École doctorale InfoMaths

Propagation de fronts structurés en biologie : modélisation et analyse mathématique

THÈSE

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Mis en page avec la classe thesul.

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1. Lire : "perdu pour moi".
2. Et de réponses, évidemment !

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Chapitre 2

Equations cinétiques de transport-réaction : Le cas d'un continuum de vitesses

En révision (2014)

Introduction

1 Motivations et cadre de la thèse

Cette thèse porte sur l'étude mathématique de phénomènes de propagation d'onde issus de la biologie, et plus précisément de la dynamique des populations structurées. La particularité de cette thèse est qu'elle s'appuie sur deux problématiques biologiques révélant une structure "microscopique" riche : une population de bactéries (*E. Coli*) structurée en vitesse, une population de crapauds (*Bufo marinus*) structurée en diffusion. A première vue, ces deux structures microscopiques sont complètement décorrélées, mais en fait elles s'avèrent pouvoir être étudiées mathématiquement de manière relativement similaire. Ceci est un point marquant de cette thèse.

Nous allons maintenant présenter en détail ces deux cadres biologiques.

1.1 Évolution Darwinienne, dynamique adaptative des phénotypes et invasion de crapauds buffles.

Le premier grand thème abordé dans cette thèse est l'évolution Darwinienne à l'échelle d'une population (asexuée dans la thèse, on se permettra des populations sexuées pour les motivations). Dans ce cadre, les populations d'individus seront structurées par un trait phénotypique. Rappelons qu'en génétique, génotype et phénotype sont des termes inventés par William Bateson au début du 20$^{\text{ème}}$ siècle. Ce dernier proposa en 1905 le terme "génétique" pour désigner la science de l'hérédité et de la variation. Le génotype détermine les caractères d'un individu, constituant le phénotype, et se transmet des parents à leurs descendants. Le phénotype est l'ensemble des caractères observables d'un individu, à toutes les échelles : macroscopique (la couleur des yeux, la longueur du cou), cellulaire (la concentration sanguine en hématies) et moléculaire (l'activité d'une enzyme).

Pour étudier ces populations structurées du point de vue de la biologie de l'évolution, on part de plusieurs concepts fondamentaux initiés par Darwin [70]. Ces principes dicteront donc les différents modèles mathématiques qui visent à décrire l'évolution Darwinienne. Il faut bien noter que nous n'avons pas la prétention de décrire la complexité des phénomènes d'évolution dans leur globalité, mais seulement de répondre à des problématiques simples. Détaillons les principes en quelques mots :

La sélection naturelle. Elle explique comment l'environnement influe sur l'évolution des espèces et des populations en sélectionnant les individus les plus adaptés. La sélection naturelle

FIGURE 1 – DIFFÉRENTS PHÉNOTYPES DE PEAU CHEZ LE POISSON SERGENT-MAJOR (*Abudefduf*). SOURCE : WIKIPÉDIA.

est essentiellement le fait que les traits qui favorisent la survie et la reproduction voient leur fréquence s'accroître d'une génération à l'autre. Elle repose sur trois principes :

La variation : Au sein d'une population, les phénotypes doivent présenter des variations (voir Figure 1).

GIRAFE ET SON GIRAFON.

L'hérédité : Les caractéristiques des individus doivent être héréditaires, c'est-à-dire qu'elles doivent pouvoir être transmises à leur descendance. Lors de la reproduction, ce sont donc les gènes qui, transmis aux descendants, entraîneront le passage de certains caractères d'une génération à l'autre. Dans le cas de reproductions asexuées, l'ADN du parent transmet directement tout le matériel génétique à son descendant. Dans le cas de reproductions sexuées (non traitées dans cette thèse), les allèles des parents se mélangent pour former le matériel génétique de l'enfant.

L'adaptation : Certains individus ont un trait phénotypique particulier qui leur permet de se reproduire davantage que les autres, dans un environnement précis. Ils disposent alors d'un avantage sélectif sur leurs congénères. Par exemple, on peut imaginer que les girafes ont un long cou pour attraper plus facilement les feuilles des arbres dans la savane. C'est dans ce principe d'adaptation uniquement qu'intervient l'environnement.

UNE GIRAFE ET UN ARBRE À GIRAFE.

Les mutations génétiques. Une mutation est une modification irréversible de l'information génétique dans le génome d'une cellule. C'est donc une modification de la séquence de l'ADN. Les mutations sont en général rares [82], et celles qui modifient vraiment les phénotypes le sont encore plus. Mais elles existent, et c'est l'une des causes principales de l'évolution des espèces, voir Figure 2.

On se pose alors la question de la dynamique adaptative au sein de ces populations structurées. Il s'agit, en prenant comme base les principes détaillés ci-dessus, de comprendre l'évolution des espèces et en particulier la répartition et l'évolution des phénotypes au cours du temps. En faisant donc l'hypothèse que les mutations sont des évènements très rares, quel phénotype sera dominant au bout d'un temps très long ?

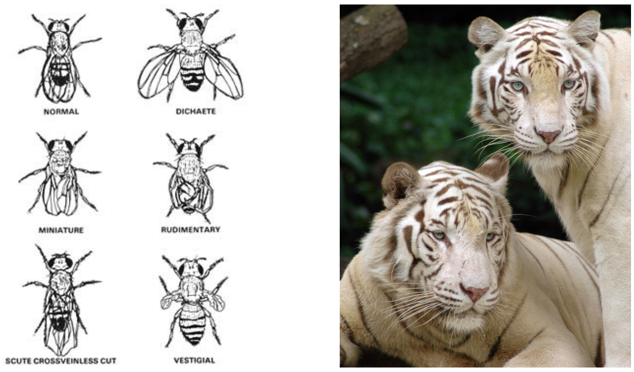


Figure 2 – Mutations chez la drosophile (gauche). Leucistisme chez le tigre (droite).

Des travaux mathématiques récents très importants (que l'on citera et sur lesquels on reviendra dans la partie suivante) établissent et étudient des modèles d'Équations aux Dérivées Partielles (EDP) ou les variables sont le temps et le trait phénotypique. L'inconnue du problème est alors une somme de masses de Dirac qui évolue avec le temps et dont le poids de chacun indique la proportion du phénotype correspondant au sein d'une certaine population d'individus. Manipuler des masses de Dirac comme solution de certaines EDP n'est pas si facile et les techniques employées dans ce cadre sont très importantes pour cette thèse.

Le bémol de ces premiers modèles est que la variable d'espace n'est pas prise en compte simultanément au trait phénotypique. Or, généralement, pour survivre dans un certain environnement, un individu doit avoir un trait phénotypique proche d'un certain trait particulier, le trait le mieux adapté, qui peut dépendre de l'espace (voir "Adaptation", ci-dessus). Par exemple, la couleur des feuilles d'un arbre d'une même espèce peut changer en fonction de la latitude ou de l'élévation, ou plus généralement les changements climatiques influent sur une population [84]. Citons [147] ainsi que les références qui s'y trouvent pour une présentation des interactions entre écologie et évolution. En conséquence de quoi, dans certaines situations en fait très communes dans la nature, la dynamique spatiale d'une population dans un environnement qui présente des variations provoque la sélection dynamique de traits phénotypiques particuliers au cours de leur déplacement.

Un exemple de ce phénomène, qui a motivé cette thèse, est l'impressionnante invasion de crapauds buffles *(Bufo Marinus)* en Australie. Ces crapauds ont été introduits aux alentours de 1930 à l'est de l'Australie (Queensland) pour débarrasser les agriculteurs d'un parasite dans les plantations de canne à sucre. En plus de ne pas avoir si clairement rempli leur mission initiale, ils ont commencé à envahir le continent Australien. Leur peau contenant des pustules de poison très puissant, ils ont relativement peu de prédateurs et peuvent donc progresser très librement. Des biologistes de terrain [199] ont mesuré sur des dizaines d'années la position et

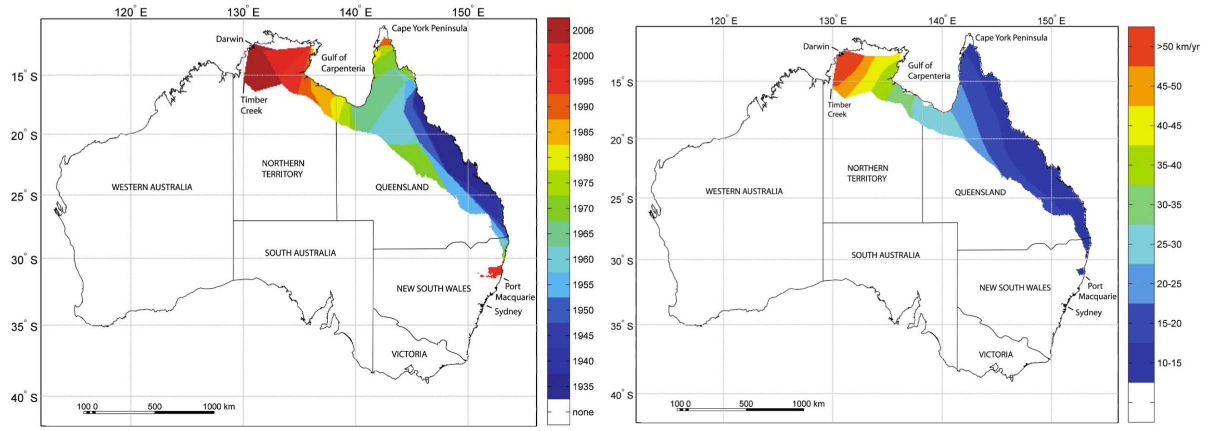


FIGURE 3 – AVANCÉE DES CRAPAUDS BUFFLES AVEC LES ANNÉES (GAUCHE). VITESSE D'INVASION DES CRAPAUDS BUFFLES LES PLUS RAPIDES (DROITE). FIGURES PRISES DE [199].

Olympic Village Effect

"These frontline toads are reproducing with each other to create what scientists call the "Olympic Village Effect." They are making fast moving babies with bigger front legs and longer back legs."

Killing cane toads - Sunshine Coast Daily (Tweed Heads, New South Wales, Australia) Dec. 2, 2008.

la vitesse de l'invasion au cours du temps, voir Figure 3. La vitesse d'invasion a été multipliée par 5 entre le début de l'invasion et aujourd'hui. Les animaux à l'avant du front d'invasion ont toujours l'endurance (que l'on pourrait assimiler à la longueur de leurs pattes) et la résistance la plus élevée au cours du temps (alors que ces caractères sont relativement homogènes à l'arrière). Les mutations génétiques seules ne peuvent pas expliquer un tel phénomène. Il s'agit d'une adaptation dynamique par une sélection à l'avant du front des individus les plus mobiles. Dans cette thèse, on étudiera des modèles (simples à énoncer mais compliqués à analyser !) qui permettent de quantifier les différentes caractéristiques de la dynamique invasive de ce désastre écologique. A noter qu'il s'agit d'un exemple parmi de nombreux autres faits du même type, voir par exemple mouches, papillons et criquets ([133, 194] et les articles qui y sont cités) et même la croissance de tumeurs [172]. Tout ceci illustre la problématique plus générale de l'évolution au sein d'un front d'invasion.

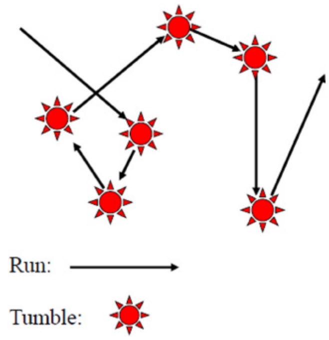
1.2 Mouvement collectif de bactéries, *e.g.* Escherichia coli.

Le second sujet qui a motivé cette thèse est le mouvement collectif de populations de bactéries. La bactérie *Escherichia coli* est un être unicellulaire qui se déplace en nageant à l'aide d'un faisceau de 5 à 6 flagelles hélicoïdal. Lors de son déplacement, la bactérie répète deux phases [27] :

– **Marche en ligne droite.** Vitesse de l'ordre de $20\mu m.s^{-1}$. Durée d'environ $1s$, "run" en anglais,

– **Changement de direction.** Un procédé interne à la bactérie fait que l'un ou plusieurs des flagelles sortent du faisceau : la bactérie change de direction. Durée : environ $0.1s$, "tumble" en anglais.

Le schéma ci-contre provient de [54].



Ainsi, en itérant ce processus, la bactérie effectue un processus erratique de sauts en vitesse qui ressemble à une marche aléatoire ("velocity jump process"). Par ailleurs, les bactéries sont capables de communiquer entre-elles via un signal chimique que chacune d'entre elles émet. Grâce à cet effet, des populations de bactéries sont capables de s'auto-organiser. Cet effet apparait dans de nombreuses situations et est fondamental en biologie du développement en général.

On s'intéresse plus particulièrement à la capacité d'une population de bactéries à suivre un signal chimique fixé à l'avance, par exemple un gradient de glucose. Des expériences effectuées par des biologistes à l'Institut Curie [186] montrent alors qu'un patch de bactéries, initialement déposé à l'extrémité gauche d'un canal étroit contenant un gradient de nutriment vers la droite, reste concentré au cours du temps, et donc que la population se propage dans le canal sous forme d'une onde solitaire. De plus, un important travail de suivi des trajectoires individuelles des bactéries a montré que dans ce mouvement macroscopique, chacune des bactéries effectue la trajectoire erratique décrite précédemment à l'échelle microscopique. Il a alors été prouvé [185] que pour décrire fidèlement le phénomène il était nécessaire de conserver la donnée microscopique des trajectoires des bactéries. Plus mathématiquement, il est nécessaire d'utiliser un modèle cinétique complet pour avoir une description fidèle de la forme et de la vitesse des fronts pulsatoires : une limite diffusive de ce dernier n'est pas pertinente. Nous renvoyons à [5, 175, 85, 151, 179, 186, 185] ainsi qu'à la section 2.3 de cette introduction pour des modèles cinétiques basés sur des processus de saut en vitesse pour modéliser des populations de bactéries à une échelle plus fine que l'échelle macroscopique. Ainsi, alors que dans le cas de l'évolution Darwinienne les populations sont structurées par le trait phénotypique, dans ce second exemple la population est structurée microscopiquement par la vitesse.

Dans le corps de cette thèse, il ne sera pas du tout question de chimiotactisme. On étudiera plus généralement des modèles cinétiques de transport-reaction pour lesquels on peut s'attendre à des phénomènes de front de propagation. Néanmoins, les modèles cinétiques pour les ondes de concentration de bactérie restent une motivation importante de ce travail.

2 État de l'art et principaux outils

Au vu du cadre et des motivations biologiques présentés précédemment, disons brièvement et une fois pour toutes que les populations seront représentées par des densités n (respectivement f) dépendantes du temps $t \in \mathbb{R}^+$, de l'espace $x \in \mathbb{R}^n$, et de la troisième variable structurante (sauf mention contraire) : le trait phénotypique, souvent noté $\theta \in \Theta$, respectivement la vitesse, souvent notée $v \in V$. Ainsi, les modèles étudiés sont des modèles de type cinétique.

Venons maintenant aux résultats déjà existants et très importants pour cette thèse. D'abord, nous allons rappeler certains points de vue existants pour étudier les phénomènes de propagation dans les équations de réaction-diffusion de type Fisher-KPP : ondes progressives, point de vue de l'optique géométrique. Ensuite, on parlera des résultats existants pour décrire la dynamique des phénotypes dans des populations ou la variable spatiale n'est pas considérée. Enfin, on présentera les modèles cinétiques décrivant les ondes de concentration de bactéries qui ont inspiré les modèles étudiés dans la thèse. On citera enfin certains travaux concernant l'étude de divers profils d'ondes cinétiques qui ont été importants pour ce travail.

2.1 Dynamique spatiale : Equations de réaction-diffusion.

Les travaux pionniers pour la propagation spatiale d'espèces indifférenciées sont dus à Fisher [97] et Kolmogorov-Petrovskii-Piskounov (KPP) [143]. Ils considèrent que les populations peuvent envahir librement un espace donné, ceci est modélisé par un opérateur de diffusion de type Laplacien avec diffusivité $D > 0$, et peuvent se reproduire avec un taux $r > 0$. Un effet de saturation par les ressources est aussi pris en compte. L'équation dite de Fisher-KPP est la suivante :

$$\frac{\partial n}{\partial t} - D\Delta n = rn(1-n), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n. \tag{1}$$

Il est possible et courant de considérer des termes de réaction différents selon la situation modélisée. Il y a essentiellement quatre types de tels termes : Le type KPP (ici présent), le type monostable, le type bistable et enfin le type ignition. Comme ces trois derniers ne seront pas utilisés dans ce mémoire nous ne les présentons pas en détail et faisons référence à [143, 127].

Revenons à (1). On observe alors que $n \equiv 1$ est une solution stationnaire : *la population est présente*, tout autant que $n \equiv 0$: *la population est absente*. Le deuxième état ($n \equiv 0$) est instable en l'absence de diffusion. On cherche alors à savoir si cette équation peut décrire un phénomène d'invasion, autrement dit l'état $n = 1$ peut il envahir l'état $n = 0$? Si oui, à quelle vitesse ?

Ondes progressives. On traite le cas unidimensionnel ($x \in \mathbb{R}$). Une solution en onde progressive (plane) de (1) est une solution n de la forme $n(t,x) = \bar{n}(x - ct)$, ou $\bar{n}$ est *le profil* et c est *la vitesse de propagation* de l'onde. Cette définition ce généralise au cas multidimensionnel en définissant une direction de propagation $e \in \mathbb{S}^{n-1}$ et le profil sous la forme $n(t,x) = \bar{n}(x \cdot e - ct)$.

