mathbb{R}^d} (v \cdot \sigma) \left(\lim_{\eta \to 0} \frac{\mathrm{Im}(\phi_\eta)}{\eta} \right) M(v) \, dv.$$

- Or you can't, since there is a "sliding hump phenomenon", and you rescale!

$$\boxed{\Phi_\eta(u) := \phi_\eta \left(\eta^{-\frac{1}{1+\beta}} u \right)}$$

and then

$$\mu(\eta) \propto \eta^{\frac{\alpha+\beta}{1+\beta}} \int_{\mathbb{R}^d} (u \cdot \sigma) \left(\lim_{\eta \to 0} \mathrm{Im}(\Phi_\eta(u)) \right) |u|^{-d-\alpha} \, dv.$$

Final comment: hypothesis are (in particular) on the size of Φ_η.

Scaling limits and hypocoercivity in kinetic theory

2 - Decay to zero and hypocoercivity

What is the question?

Claim: If the initial data has finite mass, $f^{\text{in}} \in L^1(dxdv)$, then the only integrable equilibrium (i.e. $L^1(dxdv)$) to

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L} f \,,$$
$$f(0,\cdot,\cdot) = f^{\mathrm{in}}$$

is $f \equiv 0$.

What we want to do:

→ Study quantitatively the convergence to zero. Provide rates of convergence and explicit constants.

→ Have a flexible method for this kind of problems as a preliminary brick to nonlinear problems.

A result (among a few similar others)

Theorem (B., Dolbeault, Lafleche ('20))

Let $d \geq 2$, $\alpha > 0$ and assume that β and α are such that

$$\alpha + \beta > 0\,, \quad \alpha + \beta \neq 2\,.$$

For any $k \in (0, \alpha)$,

$$\|f\|^2_{L^2(M^{-1}dx\,dv)} \lesssim \begin{cases} (1+t)^{-\frac{d}{\zeta}}\,\|f^{\rm in}\|^2_{L^2(M^{-1})\cap L^1} & \text{if} \quad \beta \leq 0\,, \\ (1+t)^{-\min\left\{\frac{d}{\zeta},\frac{k}{|\beta|}\right\}}\,\|f_0\|^2_{L^2(\lfloor v\rceil^k M^{-1})\cap L^1} & \text{if} \quad \beta > 0\,. \end{cases}$$

Remarks

- $\to$ $d = 1$ also covered with a slight change.
- $\to$ Any α, β, with $\alpha > 0$ and $\alpha + \beta > 0$ covered.
- $\to$ Use of $L^2(\lfloor v\rceil^k M^{-1} dx\,dv)$ norms only (no Sobolev).

But roughly, what is and why is hypocoercivity?

Start with the basic $L^2(M^{-1})$ energy estimate:

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2(M^{-1}dx\,dv)} \leq -\lambda \left\| \varphi - \left(\int_{\mathbb{R}^d} \varphi(v') \lfloor v' \rceil^{-\beta} dv' \right) M \right\|^2_{L^2(\lfloor v \rceil^{-\beta} M^{-1})}.$$

One cannot get any sort of decay strictly from here!

Started by Villani's memoir and with now a large literature (Villani, Mouhot, Neumann, Dolbeault, Schmeiser ....):

the intertwist between $\mathcal{L}$ and $v \cdot \nabla_x$ allows to recover a decay.

How: Re-casting the Dolbeault - Mouhot - Schmeiser method !

$$\|f\|^2_{L^2(M^{-1})} \quad \rightsquigarrow \quad \frac{1}{2}\|f\|^2_{L^2(M^{-1})} + \delta\, \mathrm{Re}\langle Af, f\rangle$$

The strategy and ingredients of the proof

1. Mode by mode analysis. Take Fourier in x.

$$\partial_t \hat{f} + i\,(v \cdot \xi)\hat{f} = \mathcal{L}\hat{f}\,,$$

2. Mode by mode hypocoercivity functional. Define a new operator A by

$$\mathrm{A}_\xi := \frac{1}{\lfloor v \rceil^2}\,\Pi\,\frac{(-\,i\,v\cdot\xi)\lfloor v \rceil^{-\beta}}{1+\lfloor v \rceil^{2\,|1-\beta|}\,|\xi|^2}$$

and the *entropy functional* by $\mathrm{H}_\xi[f] := \frac{1}{2}\|\hat{f}\|^2 + \delta\,\mathrm{Re}\langle \mathrm{A}_\xi \hat{f}, \hat{f}\rangle\,.$

3. Mode by mode and then full energy estimate

$$-\frac{d}{dt}\mathrm{H}[f] = -\,\langle f\,, \mathcal{L} f\rangle + \delta \int_{\mathbb{R}^d} \mathrm{R}_\xi[\hat{f}]\,d\xi$$

with

$$\mathrm{R}_\xi[\hat{f}] \gtrsim \mathcal{K}_1(\xi)\,\|\Pi\hat{f}\|^2 - \mathcal{K}_2(\xi)\,\|(1-\Pi)\hat{f}\|_\eta^2\,.$$

4. Conservation of weighted L^2 norms and interpolation yields to an ODE of the form:

$$H' \leq -\phi(t) H^{\smile}.$$

A summary (and other results)

▷ These results are part of other results with spatial confinement (mainly weak)

M \ V	$V \equiv 0$	$V \sim \gamma \ln(\lvert x \rvert)$	$V \sim \lvert x \rvert^\alpha$, $\alpha \in (0,1)$	$V \sim \lvert x \rvert^\alpha$, $\alpha \geq 1$
$M \lesssim e^{-\lfloor v \rceil}$	$t^{-\frac{d}{2}}$ *BDMMS*	$t^{-\frac{d-\gamma}{2}}$ *BDLS*	$e^{-\lambda t^b}$ *Cao*	$e^{-\lambda t}$ *DMS*
$M \asymp e^{-\lfloor v \rceil^\alpha}$, $\alpha \in (0,1)$	$t^{-\min\left(\frac{d}{2}, \frac{k}{\beta}\right)}$ *BDL*	Open	Open	Open
$M \asymp \lfloor v \rceil^{-d-\alpha}$, $\alpha > 0$	$t^{-\min\left(\frac{d}{\zeta}, \frac{k}{\beta_+}\right)}$ *BDL*	Open	Open	Open

Thank you for your attention !