On sait depuis Fisher [97], Kolmogorov-Petrovski-Piskunov [143], Aronson Weinberger [10] et Fife-McLeod [95] que (1) admet de telles solutions positives vérifiant

$$\bar{n}(-\infty) = 1, \qquad \bar{n}(+\infty) = 0,$$

pour toute vitesse plus grande qu'une vitesse minimale : $c \geq c^* := 2\sqrt{rD}$. Ces solutions sont uniques à translation près.

Le profil $\bar{n}$ correspondant est alors exponentiellement décroissant en l'infini. Il est important de garder en tête pour la suite de la thèse comment faire apparaître le phénomène de vitesse minimale. On part du principe que le front de propagation est tiré vers l'avant par les petites populations qui se reproduisent quasi exponentiellement. Mathématiquement parlant, le phénomène de propagation est donné par le problème linéarisé autour de $n \ll 1$. Le profil $\bar{n}$ et la vitesse c vérifient alors

$$-c\bar{n}' - D\bar{n}'' = r\bar{n}, \qquad \xi \in \mathbb{R}.$$

Ceci est une équation différentielle ordinaire, dont on peut chercher des solutions exponentiellement décroissantes $\bar{n} = e^{-\lambda\xi}$; le taux de décroissance λ vérifie alors :

$$D\lambda^2 - c\lambda + r = 0.$$

Ce polynôme du second degré n'ayant de racines réelles que si $c^2 - 4rD \geq 0$, on en déduit la vitesse minimale cherchée, associée au taux de décroissance $\lambda^* = \frac{c^*}{2D} = \sqrt{\frac{r}{D}}$. Dans ce cas précis, comme la recherche de profils d'ondes progressives se ramène à la résolution d'une EDO avec condition aux limites, un raisonnement de type "portrait de phase" permet de justifier rigoureusement leur existence [10]. Dans la suite de cette thèse, on se posera la question de l'existence d'ondes progressives pour des modèles plus complexes, pour lesquels les raisonnements de ce type ne seront plus utilisables directement. En revanche, l'heuristique du problème linéarisé sera toujours très importante.

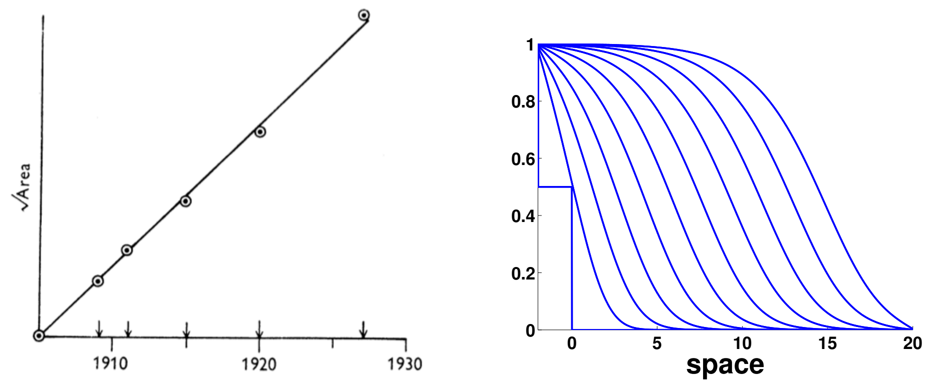


Figure 4 – (Gauche) Racine carrée de l'aire envahie par des ratons-laveurs : propagation linéaire [195]. Simulation de fronts de propagation solutions de (1) à partir d'une donnée initiale à support compact.

L'intérêt de la recherche d'ondes progressives est (notamment) le comportement en temps long de (1). Celui ci est bien compris depuis Kolmogorov-Petrovsky-Piskunov [143] pour une donnée initiale particulière et Aronson-Weinberger [11] pour le cas général. Pour une condition initiale suffisamment décroissante à l'infini, la solution se comporte asymptotiquement comme une onde progressive de vitesse minimale $s = 2\sqrt{rD}$. On sait depuis [11] que l'on a les propriétés d'invasions suivantes :

$$\forall c < c^*, \qquad \lim_{t\to\infty} \min_{|\xi|\leq ct} n(t,x) = 1,$$

et

$$\forall c > c^*, \qquad \lim_{t\to\infty} \sup_{|\xi|\geq ct} n(t,x) = 0.$$

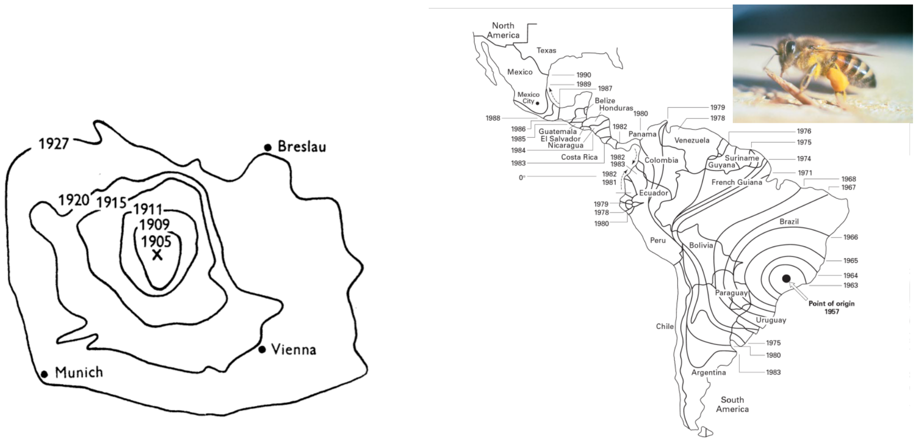


FIGURE 5 – ZONES ENVAHIES LORS DE : (GAUCHE) LA PROPAGATION DE RATONS-LAVEURS AUTOURS DE PRAGUE [195], (DROITE) L'INVASION DE L'ABEILLE AFRICANISÉE EN AMÉRIQUE [115].

Bramson [40] et plus récemment Hamel et al [121] ont démontré des estimations plus précises sur les lignes de niveau de n :

$$\lim_{t\to\infty} \min_{|\xi|\leq C} \left\| n(t,\cdot) - \bar{n}\left(\cdot - c^*t + \frac{3}{2\lambda^*}\ln(t) + \xi\right) \right\|_{L^\infty(0,+\infty)} = 0.$$

Il faut tout de même signaler que si la donnée initiale n'est pas suffisamment décroissante, la solution peut se propager plus rapidement [123]. Finalement, l'onde progressive est stable dans des espaces d'énergies à poids [104].

Point de vue de l'optique géométrique. Un autre point de vue sur la question de la propagation dans les équations de réaction-diffusion est le point de vue dit de l'optique géométrique. Expliquons et illustrons cette approche sur l'équation de Fisher-KPP (1). Une manière plus faible que l'onde progressive de caractériser une dynamique d'invasion est d'effectuer un changement d'échelle et de regarder de très loin la population se propager, et ce pendant un temps très long. Imaginons que l'on regarde l'invasion des crapauds en Australie depuis un des satellites de la Terre, on observera alors des zones, comme en Figure 5. On disposera alors de manière grossière d'une zone ou la population est présente, d'une zone ou elle est absente, et d'une frontière entre les deux : la transition entre les deux zones (qui correspond à la forme du profil dans le formalisme des ondes progressives) est réduite à une courbe. Mais ce n'est pas grave ! Car on a tout de même accès de manière raisonnable à l'évolution de l'invasion. Le formalisme mathématique est le suivant. Effectuons le *changement d'échelle hyperbolique* $(t,x) \mapsto \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$ dans l'équation (1), il vient alors :

$$\frac{\partial n}{\partial t} - \varepsilon D \Delta n = \frac{r}{\varepsilon} n(1-n), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

On s'attend à la limite $\varepsilon \to 0$ à ce que n^ε devienne singulier (au niveau de la frontière entre les deux zones). Ce n'est donc pas une quantité très informative *a priori*. Suivant [103, 88], il faut effectuer *la transformation de Hopf-Cole* suivante :

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \qquad n^\varepsilon = \exp\left(-\frac{\varphi^\varepsilon}{\varepsilon}\right). \tag{2}$$

La phase φ^ε est alors positive et satisfait l'équation de Hamilton-Jacobi diffusive :

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \qquad \partial_t \varphi^\varepsilon + D|\nabla_x \varphi^\varepsilon|^2 + r = \varepsilon D \Delta_x \varphi^\varepsilon + r\rho^\varepsilon \tag{3}$$

La théorie des solutions de viscosité des équations d'Hamilton-Jacobi [67, 14] permet de passer à la limite $\varepsilon \to 0$ dans (4.4). En effet, φ^ε converge localement uniformément vers φ^0, l'unique solution de viscosité de l'équation de *Hamilton-Jacobi avec contrainte de positivité* :

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \qquad \min\left\{\partial_t \varphi^0 + D|\nabla_x \varphi^0|^2 + r, \varphi^0\right\} = 0. \tag{4}$$

On trouvera les éléments de preuve dans [88].

La dernière étape du raisonnement est de comprendre comment décrire la propagation à partir de φ^0. On peut prouver [90, 19, 98] que la population est contenue dans l'ensemble des zéros de φ^0. On peut décrire cet ensemble en utilisant la formulation Lagrangienne de l'équation de Hamilton-Jacobi, et retrouver alors la propagation à la vitesse minimale $c^* = 2\sqrt{rD}$. Il est bon de noter qu' effectuer la transformation de Hopf-Cole revient à étudier les queues exponentielles des distributions, comme dans le formalisme des ondes progressives.

Dans cette thèse, on utilisera pleinement ce formalisme, que l'on appliquera aux modèles structurés. L'enjeu sera alors de réussir à passer à la limite, sachant que plusieurs échelles différentes seront imbriquées dans les équations. Néanmoins, la base sera la même que ce qui vient d'être rappelé ici.

2.2 Dynamique adaptative.

Dans cette sous-section, pour reprendre les notations standards de la littérature, "x" représente le trait phénotypique (la variable spatiale est absente).

La question de la dynamique adaptative de phénotypes dans une population est importante dans cette thèse. Des travaux importants fondateurs de la théorie sont ceux de Geritz et al, Metz et al [109, 110] puis Champagnat et al, Diekmann (entre autres) [56, 58, 60, 57]. Nous ne décrivons ici que certains aspects de la théorie qui a été développée depuis, qui sont importants pour ce mémoire et qui concernent la dynamique adaptative. Un point de vue asymptotique basé une nouvelle fois sur les équations de Hamilton-Jacobi contraintes a été initié dans [79]. Ils étudient une équation de sélection-mutation décrivant l'évolution de micro-organismes vivant dans un chemostat [78, 79] à deux nutriments. Prenons ici un cas contenant un seul nutriment, voir aussi l'Annexe B. L'équation est du type :

$$\begin{cases} \frac{\partial S}{\partial t} = d\big(S_{\text{in}} - S(t)\big) - S(t)\int_{-\infty}^{\infty} \eta(x) n(x,t)dx, \\ \frac{\partial n}{\partial t} = -dn(x,t) + (1-\mu)S(t)\eta(x)n(x,t) + \mu S(t)\int_{-\infty}^{\infty} K(x,y)\eta(y)n(y,t)dy. \end{cases}$$

Le nutriment qui se renouvelle perpétuellement avec un taux $d > 0$ est $S(t)$, tandis que K est le noyau de mutations. Alors, en temps long et dans la limite des mutations petites (via un noyau concentré, *e.g.* $K_\varepsilon := \frac{1}{\varepsilon}K\left(\frac{x}{\varepsilon}\right)$), la phase $\varphi^\varepsilon := -\varepsilon \ln(n^\varepsilon)$ converge vers une équation de Hamilton-Jacobi du type [79] :

$$\frac{\partial \varphi}{\partial t} = R(x, I(t)) + \mathrm{H}\left(\frac{\partial \varphi}{\partial x}\right),$$

ou l'Hamiltonien H dépend seulement de la forme du terme de mutations [79], et

$$R(x, I(t)) := -d + \frac{dS_{\text{in}}}{d + I(t)}.$$

On trouvera dans [79] des calculs similaires dans le cas de deux ressources. On s'intéresse alors à la dynamique des phénotypes et en particulier aux phénomènes de concentration et branchements. On trouvera dans [79] des simulations numériques où des branchements de phénotypes apparaissent. De manière plus générale, des modèles comme

$$\frac{\partial n}{\partial t} - \varepsilon \Delta n = \frac{n}{\varepsilon} R(x, I(t)),$$

avec

$$I(t) := \int_{\mathbb{R}} \psi(x) n(t, x) dx,$$

ont été étudiés avec le formalisme Hamilton-Jacobi dans [20, 149, 18, 178]. Les auteurs obtiennent rigoureusement des équations de Hamilton-Jacobi avec contrainte :

$$\begin{cases} \frac{\partial \varphi}{\partial t} + |\nabla_x \varphi|^2 + R(x, I(t)) = 0, \\ \min_{x \in \mathbb{R}^n} (\varphi(t, \cdot)) = 0, \qquad \forall t > 0. \end{cases}$$

Dans cette équation, $I(t)$ est un multiplicateur de Lagrange associé à la contrainte du minimum fixé.

L'information de l'évolution des phénotypes dominants est alors donnée par le lieu des points où φ atteint son minimum. Une équation canonique comme dans [77, 79] est dérivée dans les travaux [18, 149]. Il s'agit d'une équation différentielle sur la position d'un trait dominant $\bar{x}(t)$, point de minimum local de $\varphi(t, \cdot)$. Elle prend la forme :

$$\frac{d\bar{x}}{dt} = \left(-D^2 \varphi(t, \bar{x}(t))\right)^{-1} \nabla_x R(\bar{x}, I(t)).$$

Elle contient deux éléments importants que l'on reverra dans cette thèse. La Hessienne de φ qui correspond en fait à la diversité locale autour du trait dominant, et le gradient de la fitness R, qui représente l'avantage sélectif relatif d'un phénotype par un autre.

Des modèles incluant aussi une compétition dépendante du trait ont aussi été étudiés pour prendre en compte le fait que la compétition peut être plus forte entre des individus ayant des traits proches. Une équation importante dans ce cadre est *l'équation de Fisher-KPP non-locale*, voir [156, 76, 118, 119, 117, 26] ainsi que l'Annexe B de ce mémoire :

$$\frac{\partial n}{\partial t} - \lambda \frac{\partial^2 n}{\partial x^2} = n\,(1 - K * n(t, \cdot)), \qquad t \geq 0,\ x \in \mathbb{R}. \tag{5}$$

Cette équation montre des phénomènes de spéciation dans certaines configurations, voir Figure 6 ainsi que l'Annexe B.

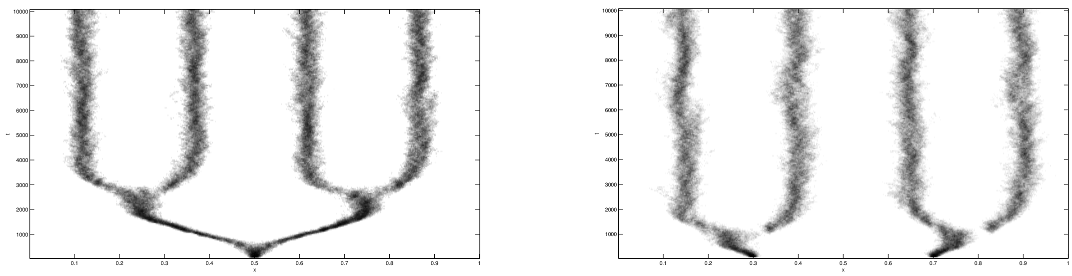


FIGURE 6 – DYNAMIQUE DE TRAITS PHÉNOTYPIQUES VIA SIMULATIONS DE MONTE-CARLO, VOIR L'ANNEXE B POUR LES DÉTAILS DES SIMULATIONS.

Finissons cette partie en ajoutant que certains travaux dont [75] utilisent des modèles intégro-différentiels pour décrire directement l'évolution de masses de Dirac sans utiliser le formalisme Hamilton-Jacobi. De plus, on pourra se référer à [162] pour de nombreux résultats et détails sur la dynamique adaptative.

2.3 Interactions chimiotactiques et ondes de concentration de bactéries : Keller-Segel et Dunbar-Alt.

Dans cette sous-section, on revient à la modélisation du déplacement collectif de certaines populations de bactéries pour préciser les modèles cinétiques dont il est question dans la partie précédente. Un modèle macroscopique pour le mouvement collectif de cellules qui interagissent via un signal chimique de cellules en déplacement est donné par le système parabolique de Keller-Segel (KS) [177, 138, 139] :

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \Delta \rho - \nabla \cdot (\chi \rho \nabla S) \ , \quad t > 0,\ x \in \Omega \subset \mathbb{R}^2, \\ \varepsilon \dfrac{\partial S}{\partial t} = \Delta S + \rho - \alpha S \ . \end{cases} \tag{6}$$

Le système de (6) présente un phénomène de masse critique. Ceci a été montré dans [81, 29]. Dans l'espace tout entier $\Omega \subset \mathbb{R}^2$ avec $\alpha, \varepsilon = 0$, pour une donnée initiale n_0 telle que $n_0 \left(|\log n_0| + (1 + |x|^2) \right) \in L^1$, la solution est globale en temps si $\chi M < 8\pi$, alors qu'elle explose en temps fini si $\chi M > 8\pi$. Ce type de résultat est raisonnable pour décrire un phénomène d'agrégation de cellules. De la même manière, (6) a été proposé comme base pour modéliser la propagation d'ondes solitaires de colonies de bactéries (voir par exemple [198]). Or il apparait que la vitesse d'advection $\chi \nabla c$ peut devenir singulière en cas d'agrégation. A l'inverse, la vitesse d'advection s'annule loin du lieu de l'onde, ce qui compromet le confinement de l'onde. Ceci motive l'utilisation de modèles cinétiques (mésoscopiques), afin de tenir compte au plus près des caractéristiques individuelles de déplacement des bactéries.

Nous avons vu au début de cette introduction que les modèles mésoscopiques prennent en compte la distribution des bactéries en vitesse à chaque position d'espace. Le modèle cinétique de Alt [5] (voir aussi Othmer, Dunbar et Alt [175]) décrit la densité f de particules ayant la vitesse $v \in V \subset \mathbb{R}^d$ en position $x \in \mathbb{R}^d$ à l'instant $t \in \mathbb{R}^+$. Dans le cadre du chimiotactisme, les cellules créent un potentiel chimique S. L'équation cinétique qui décrit le mouvement des populations prend en compte la phase de "run" via un terme de transport libre ①, la phase de "tumble" étant modélisée par un processus de saut en vitesse dont le noyau de transition dépend de signaux chimiques S et N ② :

$$\begin{cases} \underbrace{\partial_t f + v \cdot \nabla_x f}_{①} = \underbrace{\int_{v' \in V} \left(T[S,N](t,x,v,v') f(t,x,v') - T[S,N](t,x,v',v) f(t,x,v) \right) dv'}_{②}, \\ \partial_t S - \Delta S + \alpha S = \rho, \qquad (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d. \\ \partial_t N - \Delta N = -\rho N, \qquad (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d. \end{cases} \tag{7}$$

Le noyau T décrit le taux de changement de direction d'une vitesse v' à une vitesse v. Son choix est important du point de vue de la modélisation. Il est connu qu'une limite d'échelle diffusive de (7) permet de retrouver des équations d'advection-diffusion [175, 55, 85, 134, 12, 71]. Des systèmes couplés du type (7) ont été étudiés dans leur généralité (voir par exemple [38, 39]) : existence globale et phénomènes de masse critique.

La question de la modélisation des ondes pulsées a été discutée récemment par Saragosti et al. [186, 185]. Dans le premier article [186], les auteurs étudient un modèle certes de type Keller-Segel, mais dont les flux chimiotactiques proviennent clairement d'un modèle cinétique de type (7). En pratique, la vitesse d'advection est proportionnelle à la direction du gradient (et non au gradient lui-même), ce qui est justifié par la limite de diffusion de (7) pour un noyau T bien choisi. Les auteurs donnent alors théoriquement la vitesse de propagation ainsi que la forme du profil de l'onde pulsée. Les données expérimentales permettent de confirmer ces résultats théoriques. Cependant, dans le second travail [185], il est indiqué que pour décrire correctement la population de bactéries tant à l'échelle macroscopique que microscopique, il est parfois nécessaire d'étudier le modèle cinétique complet (7) (avec un noyau T bien choisi). L'existence d'agrégats non singuliers stationnaires dans un cas particulier a été récemment prouvé par Calvez *et al* [50] mais l'existence d'ondes progressives pour le modèle complet est encore ouverte.

2.4 Un peu plus d'ondes cinétiques pour des équations issues de la physique.

Dans ce mémoire, on discutera d'existence d'ondes progressives pour des modèles cinétiques. Signalons qu'au delà des questions liées au chimiotactisme, cette question a suscité beaucoup d'intérêt de la part de la communauté mathématique dans les vingt derrières années, et pour des équations diverses. Citons quelques travaux en particulier.

En ce qui concerne l'équation de Boltzmann, Caflisch et Nicolaenko en construisent des solutions faibles de type chocs en utilisant une décomposition "micro-macro" [47]. La positivité de ces chocs est une question difficile qui a été résolue par Liu et Yu vingt ans plus tard [148]. Entre temps, Golse [114] utilise la compacité donnée par les lemmes de moyenne cinétiques pour construire des ondes progressives pour le modèle de Perthame-Tadmor, qui intervient dans le cadre la formulation cinétique des lois de conservation scalaires [180]. A noter que ce procédé n'est pas constructif, et que seule la vitesse (et pas la décroissance du profil) du front est donnée dans [114] par une relation de type Rankine-Hugoniot. Dans [44], un argument de compacité similaire à celui utilisé par Golse dans [114] permit de prouver l'existence et la positivité d'ondes progressives pour une équation de type BGK (qui sont des simplifications de l'équation de Boltzmann [28]) non-linéaire. De même, Ben Adballah et Schmeiser [21] ont étudié l'existence de chocs cinétiques pour un modèle de semi-conducteurs. Enfin, la décomposition micro-macro a été utilisée pour construire des ondes dans un régime particulier d'une équation de Fisher-KPP cinétique (on y reviendra grandement dans la suite).

3 Résultats obtenus dans cette thèse

Nous présentons maintenant les résultats obtenus dans cette thèse. Nous organisons cette présentation dans le même ordre que les chapitres qui suivent cette introduction.

3.1 Un modèle cinétique à deux vitesses.

Fort du constat que les modèles cinétiques sont particulièrement bien adaptés pour décrire l'apparition d'ondes pulsées dans des populations de bactéries [3, 186, 185], nous considérons dans un premier temps la question de l'existence d'ondes progressives résultant de la combinaison de phénomènes de croissance et de transport hyperbolique. Ces modèles, contrairement au modèles cinétiques de chimiotactisme, prennent en compte des phénomènes de reproduction : l'effet de propagation n'est pas la conséquence des flux de cellules liés aux différents signaux chimiques dans l'environnement, mais par la division cellulaire, comme pour les modèles de type réaction-diffusion décrits précédemment. En effet, comme il a été rappelé au dessus, depuis Fisher et KPP [97, 143] l'invasion d'espèces biologiques a souvent été modélisée par des équations de réaction-diffusion. Dans ce premier travail, nous étudions un modèle introduit par Dunbar et Othmer [83], Hadeler [120], Holmes [131], Méndez *et al* [165, 167, 99, 173], et Fedotov [92, 93, 94] :

$$\varepsilon^2\partial_{tt}n_\varepsilon(t,x) + \left(1-\varepsilon^2F'(n_\varepsilon(t,x))\right)\partial_t n_\varepsilon(t,x) - \partial_{xx}n_\varepsilon(t,x) = F(n_\varepsilon(t,x)), \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}\,. \tag{8}$$

Dans cette équation, $\varepsilon > 0$ est un paramètre d'échelle qui sera très important dans la suite. La non-linéarité F sera de type KPP (ou monostable), on pourra penser à $F(s) = s(1-s)$. Le Chapitre 1 précise ces différentes hypothèses. Conformément au cahier des charges demandé au modèle, (8) est équivalent au système hyperbolique à deux vitesses suivant

$$\begin{cases} \partial_t f^+(t,x) + \varepsilon^{-1}\partial_x f^+(t,x) = \dfrac{\varepsilon^{-2}}{2}\left(f^-(t,x) - f^+(t,x)\right) + \dfrac{1}{2}F(\rho(t,x)) \\ \partial_t f^-(t,x) - \varepsilon^{-1}\partial_x f^-(t,x) = \dfrac{\varepsilon^{-2}}{2}\left(f^+(t,x) - f^-(t,x)\right) + \dfrac{1}{2}F(\rho(t,x))\,. \end{cases} \tag{9}$$

On étudie alors l'existence d'ondes progressives positives pour (8). Il existe comme pour l'équation de Fisher-KPP un continuum de vitesses admissibles pour cette existence, tout autant qu'une vitesse minimale d'existence. On observe néanmoins une transition entre deux régimes, l'un parabolique, l'autre hyperbolique. Dans le premier régime, donné par $\varepsilon^2F'(0) < 1$, la vitesse minimale est donnée par [92] :

$$c^*(\varepsilon) = \frac{2\sqrt{F'(0)}}{1+\varepsilon^2F'(0)}\,, \quad \text{si } \varepsilon^2F'(0) < 1\,. \tag{10}$$

Théorème 0.1 (Régime parabolique)**.** *On suppose que* $\varepsilon^2F'(0) < 1$. *Alors :*

(a) *Il n'existe aucune solution en onde progressive de vitesse $c \in [0, c^*(\varepsilon))$.*

(b) *Pour tout $c \in [c^*(\varepsilon), \varepsilon^{-1})$, il existe une solution en onde progressive de* (8) *de vitesse c.*

(c) *Pour $c = \varepsilon^{-1}$, il existe une onde progressive au sens faible (voir Chapitre 1 pour la définition précise).*

(d) *Pour $c \in (\varepsilon^{-1}, \infty)$, il existe une solution en onde progressive de* (8) *de vitesse c. Ces fronts sont plus rapides que la vitesse des caractéristiques.*

Dans le régime hyperbolique, qui correspond à $\varepsilon^2 F'(0) \geq 1$, la vitesse minimale change :

$$c^*(\varepsilon) = \varepsilon^{-1}\,, \quad \text{si } \varepsilon^2 F'(0) \geq 1\,. \tag{11}$$

Théorème 0.2 (Régime hyperbolique)**.** *Supposons $\varepsilon^2 F'(0) \geq 1$. Alors :*

(a) *Il n'existe aucune solution en onde progressive de vitesse $c \in [0, c^*(\varepsilon))$.*

(b) *Il existe une solution faible en onde progressive de vitesse $c^*(\varepsilon) = \varepsilon^{-1}$. Le profil est discontinu si $\varepsilon^2 F'(0) > 1$ et continu mais non dérivable si $\varepsilon^2 F'(0) = 1$.*

(c) *Pour $c \in (\varepsilon^{-1}, \infty)$, il existe une solution en onde progressive de* (8) *de vitesse c.*

Il est intéressant de noter que dans ce dernier régime, l'onde progressive est nulle sur une demi-droite et la vitesse de propagation n'est pas donnée par le problème linéarisé. Le front n'est pas de type "tiré" ("pulled front") mais plutôt de type "poussé" ("pushed front"). Nous avons par ailleurs prouvé que ces profils sont tous linéairement stables dans certains espaces L^2 à poids par des méthodes d'énergie, et que le profil de vitesse minimale dans le régime parabolique est non-linéairement stable. Nous renvoyons au Chapitre 1 pour les énoncés relatifs à cette stabilité.

3.2 Propagation dans des modèles cinétiques.

A la suite de ce premier travail où le modèle cinétique ne prenait en compte que deux vitesses, nous avons élargi l'étude des équations de transport-réaction à un intervalle de vitesses continu. Rappelons que dans un formalisme cinétique, la population est représentée par une densité $f(t,x,v)$ et que la densité macroscopique est notée $\rho(t,x) := \int_V f(t,x,v)dv$. L'espace des vitesses V est symétrique par rapport à l'origine, son caractère borné ou non sera fondamental dans notre étude. Le modèle, introduit par Schwetlick [188] et Cuesta, Hittmeir, Schmeiser [69] est le suivant :

$$\underbrace{\partial_t f + v\cdot\nabla_x f}_{\text{Transport libre : "Run"}} = \underbrace{(M(v)\rho - f)}_{\text{Changement de direction : "Tumble"}} + \underbrace{r\rho\,(M(v) - f)}_{\text{Croissance avec saturation}}\,. \tag{12}$$

La densité de probabilité M vérifie :

$$\int_V M(v)dv = 1, \qquad \int_V vM(v)dv = 0, \qquad \int_V v^2 M(v)dv < +\infty, \tag{13}$$

pour modéliser un mouvement sans biais à l'échelle microscopique. Le terme de réaction mis à part, on retrouve bien la structure du modèle (7) sauf que le noyau T ne dépend pas d'un signal chimique. Ici, on considère que lors d'un changement de direction, la vitesse future est redistribuée selon M, et ce indépendamment de la vitesse passée. Le coefficient $r \geq 0$ est le taux de croissance exponentiel de la population en régime linéaire.

Il est important de noter que dans ce cas précis, une limite d'échelle parabolique : $t \mapsto \frac{t}{\varepsilon}, x \mapsto \frac{x}{\varepsilon^2}, r \mapsto r\varepsilon^2$ donne l'équation de réaction-diffusion classique de Fisher-KPP (1). Ceci est prouvé par exemple dans [12]. Il est par conséquent très naturel de chercher à construire des ondes progressives solutions de (12). Du fait de la présence de la variable cinétique, absente pour l'équation (1) présentée plus haut, il est bon d'expliciter la définition d'un onde progressive pour (12) :

Définition 0.3. *On dit que $f(t,x,v)$ est une solution en onde progressive de vitesse $c \in \mathbb{R}^+$ de* (12) *si elle s'écrit $f(t,x,v) = \mu(\xi = x - ct, v)$, où le profil $\mu \in \mathcal{C}^2(\mathbb{R} \times V)$ est strictement positif, satisfait les conditions aux limites $\mu(-\infty, \cdot) = M$, $\mu(+\infty, \cdot) = 0$, et résout l'équation*

$$(v - c)\partial_\xi \mu = (M(v)\nu - \mu) + r\nu(M(v) - \mu), \qquad \xi \in \mathbb{R},\ v \in V. \tag{14}$$

où l'on note ν la densité macroscopique associée au profil μ : $\nu(\xi) = \int_V \mu(\xi, v)\, dv$.

Autrement dit, la solution se propage en espace à vitesse c, en emmenant avec elle un profil en vitesse. Un premier résultat obtenu de manière indépendante par les auteurs de [69] montre l'existence d'ondes progressives dans un régime proche du régime parabolique de Fisher-KPP, plus précisément :

Théorème 0.4 (Cuesta, Hittmeir, Schmeiser, [69])**.** *On suppose que* $\sup|V| < +\infty$. *Prenons $c \geq 2\sqrt{rD}$. On se place dans l'échelle parabolique : $t \mapsto \frac{t}{\varepsilon}, x \mapsto \frac{x}{\varepsilon}, r \mapsto r\varepsilon^2$, dans cette échelle* (12) *devient :*

$$\varepsilon \partial_t f + \varepsilon^2 v \cdot \nabla_x f = (M(v)\rho - f) + r\varepsilon^2 \rho (M(v) - f).$$

Alors pour ε suffisamment petit, il existe une solution en onde progressive de vitesse s.

Cette solution est obtenue par une méthode perturbative, en modifiant de manière cinétique une onde progressive solution de l'équation de Fisher-KPP (sa limite diffusive). La méthode repose sur une décomposition "micro-macro" dans l'esprit du travail de Caflisch et Nicolaenko pour l'équation de Boltzmann [47]. Dans cette thèse, nous suivrons une méthode alternative pour la construction d'ondes progressives pour le modèle cinétique complet, basée sur des techniques de sur- et sous- solutions venant des équations de réaction-diffusion.

Théorème 0.5 (B., Calvez, Nadin)**.** *On suppose que* $\sup|V| < +\infty$ *et que $M \in L^1(V) \cap \mathcal{C}^0(V)$ vérifie* (13). *Alors il existe une vitesse minimale $c^* \in (0, v_{max})$ telle qu'il existe une solution onde progressive de* (12) *de vitesse c pour $c \in [c^*, v_{max})$. Par ailleurs, le profil μ est décroissant par rapport à la variable en translation ξ.*

On démontre de plus que ces profils sont faiblement stables dans des espaces à poids par une méthode d'énergie, voir directement le Chapitre 2 pour les énoncés. On signale néanmoins que l'obtention d'un poids pertinent, bien que non-optimal, requiert un certain travail, et nous renvoyons au Chapitre 2.

Comme pour les équations de réaction-diffusion, il est important de voir comment obtenir la vitesse minimale, et la décroissance du profil. Pour cela, nous procédons comme pour l'équation de Fisher-KPP classique, nous regardons le problème (14) linéarisé autour de l'équilibre. Cette fois ci, dû à la nature cinétique du problème, on cherche des solutions du problème linéarisé à variables séparables du type

$$\mu(\xi, v) := e^{-\lambda\xi} Q(v),$$

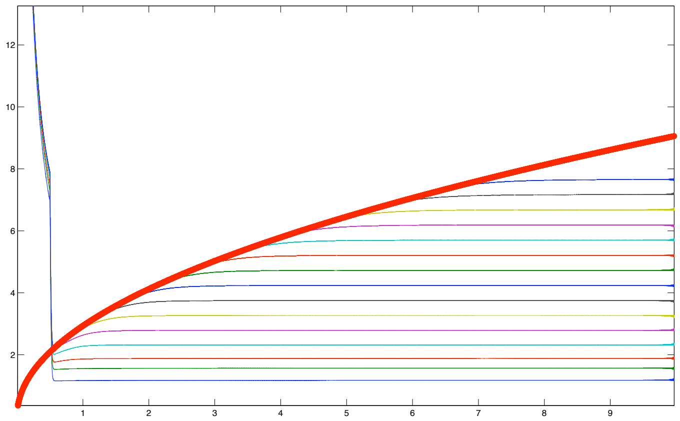


Figure 7 – Simulations numériques de (12) avec une donnée initiale de type "cloche". La densité M est une Gaussienne. Chaque tracé correspond à la vitesse du front pour une troncature particulière de l'espace des vitesses. On observe que l'enveloppe de ces courbes est très proche de $t \mapsto t^{1/2}$. Ainsi, la loi d'échelle pour la propagation avec des vitesses non-bornées est approximativement $x \sim t^{\frac{3}{2}}$. Ce taux de propagation est démontré au chapitre 2, où l'on trouvera de même plus de détails concernant ces simulations numériques.

pour un certain λ à déterminer, et pour une certaine distribution de vitesses à l'avant du front $Q(v)$ à déterminer également. Ces deux éléments sont alors déterminés par la

Proposition 0.6. *La vitesse minimale c^* est donnée par $c^* = \min_{\lambda>0} c(\lambda)$. La vitesse $c(\lambda)$ est pour tout $\lambda > 0$ une solution de la relation de dispersion suivante :*

$$\int_V \frac{(1+r)M(v)}{1+\lambda(c(\lambda)-v)}\, dv = 1\,. \tag{15}$$

Remarquons que les résultats ont été énoncés avec l'hypothèse supplémentaires de bornitude de l'espace des vitesses V. Cette hypothèse est très importante et son importance apparait dans (2.13) : pour $V = \mathbb{R}$, la relation de dispersion n'a pas de solution telle que λ et $c(\lambda)$ soient réels, ce qui indique très certainement un phénomène différent dans le cas de vitesses non bornées.

Pour comprendre plus précisément le cas de vitesses non-bornées, nous avons au préalable effectué des simulations numériques. Celles-ci, présentes en Figure 7 sont convaincantes et laissent penser à une propagation accélérée. Pour le justifier, nous avons prouvé que la propagation est effectivement accélérée en plaçant une famille de sous-solutions avançant avec une vitesse arbitrairement grande, et ce dès lors que la densité M est strictement positive.

Théorème 0.7 (B., Calvez, Nadin). *On suppose que $M(v) > 0$, pour $v \in \mathbb{R}$. Avec quelques hypothèses sur la donnée initiale (voir Chapitre 2), pour toute vitesse $c > 0$,*

$$\lim_{t\to+\infty} \sup_{x\leq ct} |M(v) - f(t,x,v)| = 0\,.$$

Ensuite, nous nous sommes focalisés sur le cas d'une densité M gaussienne pour lequel nous avons démontré que la solution se propage effectivement avec la loi d'échelle $x \sim t^{\frac{3}{2}}$:

Théorème 0.8 (B., Calvez, Nadin). *Soit* $M(v) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{v^2}{2\sigma^2}\right)$. *Alors les propriétés de propagation suivantes sont vérifiées :*

1. *Supposons qu'il existe* $1 \leq b \leq a$ *tels que*

$$(\forall (x,v) \in \mathbb{R} \times V) \quad g^0(x,v) \leq \frac{1}{b} M\left(\frac{x}{b}\right) M(v) e^{ra} .$$

Alors pour tout $\varepsilon > 0$, *on a*

$$\lim_{t \to +\infty} \left(\sup_{|x| \geq (1+\varepsilon)\sigma\sqrt{2r} t^{3/2}} \rho_f(t,x) \right) = 0 .$$

2. *Supposons qu'il existe* $\gamma \in (0,1)$, *et* $x_L \in \mathbb{R}$ *tels que*

$$(\forall (x,v) \in \mathbb{R} \times V) \quad f^0(x,v) \geq \gamma M(v) \mathbf{1}_{x < x_L},$$

Alors pour tout $\varepsilon > 0$, *on a*

$$\lim_{t \to +\infty} \left(\sup_{x \leq (1-\varepsilon)\sigma(\frac{r}{r+2} t)^{3/2}} \rho_f(t,x) \right) \geq 1 - \gamma .$$

Ce phénomène d'accélération avait été constaté par Mendez *et al* [166] mais son traitement mathématique est nouveau pour des équations cinétiques. Il est à comparer avec l'accélération dans les équations intégro-différentielles caractérisé par Garnier dans [107] et à celui dans les équations impliquant un Laplacien fractionnaire [64, 45, 46]. Une différence importante avec ce dernier est que le phénomène d'accélération apparait au niveau cinétique alors que la limite parabolique est une équation (Fisher-KPP) qui présente un phénomène de propagation de fronts à vitesse finie.

3.3 Formalisme Hamilton-Jacobi pour des équations cinétiques de transport réaction (vitesses bornées).

Dans cette sous-section nous présentons conjointement les Chapitres 3 et 4. On considère à nouveau un modèle cinétique de transport-réaction du type

$$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V, \qquad \partial_t f + v \cdot \nabla_x f = \mathrm{L}(f) + r\rho \left(M(v) - f \right) . \tag{16}$$

Ici, l'opérateur linéaire de *scattering* L n'agit que dans la variable de vitesse. On suppose à nouveau que l'opérateur homogène $L \in \mathcal{L}\left(L^2\left(\frac{dv}{M(v)}\right)\right)$ vérifie $\mathrm{Ker}(\mathrm{L}) = \mathrm{Vect}(M)$, ou la distribution $M \in \mathrm{Ker}(\mathrm{L})$ vérifie

$$\int_V M(v)dv = 1, \qquad \int_V vM(v)dv = 0, \qquad \int_V v^2 M(v)dv < +\infty.$$

On cherche maintenant à mettre en place le formalisme Hamilton-Jacobi décrit plus tôt pour les équations de réaction-diffusion dans le cadre des équations cinétiques. Tout l'enjeu est de retrouver une équation de Hamilton-Jacobi équivalente à (4) qui encode le phénomène de propagation. Mais il faut comprendre les phénomènes d'homogénéisation dans la variable de vitesse. Dans l'article préliminaire [33], nous avons traité le cas particulier où

$$\mathrm{L}(f) := M(v)\rho - f, \qquad r = 0. \tag{17}$$

Nous allons dans cette introduction schématiser la méthode à suivre de manière générale, on se référera aux Chapitres 3 et 4 pour les détails ainsi que les différentes hypothèses structurelles, qui contiennent, notamment, un principe du maximum pour l'opérateur L. Comme pour l'équation de Fisher-KPP, on effectue le changement d'échelle $(t,x,v) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$ dans (16), puis la transformation de Hopf-Cole cinétique

$$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V, \qquad f^\varepsilon(t,x,v) = M(v) e^{-\frac{\varphi^\varepsilon(t,x,v)}{\varepsilon}}. \tag{18}$$

On obtient alors

$$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V, \qquad \partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon = -\frac{\mathrm{L}\left(M(v)e^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)}{M(v)e^{-\frac{\varphi^\varepsilon}{\varepsilon}}} - r\rho^\varepsilon \left(e^{\frac{\varphi^\varepsilon}{\varepsilon}} - 1\right). \tag{19}$$

On peut réécrire cette dernière équation sous la forme

$$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V, \qquad \partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon + r = -\frac{\mathcal{L}\left(M(v)e^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)}{M(v)e^{-\frac{\varphi^\varepsilon}{\varepsilon}}} + r\rho^\varepsilon, \tag{20}$$

qui est exactement l'équivalent cinétique de (4). Reste à effectuer le passage à la limite $\varepsilon \to 0$. Afin de rester concis, nous allons présenter les éléments principaux de ce travail sur le cas particulier (17). En l'occurrence, (20) s'écrit

$$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V, \qquad \partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon = 1 - \int_V M(v') e^{\frac{\varphi^\varepsilon(v) - \varphi^\varepsilon(v')}{\varepsilon}} dv'.$$

Il semble alors naturel que φ^ε devienne indépendant de la variable v quand ε tend vers 0. C'est ici que la compacité des vitesses est cruciale. Ceci est le premier ingrédient important. Ecrivons maintenant formellement (comme pour l'homogénéisation des EDP)

$$\varphi^\varepsilon := \varphi^0(t,x) + \varepsilon \eta(t,x,v),$$

où η est communément appelé le correcteur. En négligeant les termes d'ordre plus élevé, on obtient formellement l'équation suivante :

$$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V, \qquad \partial_t \varphi^0 + v \cdot \nabla_x \varphi^0 = 1 - \int_V M(v') e^{\eta(t,x,v) - \eta(t,x,v')} dv'.$$

qui peut aussi s'écrire

$$\int_V M(v') e^{-\eta(t,x,v')} dv' + \left(v \cdot \nabla_x \varphi^0 - 1\right) e^{-\eta(t,x,v)} = \left(-\partial_t \varphi^0\right) e^{-\eta(t,x,v)},$$
$$(t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V,$$

Cette dernière équation est un problème spectral dans la variable de vitesse pour la distribution $e^{-\eta}$, à (t,x) fixés. Voilà le deuxième ingrédient très important pour le résultat de convergence. En résolvant ce problème spectral ($-\partial_t\varphi^0$ apparait comme la valeur propre de Perron d'un certain opérateur), on en déduit l'équation de Hamilton-Jacobi

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \qquad \int_V \frac{M(v)}{1-\partial_t\varphi^0 - v'\cdot\nabla_x\varphi^0}dv' = 1.$$

De plus, le vecteur propre de Perron sera important pour prouver la convergence, il servira comme correcteur pour définir une fonction test perturbée [90]. Nous renvoyons aux Chapitres 3 et 4 pour tous les détails. Le résultat de convergence générique pour (20) est le suivant :

Théorème 0.9 (B., Calvez & B.). *Soit V un sous ensemble symétrique et compact de $\mathbb{R}^n$, $M \in L^1(V)$ une densité symétrique et positive, et $r \geq 0$. Supposons que la donnée initiale soit bien préparée :*

$$\forall \varepsilon > 0, \qquad \forall (x,v) \in \mathbb{R}^n \times V, \qquad \varphi^\varepsilon(0,x,v) = \varphi_0(x),$$

et que quelques autres hypothèses structurelles présentées au Chapitre 4 soient vérifiées. Supposons de plus que le problème spectral :

"Pour tout $p \in \mathbb{R}^n$, il existe un unique $\mathcal{H}(p) \in \mathbb{R}$ tel qu'il existe un vecteur propre positif et de norme 1, $Q_p \in L^1(V)$, tel que

$$\forall v \in V, \qquad \mathcal{L}(Q_p)(v) + (v\cdot p)\,Q_p(v) = \mathcal{H}(p)Q_p(v)." \tag{21}$$

ait une solution. Alors $(\varphi^\varepsilon)_\varepsilon$ converge localement uniformément vers φ^0, qui ne dépend pas de v. De plus, φ^0 est l'unique solution de viscosité de l'une des équations de Hamilton-Jacobi ci-dessous :

(i) Si $r = 0$, alors φ^0 est la solution du problème :

$$\begin{cases} \partial_t\varphi^0 + \mathcal{H}\left(\nabla_x\varphi^0\right) = 0, & \forall (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n, \\ \varphi^0(0,x) = \varphi_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{22}$$

(ii) Si $r > 0$, alors l'équation de Hamilton-Jacobi est contrainte :

$$\begin{cases} \min\left\{\partial_t\varphi^0 + \mathcal{H}\left(\nabla_x\varphi^0\right) + r, \varphi^0\right\} = 0, & \forall (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n, \\ \varphi^0(0,x) = \varphi_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{23}$$

L'hypothèse de compacité de V est très importante pour ce résultat (comme elle l'était pour l'existence d'ondes progressives précédemment). Elle est notamment utile pour la résolution du problème spectral indiqué dans le théorème précédent. On renvoie à la fin du Chapitre 4 et aux perspectives de cette thèse puis à l'Annexe A pour des indications sur le cas des vitesses non bornées et notamment pour l'introduction de nouveaux changements d'échelle pour des fronts accélérés (dus au fait qu'il n'y a pas d'homogénéisation en vitesse dans ce cas) et les passages à la limite qui s'en suivent. Terminons par signaler que dans le cas de données initiales non préparées, il pourrait exister une couche limite non étudiée à ce jour.

3.4 Dynamique de populations structurées en espace-trait : Invasion des crapauds buffles.

Nous présentons ici les travaux des Chapitres 5 et 6. Dans ces travaux, on s'intéresse à la modélisation de fronts d'invasion pour des populations structurées à la fois en trait phénotypique et en variable d'espace. La motivation pour de tels modèles à été présentée au début de cette introduction. De manière générale, les fronts d'invasion en écologie ont été largement étudiés mais peu de résultats mathématiques existent pour le cas d'une population structurée par rapport au coefficient de diffusion. Au vu de l'invasion des crapauds buffles en Australie, il parait naturel de considérer des modèles prenant en compte un telle variabilité : des crapauds plus endurants peuvent se disperser plus loin dans l'espace. À partir d'un modèle de réaction-diffusion relativement simple en apparence, nous expliquons au moins formellement le phénomène d'accélération du front observé par les biologistes de terrain [199, 181]. Par ailleurs, nous construisons des solutions en ondes progressives qui sélectionnent les individus les plus mobiles lorsque la mobilité est bornée. Le modèle que nous regardons dans les deux Chapitres 5 et 6, issu de [43] (voir aussi [74, 9] et la version probabiliste [61]) est le suivant :

$$\partial_t n(t,x,\theta) = \underbrace{\theta\partial^2_{xx} n(t,x,\theta)}_{①} + \underbrace{\alpha\partial^2_{\theta\theta} n(t,x,\theta)}_{③} + \underbrace{rn(t,x,\theta)}_{②} - \underbrace{rn(t,x,\theta)\rho(t,x)}_{④}, \qquad x \in \mathbb{R},\ \theta \in \Theta \subset \mathbb{R}^{+*}. \quad (24)$$

L'équation est complétée par des conditions aux bords de Neumann dans la variable θ : l'ensemble des traits accessibles est l'intervalle Θ. Les termes de l'équation représentent respectivement les éléments suivants :

① Diffusion spatiale avec mobilité variable.

② Chaque crapaud donne en moyenne son trait à son descendant.

③ Mutations génétiques, modélisées ici par une diffusion dans la variable de trait (déviation sans biais de trait moyen). Un opérateur à noyau pourrait aussi être considéré (voir les perspectives à la fin de cette introduction).

④ Les individus sont en compétition pour les ressources indépendamment de leur trait phénotypique.

Dans ce formalisme ρ représente aussi la densité macroscopique :

$$\rho(t,x) = \int_\Theta n(t,x,\theta)\,d\theta\,.$$

Dans le cas de traits bornés, on s'attend à l'existence d'ondes progressives (propagation à vitesse finie). Le point clé pour la construction des fronts est, comme dans le cas des fronts cinétiques, la relation de dispersion qui relie la vitesse de l'onde avec la décroissance en espace à l'avant du front. En s'intéressant au problème linéarisé, comme pour l'équation de Fisher-KPP puis pour les fronts cinétiques introduits plus tôt dans cette thèse, on obtient la relation de dispersion donnée par *le problème spectral* suivant :

"Etant donné un taux de décroissance $\lambda < 0$, trouver $c(\lambda)$ et un vecteur propre positif $Q_\lambda(\theta)$ tels que

$$\begin{cases} \alpha \partial^2_{\theta\theta} Q_\lambda(\theta) + \left(\lambda c(\lambda) + \theta\lambda^2 + r\right) Q_\lambda(\theta) = 0\,, \\ \partial_\theta Q_\lambda(\theta_{\min}) = \partial_\theta Q_\lambda(\theta_{\max}) = 0\,, \qquad \int_\Theta Q_\lambda(\theta)\, d\theta = 1\,." \end{cases} \tag{25}$$

La vitesse de l'onde $c(\lambda)$ est caractérisée par le fait que 0 soit la valeur propre principale de ce problème spectral. Nous retrouvons à nouveau la même structure que pour l'équation de Fisher-KPP :

Théorème 0.10 (B., Calvez). *Soit $c^* := \inf_{\lambda>0} c(\lambda)$. Alors il existe une solution positive en onde progressive de* (24) *de vitesse c^* : $n(t,x,\theta) = N(x - ct, \theta)$.*

La distribution phénotypique à l'avant du front est donnée par $Q_\lambda(\theta)$. Nous vérifions que $Q_\lambda(\cdot)$ est bien croissante : Les individus les plus mobiles sont en majorité à l'avant du front. Par ailleurs, dans le cas de mutations très petites ($\alpha \sim 0$), la distribution $Q_\lambda(\cdot)$ se concentre au voisinage de $\max(\Theta)$. Remarquons enfin que dû à la non-localité présente dans le problème, aucun principe du maximum n'est applicable directement sur l'équation complète. L'existence des fronts est alors démontrée de manière complètement abstraite par un argument de type degré topologique. L'estimation sur les points fixes nécessaire pour ce raisonnement n'est pas simple à obtenir : On combine des arguments d'énergie (pour gagner de la régularité dans la variable θ) à des arguments de type principe du maximum. Nous renvoyons au Chapitre 6 pour la mise en place de ces éléments.

Lorsque la mobilité n'est pas bornée nous proposons des changements d'échelles à la fois naturels et pertinents pour une propagation accélérée, en l'occurrence

$$(t,x,\theta) \mapsto \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\frac{3}{2}}}, \frac{\theta}{\varepsilon}\right).$$

Nous montrons formellement par une approche Hamilton-Jacobi que la position du front suit une loi d'échelle en $x(t) \sim t^{3/2}$: Les lignes de niveau zéro de la phase font apparaître ce phénomène, voir le Chapitre 5. La propagation est bien accélérée. Nous n'avons pas encore réussi à justifier le passage à limite dans notre approche, notamment à cause du manque d'estimations *a priori* uniformes et la formulation compliquée du problème limite. L'enchainement au moins formel des idées est néanmoins très éclairant. Ceci est à rapprocher de la propagation accélérée constatée précédemment pour les équations cinétiques de transport-réaction dans le cas de vitesses non-bornées.

3.5 Approche Hamilton-Jacobi pour des populations structurées en espace-trait.

Nous terminons la présentation des résultats de cette thèse avec le Chapitre 7 dans lequel nous justifions rigoureusement le passage à la limite de Hamilton-Jacobi dans un modèle de réaction-diffusion non-local général. Le modèle peut être obtenu à partir de modèles stochastiques individus-centrés (voir [61]). Ce modèle ne prend toutefois pas en compte de mobilité variable. On s'attend à une propagation spatiale (*i.e.* dans la variable x), au cours de laquelle la population atteint une certaine distribution en trait. Avec les mêmes notations que dans la sous-section précédente, le modèle s'écrit de la manière suivante :

$$\begin{cases} \partial_t n(t,x,\theta) = \underbrace{D\Delta_x n(t,x,\theta)}_{①} + \underbrace{\alpha\Delta_\theta n(t,x,\theta)}_{③} + \underbrace{rn(t,x,\theta)\left(a(x,\theta)-\rho(t,x)\right)}_{②-④}, \\ \qquad\qquad\qquad\qquad (t,x,\theta) \in (0,\infty)\times\mathbb{R}^d\times\Theta, \\ \dfrac{\partial n}{\partial \mathbf{n}} = 0 \qquad \text{on } (0,\infty)\times\mathbb{R}^d\times\partial\Theta, \\ n(0,x,\theta) = n^0(x,\theta), \qquad (x,\theta)\in\mathbb{R}^d\times\Theta. \end{cases} \tag{26}$$

L'hétérogénéité en espace apparait dans la fitness $a(x,\theta)$. Cette fonction est typiquement quadratique en la variable de trait θ, tend vers $-\infty$ en l'infini, avec un maximum qui dépend de la position x. Cela rend compte du fait que le trait le mieux adapté du point de vue de la reproduction peut varier avec la position spatiale. Clarifions chacun des termes :

① Diffusion spatiale, ici avec diffusivité D constante.

② La reproduction des individus est hétérogène en trait et en espace.

③ Mutations génétiques, modélisées ici par une diffusion dans la variable de trait (déviation sans biais de trait moyen).

④ Les individus sont en compétition pour les ressources indépendamment de leur trait phénotypique (ce qui introduit une non-localité).

Le modèle n'étant pas invariant par translation en général, la question de l'existence d'ondes progressives est compliquée. Il faudrait recourir à une notion généralisée de front de transition [25] ce que nous ne ferons pas ici. Dans [4], la fitness a a une forme très particulière qui permet, après un changement de coordonnées, de construire des ondes progressives. Un intérêt d'utiliser ici une méthode Hamilton-Jacobi est que l'on peut considérer des taux de croissance a relativement généraux. On s'intéresse ici à la limite des petites mutations en temps long (attention, ce n'est pas exactement la même chose que temps long / grandes distances) :

$$t \mapsto \frac{t}{\varepsilon}, \qquad D \mapsto \varepsilon^2 D.$$

Après une transformation de Hopf-Cole déjà introduite auparavant (attention au changement de convention pour rester fidèle au Chapitre 7) :

$$u_\varepsilon := \varepsilon \ln n_\varepsilon, \qquad \text{ou alors,} \qquad n_\varepsilon = \exp\left(\frac{u_\varepsilon}{\varepsilon}\right). \tag{27}$$

on cherche à passer à la limite $\varepsilon \to 0$ dans

$$\begin{cases} \partial_t u_\varepsilon = \varepsilon D\Delta_{xx} u_\varepsilon + \frac{\alpha}{\varepsilon}\Delta_{\theta\theta} u_\varepsilon + D|\nabla_x u_\varepsilon|^2 + \frac{\alpha}{\varepsilon^2}|\nabla_\theta u_\varepsilon|^2 + r(a(x,\theta)-\rho_\varepsilon), \\ \qquad\qquad\qquad\qquad (t,x,\theta)\in(0,\infty)\times\mathbb{R}^d\times\Theta, \\ \dfrac{\partial u_\varepsilon}{\partial \mathbf{n}} = 0 \qquad \text{on } (0,\infty)\times\mathbb{R}^d\times\partial\Theta, \\ u_\varepsilon(0,x,\theta) = u_\varepsilon^0(x,\theta) \qquad (x,\theta)\in\mathbb{R}^d\times\Theta. \end{cases} \tag{28}$$

Nous avons essentiellement la même situation que dans le modèle utilisé pour l'invasion des crapauds buffles (24). Cependant, l'absence de diffusivité variable nous permet d'obtenir plus facilement les estimations *a priori* nécessaires au passage à la limite, à commencer par le fait que ρ^ε est uniformément borné. Nous indiquons maintenant toute l'heuristique nécessaire à la compréhension du résultat final. Commençons par écrire un développement en ε de u^ε :

$$u_\varepsilon(t,x,\theta) = u_0(t,x,\theta) + \varepsilon u_1(t,x,\theta) + \mathcal{O}(\varepsilon^2).$$

et insérons ce développement dans (28). En ne gardant que les termes d'ordre ε^{-2}, on obtient :

$$|\nabla_\theta u_0(t,x,\theta)|^2 = 0.$$

Ainsi la limite u_0 ne devrait pas dépendre de θ : $u_0(t,x,\theta) = u_0(t,x)$. Ensuite, en ne gardant que les termes d'ordre 0 :

$$-\alpha\left(\Delta_\theta u_1 + |\nabla_\theta u_1|^2\right) - ra(x,\theta) = \left[-\partial_t u_0 + D|\nabla_x u_0|^2 - r\rho_0\right](t,x). \tag{29}$$

On reconnait ici un problème spectral dans la variable θ à (t,x) fixés, de valeur propre dominante $H(x)$, qui permet d'obtenir l'équation de Hamilton-Jacobi limite :

$$\left[\partial_t u_0 - |\nabla_x u_0|^2 + r\rho_0\right](t,x) = H(x) \quad \text{et} \quad u_1(t,x,\theta) = \ln Q(x,\theta) + \mu(t,x).$$

Il nous reste à comprendre ρ^0 au moins formellement. On s'attend à

$$\begin{cases} \rho_0(t,x) = 0 \quad \Longrightarrow \quad \partial_t u_0(t,x) - D|\nabla_x u_0|^2(t,x) - H(x) = 0, \\ \rho_0(t,x) > 0 \quad \Longrightarrow \quad u_0(t,x) = 0 \quad \text{et} \quad r\exp(\mu(t,x)) = r\rho_0(t,x) = H(x), \end{cases}$$

ce qui nous amène directement à la formulation variationnelle :

$$\max\left(\partial_t u_0 - D|\nabla_x u_0|^2 - H(x)\,,\, u_0\right) = 0.$$

Ecrivons alors le résultat principal obtenu dans le Chapitre 7 :

Théorème 0.11 (B. & Mirrahimi). *En supposant quelques hypothèses structurelles naturelles détaillées au Chapitre 7,*

(i) La famille $(u_\varepsilon)_\varepsilon$ converge localement uniformément vers $u : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}$, l'unique solution de viscosité de :

$$\begin{cases} \max(\partial_t u - D|\nabla_x u|^2 - H, u) = 0, & \text{in } (0,\infty)\times\mathbb{R}^d, \\ u(0,\cdot) = u_0(\cdot) & \text{in } \mathbb{R}^d. \end{cases} \tag{30}$$

On obtient de plus le comportement qualitatif suivant (non optimal dû majoritairement au manque de principe du maximum) :

(ii) Uniformément sur les compacts de Int $\{u<0\}\times\Theta$, $\lim_{\varepsilon\to 0} n^\varepsilon = 0$,

(iii) Sur tout compact de Int $(\{u(t,x)=0\}\cap\{H(x)>0\})$, il existe $\overline{C}>1$ telle que,

$$\liminf_{\varepsilon\to 0} \rho_\varepsilon(t,x) \geq \frac{H(x)}{r\overline{C}}, \qquad \text{uniformément sur } K. \tag{31}$$

Pour montrer ce résultat nous utilisons en particulier la méthode des semi-limites relaxées de Barles et Perthame [14] ainsi qu'un principe d'unicité pour des solutions de viscosité *a priori* discontinues. En effet, il n'est pas possible d'obtenir aisément des estimations sur les dérivées spatiales de u^ε du fait de la présence de la non-localité. En dérivant (28) par rapport à la variable d'espace, la quantité $\nabla_x \rho^\varepsilon$ apparait : on ne sait pas la contrôler facilement. Un théorème d'Ascoli classique n'est pas applicable directement pour extraire une sous-suite convergente. On précise que le signe de H dénote en quelque sorte la capacité d'invasion, puisque l'on peut prouver que :

$$(t,x) \in \operatorname{Int}\{u(t,x)=0\} \quad \Longrightarrow \quad H(x) \geq 0.$$

Un point important de cette thèse apparait ici : La méthode pour attaquer le problème de passage à la limite est très similaire à celle utilisée dans le cadre des équations de transport-réaction cinétiques aussi étudiées dans cette thèse. Par ailleurs, l'étude du problème spectral, et notamment le comportement de Q lorsque la fitness a change sont intéressants. Nous renvoyons à la fin du Chapitre 7 pour une discussion numérique. Nous montrons en Figure 8 une illustration numérique du type de comportement qualitatif attendu.

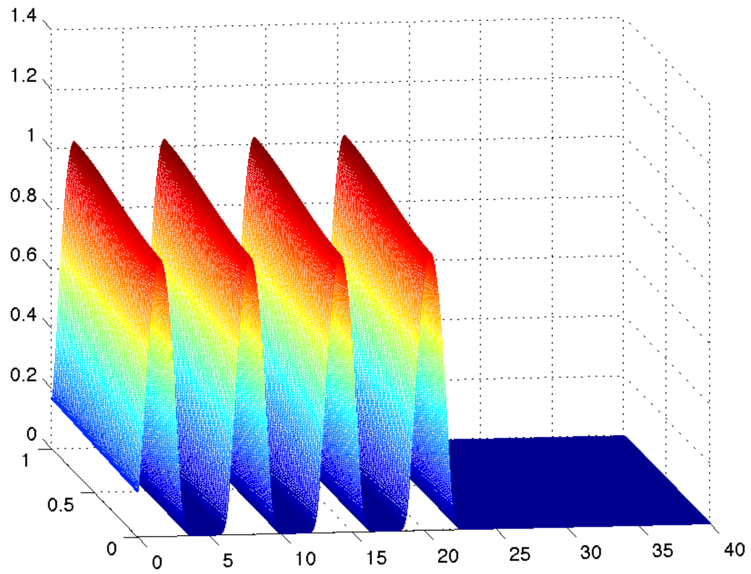


FIGURE 8 – Onde pulsée obtenue avec la fitness $a(x,\theta) = \frac{1}{4} + \frac{\theta}{2} + \left(\sin(x) - \frac{1}{2}\right)$. Voir la fin du Chapitre 7 pour plus de détails.

4 Perspectives et travaux en cours

Dans cette dernière section nous détaillons quelques perspectives et travaux que nous aimerions poursuivre à la suite de cette thèse. Certains sujets sont plus explorés que d'autres au moment de la rédaction.

Généralisation de l'approche Hamilton-Jacobi cinétique à un espace de vitesses non borné. Une question importante dans cette thèse qui a suivi les résultats d'existence d'ondes progressives cinétiques du Chapitre 2 ainsi que la mise en place de l'approche Hamilton-Jacobi pour les équations de type (20) aux Chapitres 3 et 4 est la question de l'espace des vitesses non borné. Une réponse en terme de phénomène d'accélération a été donnée au Chapitre 2. Nous aimerions mettre en place la technique Hamilton-Jacobi dans ce cadre. Un premier cas est le cas ou le problème spectral du Théorème 0.9 a une solution même si l'espace des vitesses n'est pas borné. C'est le cas par exemple pour l'équation de Vlasov-Fokker-Planck

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot \left(\sigma^2 \nabla_v f + v f\right), \qquad (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n,$$

pour laquelle il s'écrit

$$\nabla_v \cdot \left(\sigma^2 \nabla_v Q_p + v Q_p\right) + (v \cdot p)\, Q_p = \mathcal{H}(p) Q_p,$$

et a pour solution

$$\mathcal{H}(p) = \sigma^2 |p|^2, \qquad Q_p(v) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(v-\sigma^2 p)^2}{2\sigma^2}\right).$$

Nous pouvons par ailleurs prouver par des estimations *a priori* de type Bernstein (non présentée dans ce texte) que la phase de Hamilton-Jacobi va effectivement converger localement uniformément (à extraction près) vers une limite indépendante de v. Malheureusement, le raisonnement de passage à la limite effectué au Chapitre 4 ne peut pas être utilisé directement car le correcteur utilisé pour créer la fonction test perturbée n'est pas borné en vitesses. Nous sommes néanmoins convaincus qu'en étant capable de prouver une estimation plus précise sur la convergence de la phase en domaine borné et en tronquant le domaine en vitesse il sera possible de montrer la convergence de la phase vers la solution de l'équation de Hamilton-Jacobi attendue.

Compréhension de la dispersion cinétique : Grandes déviations pour BGK dans tout l'espace. Il n'est en revanche pas simple de comprendre comment étudier les grandes déviations dans le cas d'un espace de vitesses non-borné, par exemple $V = \mathbb{R}$ lorsque le problème spectral n'a pas de solutions. C'est précisément le cas de la simple équation BGK étudiée dans le Chapitre 3. Les travaux de 3 montrent alors que dans le cas de vitesses non bornées le changement d'échelle hyperbolique n'est plus adapté : il faut aussi changer d'échelle en vitesse. Étant inspirés par le travail du Chapitre 2, nous proposons avec Vincent Calvez, Emmanuel Grenier, et Grégoire Nadin, d'effectuer de changement d'échelle

$$(t,x,v) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\frac{3}{2}}}, \frac{v}{\varepsilon^{\frac{1}{2}}}\right).$$

Ce changement semble pertinent (et il semble être le seul) du fait que l'on attend une propagation $x \sim t^{\frac{3}{2}}$. Effectuons alors la transformée de Hopf-Cole cinétique :

$$f^\varepsilon(t,x,v) = \exp\left(-\frac{\varphi^\varepsilon}{\varepsilon}\right).$$

La nouvelle équation dans laquelle on cherche à passer à la limite $\varepsilon \to 0$ est

$$\partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon = 1 - \frac{1}{\sqrt{\pi\varepsilon}} \int_V \exp\left(\frac{\varphi^\varepsilon(v) - \varphi^\varepsilon(v') - v^2}{\varepsilon}\right) dv'.$$

Nous tenons à présenter en Annexe A quelques premiers résultats concernant le passage à la limite, notamment concernant la formulation du système limite. Certains points sont formels, ils sont indiqués clairement dans l'Annexe. Terminer la compréhension de ce passage à la limite ainsi qu'interpréter tous les résultats est une perspective à court terme.

Etude qualitative de fronts de propagation pour des modèles cinétiques avec vitesses non bornées. Il est tout de même possible de construire des ondes progressives pour des équations cinétiques avec un domaine de vitesse non-bornées. C'est le cas notamment pour le modèle de Perthame-Tadmor, voir [114], ou encore pour des équations de type BGK non-linéaires qui "ressemblent" à des lois de conservation scalaires [44]. Dans ce dernier exemple, le modèle considéré est

$$\partial_t f + v \cdot \nabla_x f = M(\rho, v) - f, \qquad (t, x, v) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}, \tag{32}$$

où la Maxwellienne est positive, généralement non linéaire et vérifie :

$$\int_V M(\rho, v) dv = \rho, \quad \int_V v M(\rho, v) dv = a(\rho),$$

où $a(\rho)$ est le flux macroscopique, appellation venant du fait que à des termes d'ordre $\mathcal{O}(\varepsilon^2)$ près, la limite parabolique de (32) est

$$\partial_t \rho + \nabla_x (a(\rho)) = \varepsilon \nabla_x \cdot (D(\rho) \nabla_x \rho), \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R},$$

c'est à dire une loi de conservation scalaire dans la limite $\varepsilon \to 0$. La diffusivité pourrait jouer un rôle :

$$D(\rho) = \int_V (v - a'(\rho))^2 \partial_\rho M(\rho, v) dv.$$

Pour des raisons d'entropie, on doit avoir

$$\partial_\rho M(\rho, v) > 0.$$

Dans l'article [44], les auteurs prouvent l'existence de profils de chocs cinétiques par un argument de compacité basé sur un lemme de moyenne cinétique en dimension 1. Á part la vitesse de propagation c du profil qui est donnée à l'avance par une relation de Rankine-Hugoniot, aucune information sur la décroissance à l'infini du profil n'est indiquée. C'est notamment ce point qu'il nous semble intéressant d'étudier. Cette question est par ailleurs très reliée à la compréhension de la dispersion donnée par l'opérateur de transport cinétique qui fait partie d'une des perspectives précédentes. Nous pensons étudier en parallèle la même question pour l'équation cinétique bistable suivante (effet Allee) :

$$\partial_t f + v \cdot \nabla_x f = M(v)\rho - f + r\rho(\rho - \alpha)(M(v) - f), \qquad (t, x, v) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}, \tag{33}$$

Pour ce modèle, il parait naturel que l'effet d'accélération de l'onde présent dans le cas monostable n'aura pas lieu car l'effet Allee élimine les petites populations responsables de cette accélération. Ainsi, on s'attend à une propagation à vitesse finie.

La question de la décroissance du profil est intéressante car nous sommes convaincus qu'il n'est pas possible en général que ces fronts soient à décroissance exponentielle (le problème linéarisé à l'avant du front ne le permet pas). On conjecture la décroissance sous-exponentielle :

Conjecture 0.12. *Soit* $M(v) := \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{v^2}{2\sigma^2}\right)$. *Soit* f *une solution en onde progressive de* (32) *ou* (33). *Alors pour une certaine constante* $C > 0$, $-\ln(\rho) \sim C x^{\frac{2}{3}}$ *lorsque* $x \to +\infty$.

Des simulations numériques présentes en Figure 9 appuient cette décroissance sous-exponentielle pour (33). Des simulations numériques très similaires sont obtenues pour (32) (non présentées ici). On présente de plus en Figure 10 une simulation qui révèle à l'avant du front la forme de la solution fondamentale présentée au dans l'Annexe A. Ceci appuie et justifie l'intérêt de certains éléments liés à la compréhension de la dispersion cinétique provenant de l'Annexe A.

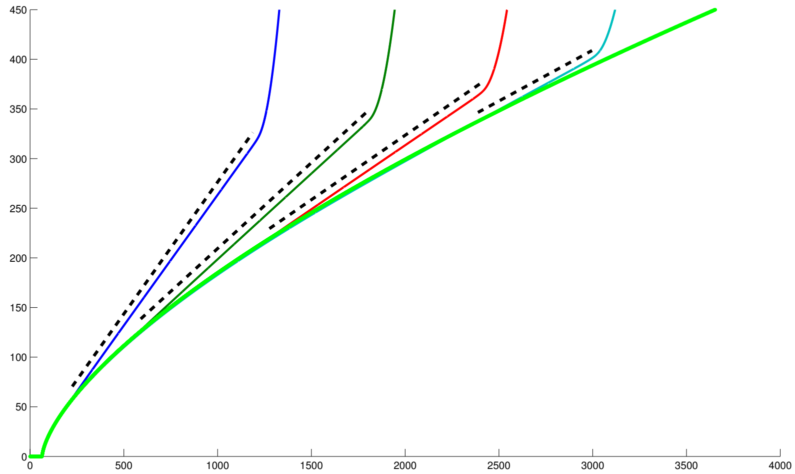


FIGURE 9 – En trait fin : Tracé de $-\ln(\rho)$, où ρ est la densité macroscopique relative à l'équation cinétique bistable (33). Les courbes sont ordonnées par vitesse maximale croissante de la gauche vers la droite. Pour chaque troncature, on s'attend à un front à décroissance exponentielle, ce qui est confirmé par le comportement linéaire de chaque courbe en trait fin : nous avons inséré des segments en trait pointillé pour souligner ce comportement. La partie croissante extrêmement rapide suivant chaque partie linéaire est juste un régime transitoire dû à la condition initiale à support compact. On observe que l'enveloppe inférieure de ces courbes (voir 7 pour le même type de simulations effectuées pour l'équation de transport monostable), correspond bien à la courbe en trait vert épais $x \mapsto x^{\frac{2}{3}}$: Le front existant pour des vitesses admissibles infini est à décroissance sous-exponentielle, la vitesse de décroissance dépendant fortement de la décroissance de M à l'infini.

Extensions du modèle pour l'invasion des crapauds : Ondes progressives. Nous souhaitons nous pencher sur plusieurs variantes du modèle (24) pour lesquels on s'attend à une

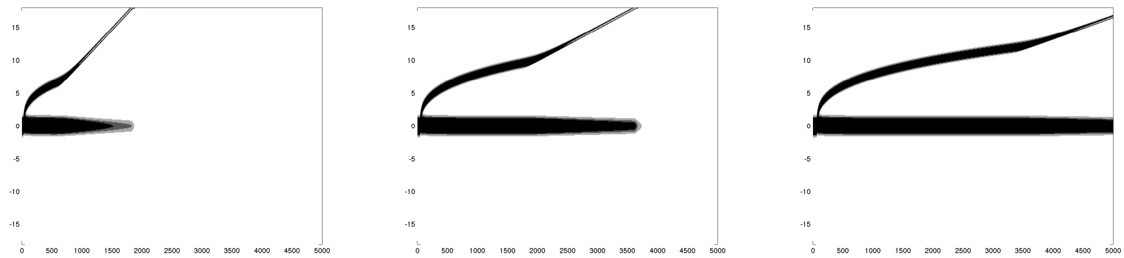


FIGURE 10 – Tracé de $-\ln(f)$, où f est la densité cinétique solution de l'équation (33). On observe à l'avant du front la structure de la solution fondamentale présentée dans l'Annexe A : Une transition le long de la droite $v = \frac{x}{t}$, qui dévoile une courbe stationnaire (en $x^{\frac{1}{3}}$).

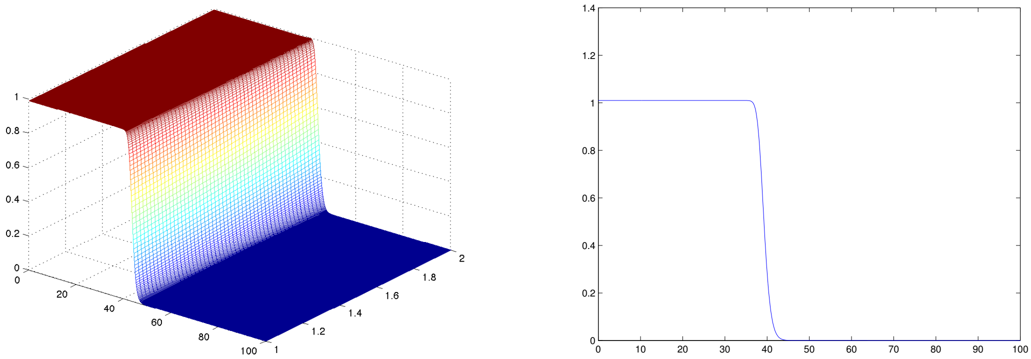
propagation à vitesse finie. Premièrement, le cas de traits non bornés mais avec une fitness quadratique :

$$\partial_t n - d(\theta)\partial_{xx} n - \alpha \Delta_\theta n = r\, n\, (a(\theta) - \rho)\,, \qquad (t,x,\theta) \in \mathbb{R}^+ \times \mathbb{R} \times \Theta.$$

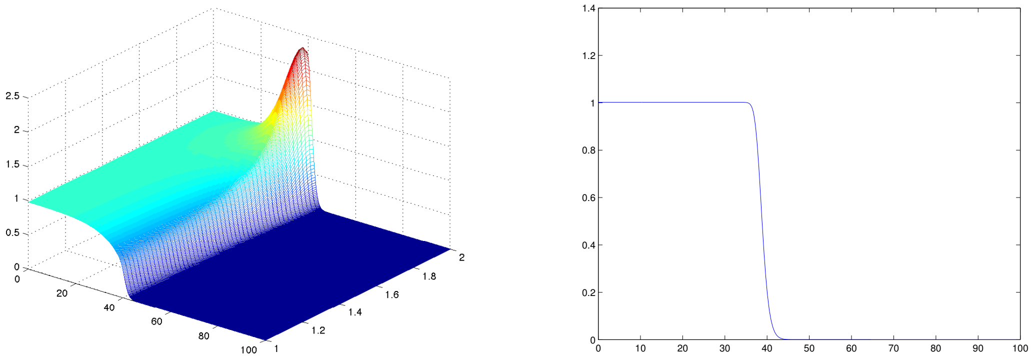
Ici d est une fonction croissante et minorée par $d_{\min} > 0$ sur $\Theta := (-\infty, +\infty)$ et $a(\theta)$ est une fonction de fitness majorée et quadratique à l'infini. Nous pouvons d'ores et déjà montrer que l'équivalent de l'estimation uniforme sur les points fixes nécessaire pour résoudre un problème sur un "slab" (de la même façon qu'au Chapitre 6) est vérifiée. Deuxièmement, on souhaite s'intéresser au cas ou les mutations dans (24) ne sont plus modélisées par un opérateur Laplacien mais par un opérateur de convolution. Ceci est important du point de vue de la modélisation car les modèles à noyau permettent de s'affranchir de l'hypothèse de petites mutations. Ceci est aussi motivé mathématiquement par le fait que les estimations obtenues dans le Chapitre 6 utilisaient fortement la régularité elliptique. L'équation s'écrit alors (dans un domaine de trait borné dans un premier temps) :

$$\partial_t n - d(\theta)\partial_{xx} n - \alpha \left(K \star_\theta n - n\right) = r\, n\, (1 - \rho)\,, \qquad (t,x,\theta) \in \mathbb{R}^+ \times \mathbb{R} \times \Theta.$$

La méthode utilisée au Chapitre 6 pour obtenir les estimations *a priori* ne fonctionne plus dans ce cadre lorsque K est un noyau régulier (par exemple : de type Gaussien). Il serait possible de considérer un noyau singulier en 0 pour récupérer de la régularité mais ce n'est pas notre but. La recherche d'états stationnaires pour ce problème a été traitée par Arnold, Desvillettes et Prévost [74, 8]. Ils montrent qu'en général il existe un état stationnaire au sens faible, régulier par rapport à la variable d'espace, mais étant en général une mesure dans la variable de trait θ. Néanmoins, lorsque la diffusion en trait est suffisamment importante (une relation algébrique sur les paramètres est indiquée dans [8]), une estimation d'énergie permet l'éliminer la présence d'éventuelles mesures. La description générale n'est pas achevée. Ainsi, la question de l'existence et de la régularité d'éventuelles ondes progressives est intéressante. On trouvera en Figure 11 des simulations laissant à penser qu'il est possible d'avoir une mesure singulière en trait lorsque α est petit (dans un certain sens à déterminer, par rapport à r).



(a) $\alpha = 5$, $r = 10$.



(b) $\alpha = 1$, $r = 10$.

Figure 11 – Profil pour diverses valeurs de α et r. On voit une concentration en θ lorsque α est petit que l'on ne constate pas pour α assez grand. En revanche la densité macroscopique ρ reste régulière.

Formalisme Hamilton-Jacobi pour le modèle (24) **et quelques variations.** Après avoir traité un cas sans diffusivité variable au Chapitre 7, il est naturel de s'intéresser à la question du passage à limite de Hamilton-Jacobi pour le modèle (24). La difficulté très importante est à nouveau de réussir à trouver des estimations uniformes[3]. Néanmoins, nous souhaitons dans le futur pouvoir justifier entièrement le phénomène d'accélération dans le cas Θ non borné grâce au formalisme Hamilton-Jacobi, formellement indiqué au Chapitre 5, et qui n'est pas une conséquence de la question déjà résolue.

Par ailleurs, nous souhaitons, avec les co-auteurs du Chapitre 5, nous intéresser à la sélection des traits phénotypiques lorsque les individus sont confinés dans un domaine borné Ω de l'espace. Dockery *et al.* ont étudié un modèle de réaction-diffusion discret en trait, dans un milieu hétérogène [80]. Ceci suit notamment certains travaux d'Hastings [124], voir aussi les autres travaux de Holt et McPeek ([132, 152], entre autres). Ils montrent alors que dans cette situation le trait sélectionné en temps long est le trait le moins dispersant. Nous voudrions prouver ce phénomène rigoureusement dans le cas d'un modèle continu. Le modèle est le suivant

$$\begin{cases} \varepsilon\partial_t n_\varepsilon - \theta\Delta_x n_\varepsilon - \alpha\varepsilon^2\partial_{\theta\theta} n_\varepsilon = n_\varepsilon(k(x)-\rho_\varepsilon)\,, \qquad (t,x,\theta)\in\mathbb{R}^+\times\mathbb{R}^n\times\Theta, \\ \partial_\theta n_\varepsilon(t,x,\theta_{\min}) = \partial_\theta n_\varepsilon(t,x,\theta_{\max}) = 0\,, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^n. \\ \partial_{\mathbf{n}} n_\varepsilon(t,x,\theta) = 0, \qquad (t,x)\in\mathbb{R}^+\times\Omega. \end{cases} \tag{34}$$

La nouvelle difficulté est la dépendance en x restant dans le terme non-local : Un problème spectral sous-jacent existe, mais il dépend fonctionnellement de la limite éventuelle de ρ^ε, ce qui va compliquer la tâche de passage à la limite.

Etude via Hamilton-Jacobi de la maladaptation des populations structurées en âge. Nous terminons cette série de perspectives en indiquant qu'un travail en cours avec Jimmy Garnier (Univ. Savoie), Vincent Calvez (ENS de Lyon) et Thomas Lepoutre (INRIA Lyon) est l'étude via le formalisme Hamilton-Jacobi cinétique de l'évolution de populations (sexuées ou asexuées) structurées en âge.

Dans le cas de populations asexuées, on considère un modèle continu de population structurée en âge $a \geq 0$ et en phénotype $z \in \mathbb{R}$. On suppose pour simplifier que le trait n'agit que sur la probabilité de survie, ici le taux de mort dépendant de l'âge $\mu(a,z)$. Le modèle est une équation de renouvellement sur la densité d'individus $f(t,a,z)$:

$$\begin{cases} \partial_t f(t,a,z) + \partial_a f(t,a,z) + \mu(a,z) f(t,a,z) = 0, \\ f(t,0,z) = \displaystyle\int_0^\infty \int_{\mathbb{R}} \beta(a) f(t,a,z') K_\varepsilon(z-z')\, dz' da\,. \end{cases} \tag{35}$$

On suppose que le noyau de mutations est de la forme $K_\varepsilon = \frac{1}{\varepsilon}K\left(\frac{\cdot}{\varepsilon}\right)$. Autrement dit on suppose que la distribution des traits-enfants est concentrée autour du trait-parent : les mutations sont supposées de faible amplitude. On s'intéresse alors à la dynamique en temps long de la population et notamment à sa croissance exponentielle (le modèle est linéaire) et à la distribution des âges. On retrouve notamment les conclusions de Hamilton (1966) et Lande (1982) et on

3. Nous avons appris lors de la phase de rédaction de ce manuscrit que cette question a été résolue par Olga Turanova dans le cas Θ borné (communication personnelle).

souhaite retrouver les conclusions des biologistes Cotto et Ronce [63] concernant la maladaptation et l'évolution de la sénescence de populations structurées en âge dans un environnement variable.

Dans le cas de populations sexuées, l'objectif est de comprendre l'effet de la reproduction sexuée sur la distribution phénotypique d'une population structurée en âge. On se place dans le cadre de modèles de populations structurées en âge avec reproduction sexuée, introduits par Barfield [13] ainsi que par Mirrahimi et Raoul [161]. Pour une population non structurée en âge, la densité de population peut être décrite par le modèle infinitésimal suivant :

$$\partial_t f(t,z) = \frac{1}{\int_{\mathbb{R}} f(t,z')\,dz'} \iint_{\mathbb{R}^2} f(t,z_*) f(t,z'_*) G\left(z - \frac{z_* + z'_*}{2}\right) dz_* dz'_*\,, \tag{36}$$

et le noyau de redistribution des gamètes G est une Gaussienne de variance $\frac{V_{LE}}{2}$. Bülmer (1980) et Turelli-Barton (1994) ont montré que le seul état stationnaire à croissance exponentielle près de ce modèle est une Gaussienne de variance V_{LE}. On s'intéresse désormais une population structurée en âges dont la reproduction est sexuée. On suppose comme dans le modèle avec reproduction asexuée (35), que le trait n'agit que sur le taux de mortalité $\mu(a,z)$. Le taux de reproduction $\beta(a)$ ne dépend que de la classe d'âge. Ainsi, la dynamique de la densité de population f de trait z et d'âge a à l'instant t peut être par l'équation suivante :

$$\begin{cases} \partial_t f(t,a,z) + \partial_a f(t,a,z) + \mu(a,z) f(t,a,z) = 0, \\ f(t,0,z) = \dfrac{\int_{\mathbb{R}^2 \times (\mathbb{R}^+)^2} \beta(a) f(t,a,z_*) \beta(a') f(t,a',z'_*) G\left(z - \frac{z_* + z'_*}{2}\right) dz_* dz'_* da da'}{\int_{\mathbb{R} \times \mathbb{R}^+} \beta(a) f(t,a,z')\,dz' da}. \end{cases} \tag{37}$$

On s'attend à une croissance exponentielle de cette densité d'individus. En conséquence de quoi il est légitime de chercher une distribution stable des phénotypes et des âges à croissance exponentielle près, comme pour le cas asexué. Cette distribution stable serait l'analogue de la distribution stationnaire Gaussienne du modèle de reproduction sexuée homogène (36).

Première partie

Phénomènes de propagation pour des équations cinétiques

Chapitre 1

Etude d'ondes progressives pour un modèle à deux vitesses

Dans ce travail en collaboration avec Vincent Calvez et Grégoire Nadin, nous analysons un modèle hyperbolique très proche de l'équation de Fisher-KPP. Dans ce modèle, les particules se déplacent à la vitesse $\pm\varepsilon^{-1}$ ($\varepsilon > 0$), et se reproduisent selon une croissance monostable. On se pose la question de l'existence et la stabilité d'ondes progressives pour ce modèle. Nous exhibons une transition qui dépend du paramètre ε. Pour une petite valeur de ce paramètre, le comportement qualitatif est très proche de celui de l'équation de Fisher-KPP (régime parabolique). En revanche, lorsque ε est grand, l'onde de vitesse minimale est discontinue et voyage à la vitesse maximale ε^{-1}. Nous prouvons de plus la stabilité linéaire des différents fronts dans des espaces à poids bien choisis ainsi que la stabilité non-linéaire du front de vitesse minimale dès lors que ε est plus petit que le paramètre critique.

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1.1 Introduction

We consider the problem of travelling fronts driven by growth (*e.g.* cell division) together with cell dispersal, where the motion process is given by a hyperbolic equation. This is motivated by the occurence of travelling pulses in populations of bacteria swimming inside a narrow channel [3, 186]. It has been demonstrated that kinetic models are well adapted to this problem [185]. We will focus on the following model introduced by Dunbar and Othmer [83], Hadeler [120], Holmes [131], Méndez and co-authors [165, 167, 99, 173], and Fedotov [92, 93, 94] (see also the recent book [155]),

$$\varepsilon^2 \partial_{tt}\rho_\varepsilon(t,x) + \left(1 - \varepsilon^2 F'(\rho_\varepsilon(t,x))\right)\partial_t \rho_\varepsilon(t,x) - \partial_{xx}\rho_\varepsilon(t,x) = F(\rho_\varepsilon(t,x)) \qquad t > 0\,, \quad x \in \mathbb{R}\,. \tag{1.1}$$

The cell density is denoted by $\rho_\varepsilon(t,x)$. The parameter $\varepsilon > 0$ is a scaling factor. It accounts for the ratio between the mean free path of cells and the space scale. The growth function F is subject to the following assumptions (the so-called monostable nonlinearity)

$$\begin{cases} F \in \mathcal{C}^3([0,1])\,, \quad F \text{ is uniformly strictly concave : } \inf_{[0,1]}(-F'') =: \alpha > 0\,, \\ F(0) = F(1) = 0\,, \quad F(\rho) > 0 \text{ if } \rho \in (0,1)\,. \end{cases} \tag{1.2}$$

For the sake of clarity we will sometimes take as an example the logistic growth function $F(\rho) = \rho(1-\rho)$.

Equation (1.1) is equivalent to the hyperbolic system

$$\begin{cases} \partial_t \rho_\varepsilon + \varepsilon^{-1}\partial_x (j_\varepsilon) = F(\rho_\varepsilon) \\ \varepsilon \partial_t j_\varepsilon + \partial_x \rho_\varepsilon = -\varepsilon^{-1} j_\varepsilon\,. \end{cases} \tag{1.3}$$

The expression of j_ε can be computed explicitly in terms of ρ_ε as follows,

$$j_\varepsilon(t,x) = -\frac{1}{\varepsilon}\int_0^t \partial_x\rho_\varepsilon(s,x)\exp\left(\frac{s-t}{\varepsilon^2}\right)ds + j_\varepsilon(0,x)\,, \tag{1.4}$$

but this expression will not be directly used afterwards. We will successively use the formulation (1.1) or the equivalent formulation (1.3).

Since the pioneering work by Fisher [97] and Kolmogorov-Petrovskii-Piskunov [143], dispersion of biological species has been usually modelled by mean of reaction-diffusion equations. The main drawback of these models is that they allow infinite speed of propagation. This is clearly irrelevant for biological species. Several modifications have been proposed to circumvent this issue. It has been proposed to replace the linear diffusion by a nonlinear diffusion of porous-medium type [192, 163, 176]. This is known to yield propagation of the support at finite speed [170, 169]. The density-dependent diffusion coefficient stems for a pressure effect among individuals which influences the speed of diffusion. Pressure is very low when the population is sparse, whereas it has a strong effect when the population is highly densified. Recently, this approach has been developped for the invasion of glioma cells in the brain [72]. Alternatively, some authors have proposed to impose a limiting flux for which the nonlinearity involves the gradient of the concentration [7, 51, 6].

The diffusion approximation is generally acceptable in ecological problems where space and time scales are large enough. However, kinetic equations have emerged recently to model self-organization in bacterial population at smaller scales [5, 175, 85, 151, 179, 186, 185]. These models are based on velocity-jump processes. It is now standard to perform a drift-diffusion limit to recover classical reaction-diffusion equations [175, 55, 85, 134]. However it is claimed in [185] that the diffusion approximation is not suitable, and the full kinetic equation has to be handled with. Equation (1.1) can be reformulated as a kinetic equation with two velocities only $v = \pm\varepsilon^{-1}$ (see (1.8) below). This provides a clear biological interpretation of equation (1.1) as a simple model for bacteria colonies where bacteria reproduce themselves, and move following a run-and-tumble process. We also emphasize that model (1.1) arises in the biological issue of species range expansion [131, 173], and in particular the human Neolithic Transition [99].

Hyperbolic models coupled with growth have already been studied in [83, 120, 105, 69]. In [120] it is required that the nonlinear function in front of the time first derivative $\partial_t\rho_\varepsilon$ is positive (namely here, $1-\varepsilon^2F'(\rho) > 0$). Indeed, this enables to perform a suitable change of variables in order to reduce to the classical Fisher-KPP problem. In our context this is equivalent to $\varepsilon^2F'(0) < 1$ since F is concave. In [105] this nonlinear contribution is replaced by 1 : the authors study the following equation (damped hyperbolic Fisher-KPP equation),

$$\varepsilon^2\partial_{tt}\rho_\varepsilon(t,x) + \partial_t\rho_\varepsilon(t,x) - \partial_{xx}\rho_\varepsilon(t,x) = F(\rho_\varepsilon(t,x))\,.$$

We also refer to [69] where the authors analyse a kinetic model more general than (1.1). They develop a perturbative approach, close to the diffusive regime $\varepsilon \ll 1$.

It is worth recalling some basic results related to reaction-diffusion equations. First, as $\varepsilon \to 0$ the density ρ_ε solution to (1.1) formally converges to a solution of the Fisher-KPP equation [69] :

$$\partial_t\rho_0(t,x) - \partial_{xx}\rho_0(t,x) = F(\rho_0(t,x))\,.$$

The long time behaviour of such equation is well understood since the pioneering works by Kolmogorov-Petrovsky-Piskunov [143] and Aronson-Weinberger [11]. For nonincreasing initial data with sufficient decay at infinity the solution behaves asymptotically as a travelling front moving at the speed $s = 2\sqrt{F'(0)}$. Moreover the travelling front solution with minimal speed is stable in some L^2 weighted space [104].

In this work we prove that analogous results hold true in the *parabolic regime* $\varepsilon^2 F'(0) < 1$. Namely there exists a continuum of speeds $[s^*(\varepsilon), \varepsilon^{-1})$ for which (1.1) admits smooth travelling fronts. The minimal speed is given by [92]

$$s^*(\varepsilon) = \frac{2\sqrt{F'(0)}}{1+\varepsilon^2 F'(0)}, \quad \text{if } \varepsilon^2 F'(0) < 1\,. \tag{1.5}$$

Obviously we have $s^*(\varepsilon) \le \min(2\sqrt{F'(0)}, \varepsilon^{-1})$. There also exists *supersonic* travelling fronts, with speed $s > \varepsilon^{-1}$. This appears surprising at first glance since the speed of propagation for the hyperbolic equation (1.1) is ε^{-1} (see formulation (1.3) and Section 1.2). These fronts are essentially driven by growth, since they travel faster than the maximum speed of propagation. The results are summarized in the following Theorem.

Theorem 1.1 (Parabolic regime)**.** *Assume that $\varepsilon^2 F'(0) < 1$. The following alternatives hold :*

(a) *There exists no smooth or weak travelling front of speed $s \in [0, s^*(\varepsilon))$.*

(b) *For all $s \in [s^*(\varepsilon), \varepsilon^{-1})$, there exists a smooth travelling front solution of* (1.1) *with speed s.*

(c) *For $s = \varepsilon^{-1}$ there exists a weak travelling front.*

(d) *For all $s \in (\varepsilon^{-1}, \infty)$ there also exists a smooth travelling front of speed s.*

We also obtain that the minimal speed travelling front is nonlinearly locally stable in the parabolic regime $\varepsilon^2 F'(0) < 1$ (see Section 1.5, Theorem 1.26).

There is a transition occuring when $\varepsilon^2 F'(0) = 1$. In the *hyperbolic regime* $\varepsilon^2 F'(0) \ge 1$ the minimal speed speed becomes :

$$s^*(\varepsilon) = \varepsilon^{-1}, \quad \text{if } \varepsilon^2 F'(0) \ge 1\,. \tag{1.6}$$

On the other hand, the front travelling with minimal speed $s^*(\varepsilon)$ is discontinuous as soon as $\varepsilon^2 F'(0) > 1$. In the critical case $\varepsilon^2 F'(0) = 1$ there exists a continuous but not smooth travelling front with minimal speed $s^* = \sqrt{F'(0)}$.

Theorem 1.2 (Hyperbolic regime)**.** *Assume that $\varepsilon^2 F'(0) \ge 1$. The following alternatives hold :*

(a) *There exists no smooth or weak travelling front of speed $s \in [0, s^*(\varepsilon))$.*

(b) *There exists a weak travelling front solution of* (1.1) *with speed $s^*(\varepsilon) = \varepsilon^{-1}$. The wave is discontinuous if $\varepsilon^2 F'(0) > 1$.*

(c) *For all $s \in (\varepsilon^{-1}, \infty)$ there exists a smooth travelling front of speed s.*

We conclude this introduction by giving the precise definition of travelling fronts (smooth and weak) that will be used throughout the paper.

Definition 1.3. *We say that a function $\rho(t,x)$ is a smooth travelling front solution with speed s of equation* (1.1) *if it can be written $\rho(t,x) = \nu(x-st)$, where $\nu \in \mathcal{C}^2(\mathbb{R})$, $\nu \geq 0$, $\nu(-\infty) = 1$, $\nu(+\infty) = 0$ and ν satisfies*

$$(\varepsilon^2 s^2 - 1)\nu''(z) - \left(1 - \varepsilon^2 F'(\nu(z))\right) s\nu'(z) = F(\nu(z)) \,, \quad z \in \mathbb{R} \,. \tag{1.7}$$

We say that ρ is a weak travelling front with speed s if it can be written $\rho(t,x) = \nu(x-st)$, where $\nu \in L^\infty(\mathbb{R})$, $\nu \geq 0$, $\nu(-\infty) = 1$, $\nu(+\infty) = 0$ and ν satisfies (1.7) *in the sense of distributions :*

$$\forall \varphi \in \mathcal{D}(\mathbb{R}) \,, \quad \int_{\mathbb{R}} \Big((\varepsilon^2 s^2 - 1)\nu\varphi'' + \left(\nu - \varepsilon^2 F(\nu)\right) s\varphi' - F(\nu)\varphi \Big) dx = 0 \,.$$

In the following Section 1.2 we show some numerical simulations in order to illustrate our results. Section 1.3 is devoted to the proof of existence of the travelling fronts in the various regimes (resp. parabolic, hyperbolic, and supersonic). Finally, in Section 1.4 and Section 1.5 we prove the stability of the travelling fronts having minimal speed $s^*(\varepsilon)$. We begin with linear stability (Section 1.4) since it is technically better tractable, and it let us discuss the case of the hyperbolic regime. We prove the full nonlinear stability in the range $\varepsilon \in (0, 1/\sqrt{F'(0)})$ (parabolic regime) in Section 1.5.

1.2 Numerical simulations

In this Section we perform numerical simulations of (1.1). We choose a logistic reaction term : $F(\rho) = \rho(1-\rho)$. We first symmetrize the hyperbolic system (1.3) by introducing $f^+ = \frac{1}{2}(\rho + j)$ and $f^- = \frac{1}{2}(\rho - j)$. This results in the following system :

$$\begin{cases} \partial_t f^+(t,x) + \varepsilon^{-1}\partial_x f^+(t,x) = \dfrac{\varepsilon^{-2}}{2}\left(f^-(t,x) - f^+(t,x)\right) + \dfrac{1}{2}F(\rho(t,x)) \\ \partial_t f^-(t,x) - \varepsilon^{-1}\partial_x f^-(t,x) = \dfrac{\varepsilon^{-2}}{2}\left(f^+(t,x) - f^-(t,x)\right) + \dfrac{1}{2}F(\rho(t,x)) \,. \end{cases} \tag{1.8}$$

In other words, the population is split into two subpopulations : $\rho = f^+ + f^-$, where the density f^+ denotes particles moving to the right with velocity ε^{-1}, whereas f^- denotes particles moving to the left with the opposite velocity.

We discretize the transport part using a finite volume scheme. Since we want to catch discontinuous fronts in the hyperbolic regime $\varepsilon^2 F'(0) > 1$, we aim to avoid numerical diffusion. Therefore we use a nonlinear flux-limiter scheme [113, 73]. The reaction part is discretized following the Euler explicit method.

$$f^+_{n+1,i} = f^+_{n,i} - \varepsilon^{-1}\frac{\Delta t}{\Delta x}\left(f^+_{n,i} + p_i \frac{\Delta x}{2} - f^+_{n,i-1} - p_{i-1}\frac{\Delta x}{2} \right) + \varepsilon^{-2}\frac{\Delta t}{2}\left(f^-_{n,i} - f^+_{n,i} \right) + \frac{\Delta t}{2} F(\rho_{n,i}) \,.$$

The non-linear reconstruction of the slope is given by

$$p_i = \operatorname{minmod}\left(\frac{f^+_{n,i} - f^+_{n,i-1}}{\Delta x}, \frac{f^+_{n,i+1} - f^+_{n,i}}{\Delta x} \right),$$

$$\textit{where} \quad \operatorname{minmod}(p,q) = \begin{cases} 0 & \text{if } \operatorname{sign}(p) \neq \operatorname{sign}(q) \\ \min(|p|,|q|)\operatorname{sign}(p) & \text{if } \operatorname{sign}(p) = \operatorname{sign}(q) \end{cases} .$$

We compute the solution on the interval (a, b) with the following boundary conditions : $f^+(a) = 1/2$ and $f^-(b) = 0$. The discretization of the second equation for f^- (1.8) is similar. The CFL condition reads $\Delta t < \varepsilon \Delta x$. It degenerates when $\varepsilon \searrow 0$, but we are mainly interested in the hyperbolic regime when ε is large enough. Other strategies should be used in the diffusive regime $\varepsilon \ll 1$, e.g. asymptotic-preserving schemes (see [96, 53] and references therein).

Results of the numerical simulations in various regimes (parabolic and hyperbolic) are shown in Figure 1.1.

1.3 Travelling wave solutions : Proof of Theorems 1.1 and 1.2

1.3.1 Characteristic equation

We begin with a careful study of the linearization of (1.7) around $\nu \approx 0$. We expect an exponential decay $e^{-\lambda z}$ as $z \to +\infty$. The characteristic equation reads as follows,

$$(\varepsilon^2 s^2 - 1)\lambda^2 + (1 - \varepsilon^2 F'(0))s\lambda - F'(0) = 0\,. \tag{1.9}$$

The discriminant is $\Delta = \left(\varepsilon^2 F'(0) + 1\right)^2 s^2 - 4F'(0)$. Hence we expect an oscillatory behaviour in the case $\Delta < 0$, i.e. $s < s^*(\varepsilon)$. We assume henceforth $s \geq s^*(\varepsilon)$. In the case $s < \varepsilon^{-1}$ (*subsonic fronts*) we have to distinguish between the *parabolic regime* $\varepsilon^2 F'(0) < 1$ and the *hyperbolic regime* $\varepsilon^2 F'(0) > 1$. In the former regime equation (1.9) possesses two positive roots, accounting for a damped behaviour. In the latter regime equation (1.9) possesses two negative roots. In the case $s > \varepsilon^{-1}$ (*supersonic fronts*) we get two roots having opposite signs.

Next we investigate the linear behaviour close to $\nu \approx 1$. We expect an exponential relaxation $1 - e^{\lambda' z}$ as $z \to -\infty$. The characteristic equation reads as follows,

$$(\varepsilon^2 s^2 - 1)\lambda'^2 - (1 - \varepsilon^2 F'(1))s\lambda' - F'(1) = 0\,. \tag{1.10}$$

We have $\Delta' = [\varepsilon^2 F'(1) + 1]^2 s^2 - 4F'(1) > 0$. In the case $s < \varepsilon^{-1}$ equation (1.10) possesses two roots having opposite signs. In the case $s > \varepsilon^{-1}$ it has two positive roots.

We summarize our expectations about the possible existence of nonnegative travelling fronts in Table 1.1.

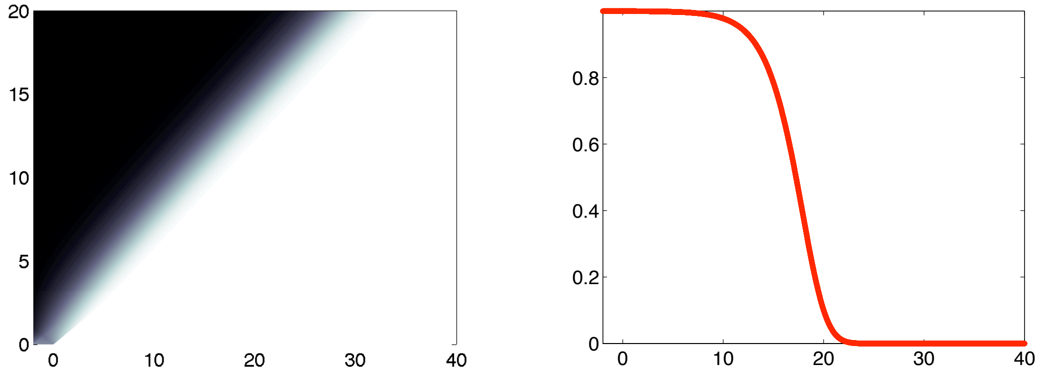
1.3.2 Proof of Theorems 1.1.(a) and 1.2.(a) : Obstruction for $s < s^*(\varepsilon)$

In this section we prove that no travelling front solution exists if the speed is below $s^*(\varepsilon)$.

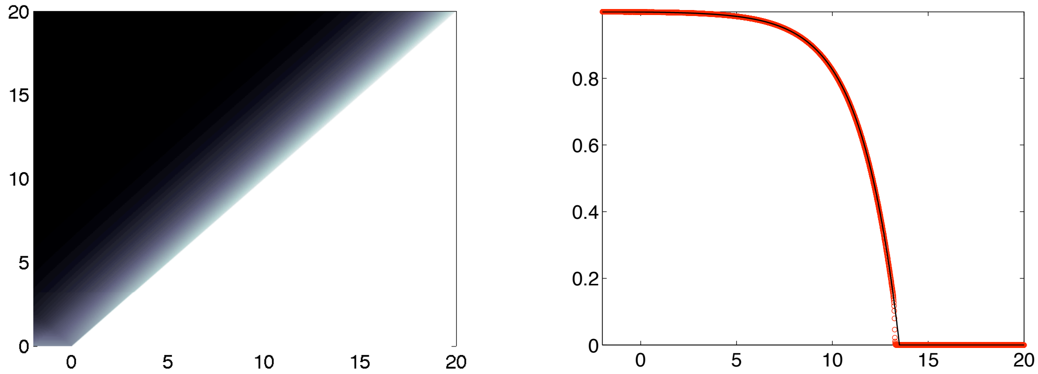
Proposition 1.4. *There exists no travelling front with speed s for $s < s^*(\varepsilon)$, where $s^*(\varepsilon)$ is given by* (1.5)-(1.6).

Remark 1.5. *Note that the proof below works in both cases $\varepsilon^2 F'(0) < 1$ and $\varepsilon^2 F'(0) \geq 1$.*

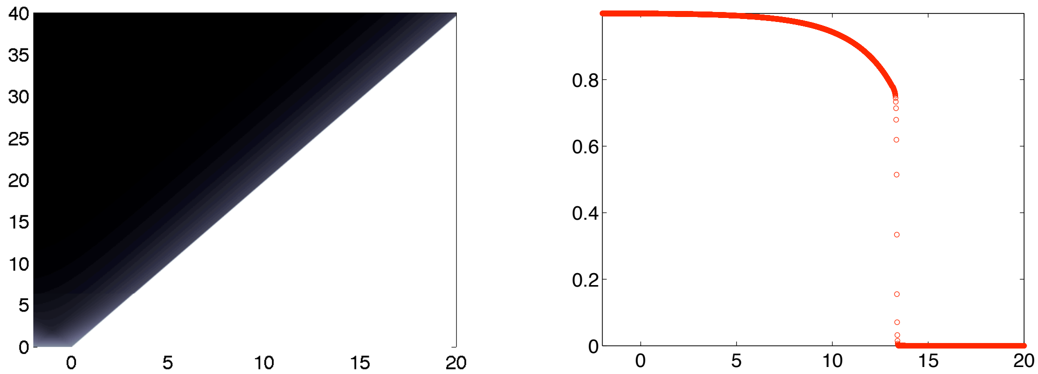
Proof 1.6. *We argue by contradiction. The obstruction comes from the exponential decay at $+\infty$. Assume that there exists such a travelling front $\nu(z)$. As $s < s^*(\varepsilon)$, one has $s < \varepsilon^{-1}$ in the parabolic as well as in the hyperbolic regime. Hence, as ν is bounded and satisfies the elliptic equation* (1.7) *in the*



(a) $\varepsilon = 1/2$



(b) $\varepsilon = 1$



(c) $\varepsilon = 2$

FIGURE 1.1 – Numerical simulations of the equation (1.1) for $F(\rho) = \rho(1-\rho)$ and for different values of $\varepsilon = 0.5, 1, 2$. Numerical method is described in Section 1.2. The initial data is the step function $f^+(x<0) = 1$, $f^+(x>0) = 0$, and $f^- \equiv 0$. For each value of ε we plot the density function $\rho = f^+ + f^-$ in the (x,t) space, and the density $\rho(t_0,\cdot)$ at some chosen time t_0. We clearly observe in every cases a front travelling asymptotically at speed $s^*(\varepsilon)$ as expected. We also observe the transition between a smooth front and a discontinuous one. The transition occurs at $\varepsilon = 1$. In the case $\varepsilon = 1$ we have superposed the expected profile $\nu(z) = \left(1 - e^{z/2}\right)_+$ in black, continuous line, for the sake of comparison.

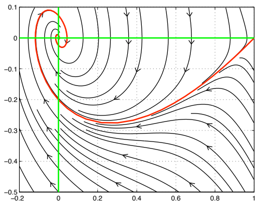
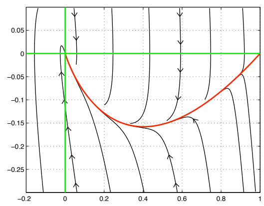
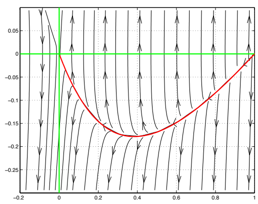
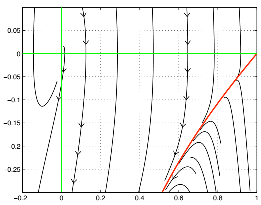
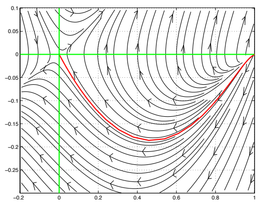
	$s < \varepsilon^{-1}$		$s > \varepsilon^{-1}$
parabolic	if $s < s^*(\varepsilon)$, NO 	if $s \geq s^*(\varepsilon)$, YES 	YES 
hyperbolic	NO 		YES 

Table 1.1 – Phase plane dynamics depending on the regime (parabolic vs. hyperbolic) and the value of the speed with respect to $s^*(\varepsilon)$ and ε^{-1}. In every picture the red line represents the travelling front trajectory, and the green lines are the axes $\{u = 0\}$ and $\{v = 0\}$. We do not consider the case $s = \varepsilon^{-1}$ since the dynamics are singular in this case and should be considered separately (see Section 1.3.5).

sense of distributions, classical regularity estimates show that ν is smooth. It is necessarily decreasing as soon at it is below 1. Otherwise, it would reach a local minimum at some point $z_0 \in \mathbb{R}$, for which $\nu(z_0) < 1$, $\nu'(z_0) = 0$ and $\nu''(z_0) \geq 0$. It would then follow from (1.7) that $F(\nu(z_0)) \leq 0$ and thus $\nu(z_0) = 0$. As $F \in \mathcal{C}^1([0,1])$, the Cauchy-Lipschitz theorem would imply $\nu \equiv 0$, a contradiction.

Next, we define the exponential rate of decay at $+\infty$:

$$\lambda := \liminf_{z\to+\infty} \frac{-\nu'(z)}{\nu(z)} \geq 0\,.$$

Consider a sequence $z_n \to +\infty$ such that $-\nu'(z_n)/\nu(z_n) \to \lambda$ and define the renormalized shift :

$$\nu_n(z) := \frac{\nu(z+z_n)}{\nu(z_n)}\,.$$

This function is locally bounded by classical Harnack estimates. It satisfies

$$(\varepsilon^2 s^2 - 1)\nu_n''(z) + \left(\varepsilon^2 F'(\nu(z+z_n)) - 1\right) s\nu_n'(z) = \frac{1}{\nu(z_n)} F(\nu(z_n)\nu_n(z))\,, \quad z \in \mathbb{R}\,.$$

As $F \in \mathcal{C}^1([0,1])$, $F(0) = 0$ and F is concave, the functions $z \mapsto \left(\varepsilon^2 F'(\nu(z+z_n)) - 1\right) s$ and $z \mapsto \frac{1}{\nu(z_n)} F(\nu(z_n)\nu_n(z))$ are uniformly bounded, uniformly in n. Hence, Schauder elliptic regularity estimates yield that the sequence $(\nu_n)_n$ is locally bounded in the Hölder space $\mathcal{C}^\alpha(K)$ for any compact subset $K \subset \mathbb{R}$ and any $\alpha \in (0,1)$. The Ascoli theorem and a diagonal extraction process give an extraction, that we still denote $(\nu_n)_n$, such that $(\nu_n)_n$ converges to some function ν_∞ in $\mathcal{C}^\alpha(K)$ for any compact subset $K \subset \mathbb{R}$ and any $\alpha \in (0,1)$. The limiting function is a solution in the sense of distributions of

$$(\varepsilon^2 s^2 - 1)\nu_\infty''(z) + \left(\varepsilon^2 F'(0) - 1\right) s\nu_\infty'(z) = F'(0)\nu_\infty(z)\,, \quad z \in \mathbb{R}\,. \tag{1.11}$$

As this equation is linear, one has $\nu_\infty \in \mathcal{C}^\infty(\mathbb{R})$. If $\nu_\infty(z_0) = 0$, then as ν_∞ is nonnegative, one would get $\nu_\infty'(z_0) = 0$ and thus $\nu_\infty \equiv 0$ by uniqueness of the Cauchy problem, which would be a contradiction since $\nu_\infty(0) = \lim_{n\to+\infty} \nu_n(0) = 1$. Thus ν_∞ is positive.

Define $V = \nu_\infty'/\nu_\infty$. The definition of λ yields $\min_{\mathbb{R}} V = V(0) = -\lambda$. Thus $V'(0) = 0$. Hence we deduce from (1.11) that $\nu_\infty(z) = \nu_\infty(0)e^{-\lambda z}$. Plugging this into (1.11), we obtain that λ satisfies the following second order equation,

$$(\varepsilon^2 s^2 - 1)\lambda^2 - (\varepsilon^2 F'(0) - 1)s\lambda - F'(0) = 0\,.$$

We know from Section 1.3.1 that, both in the parabolic and hyperbolic regimes, there is no real root in the case $s < s^*(\varepsilon)$.

1.3.3 Proof of Theorem 1.1.(b) : Existence of smooth travelling fronts in the parabolic regime $s \in [s^*(\varepsilon), \varepsilon^{-1})$

In [120] the author proves the existence of travelling front, by reducing the problem to the classical Fisher-KPP problem. It is required that the nonlinear function $1 - \varepsilon^2 F'(\rho)$ remains positive, which reads exactly $\varepsilon^2 F'(0) < 1$ in our context. We present below a direct proof based on the method of sub- and supersolutions, following the method developed by Berestycki and Hamel [126].

The linearized problem

Proposition 1.7. *Let λ_s be the smallest (positive) root of the characteristic polynomial* (1.9). *Then* $\overline{v}(z) = \min\{1, e^{-\lambda_s z}\}$ *is a supersolution of (1.7).*

Proof 1.8. *Let $r(z) = e^{-\lambda_s z}$. Then as r is decreasing and F is concave, it is easy to see that r is a supersolution of (1.7). On the other hand, the constant function* 1 *is clearly a solution of (1.7). We conclude since the minimum of two supersolutions is a supersolution.*

Resolution of the problem on a bounded interval

Proposition 1.9. *For all $a > 0$ and $\tau \in \mathbb{R}$, there exists a solution $v_{a,\tau}$ of*

$$\begin{cases} (\varepsilon^2 s^2 - 1)v''_{a,\tau} + (\varepsilon^2 F'(v_{a,\tau}) - 1)s v'_{a,\tau} = F(v_{a,\tau}) \text{ in } (-a,a), \\ v_{a,\tau}(-a) = \overline{v}(-a+\tau), \\ v_{a,\tau}(a) = \overline{v}(a+\tau). \end{cases} \tag{1.12}$$

Moreover, this function is nonincreasing over $(-a, a)$ and it is unique in the class of nonincreasing functions.

In order to prove this result, we consider the following sequence of problems : $v_0(z) = \overline{v}(z+\tau)$, and v_{n+1} is solution to

$$\begin{cases} (\varepsilon^2 s^2 - 1)v''_{n+1} + (\varepsilon^2 F'(v_n) - 1)s v'_{n+1} + M v_{n+1} = F(v_n) + M v_n \text{ in } (-a,a), \\ v_{n+1}(-a) = \overline{v}(-a+\tau), \\ v_{n+1}(a) = \overline{v}(a+\tau), \end{cases} \tag{1.13}$$

where $\overline{v}$ is defined in Proposition 1.7 and $M > \frac{s^2}{2}\left(\varepsilon^2 F'(0) - 1\right)$ is large enough so that $s \mapsto F(s) + Ms$ is increasing.

Lemma 1.10. *The sequence $(v_n)_n$ is well-defined. The functions $z \mapsto v_n(z)$ are nonincreasing and for all $z \in (-a, a)$, the sequence $(v_n(z))_n$ is nonincreasing.*

Proof 1.11. *We prove this Lemma by induction. Clearly, v_0 is nonincreasing. First, one can find a unique weak solution $v_1 \in \mathcal{C}^0([-a, a])$ of*

$$\begin{cases} (\varepsilon^2 s^2 - 1)v''_1 + (\varepsilon^2 F'(v_0) - 1)s v'_1 + M v_1 = F(v_0) + M v_0 \text{ in } (-a,a), \\ v_1(-a) = \overline{v}(-a+\tau), \\ v_1(a) = \overline{v}(a+\tau), \end{cases} \tag{1.14}$$

using the Lax-Milgram theorem and noticing that the underlying operator is coercive since $M > \frac{s^2}{2}\left(\varepsilon^2 F'(0) - 1\right)$ and $s < \varepsilon^{-1}$.

Let $w_0 = v_1 - v_0$. As v_0 is a supersolution of equation (1.7), one has

$$\begin{cases} (\varepsilon^2 s^2 - 1)w''_0 + (\varepsilon^2 F'(v_0) - 1)s w'_0 + M w_0 \le 0 \text{ in } (-a,a), \\ w_0(-a) = w_0(a) = 0. \end{cases}$$

As $M > 0$, the weak maximum principle gives $w_0 \le 0$, that is, $v_1 \le v_0$.

Define the constant function $\underline{\nu} = \overline{v}(a+\tau)$. *It satisfies*

$$(\varepsilon^2 s^2 - 1)\underline{\nu}'' + (\varepsilon^2 F'(\nu_0) - 1)s\underline{\nu}' + M\underline{\nu} = M\underline{\nu} \leq F(\underline{\nu}) + M\underline{\nu} \leq F(\nu_0) + M\nu_0$$

in $(-a,a)$ *since* $s \mapsto F(s) + Ms$ *is increasing and* $\nu_0(z) = \overline{v}(z+\tau) \geq \overline{v}(a+\tau) = \underline{\nu}$ *by monotonicity of* $\overline{v}$. *The same arguments as above lead to* $\nu_1 \geq \underline{\nu}$.

Assume that Lemma 1.10 is true up to rank n. The existence and the uniqueness of ν_{n+1} *follow from the same arguments as that of* ν_1. *Let* $w_n = \nu_{n+1} - \nu_n$. *As F is concave and* $\nu_{n-1} \geq \nu_n$, *we know that* $F'(\nu_{n-1}) \leq F'(\nu_n)$. *As* ν_n *is nonincreasing, we thus get*

$$\begin{cases} (\varepsilon^2 s^2 - 1)w_n'' + (\varepsilon^2 F'(\nu_n) - 1)sw_n' + Mw_n \leq 0 \text{ in } (-a,a), \\ w_n(-a) = w_n(a) = 0. \end{cases}$$

Hence, $w_n \leq 0$ *and thus* $\nu_{n+1} \leq \nu_n$. *Similarly, one easily proves that* $\nu_{n+1} \geq \underline{\nu}$ *in* $(-a,a)$.

Differentiating (1.13) and denoting $v = \nu_{n+1}'$, *one gets*

$$(\varepsilon^2 s^2 - 1)v'' + \big(\varepsilon^2 F'(\nu_0) - 1\big)sv' + \big(M + \varepsilon^2 F''(\nu_0)\nu_0'\big)v = \big(F'(\nu_0) + M\big)\nu_0' \leq 0 \text{ in } (-a,a)$$

since $s \mapsto F(s) + Ms$ *is increasing and* ν_0 *is nonincreasing. As F is concave, the zeroth-order term is positive and thus the elliptic maximum principle ensures that v reaches its maximum at* $z = -a$ *or at* $z = a$. *But as* $\overline{v}(a+\tau) \leq \nu_{n+1}(z) \leq \overline{v}(z+\tau)$ *for all* $z \in (-a,a)$, *one has*

$$v(-a) \leq \limsup_{z \to -a^+} \frac{\nu_{n+1}(z) - \nu_{n+1}(-a)}{z+a} \leq \limsup_{z \to -a^+} \frac{\overline{v}(z+\tau) - \overline{v}(-a+\tau)}{z+a} \leq 0$$

and similarly $v(a) \leq 0$. *Thus* $v \leq 0$, *meaning that* ν_{n+1} *is nonincreasing.*

Proof 1.12 (Proof of Proposition 1.9.). *As the sequence* $(\nu_n)_n$ *is decreasing and bounded from below, it admits a limit* $\nu_{a,\tau}$ *as* $n \to +\infty$. *It easily follows from the classical regularity estimates that* $\nu_{a,\tau}$ *satisfies the properties of Proposition 1.9.*

If ν_1 *and* ν_2 *are two nondecreasing solutions of (1.12), then the same arguments as before give that* $\nu_1^\mu < \nu_1$ *in* Σ_μ *for all* $\mu \in (0, 2a)$. *Hence,* $\nu_1 \leq \nu_2$ *and a symmetry argument gives* $\nu_1 \equiv \nu_2$.

Lemma 1.13. *For all* $a > 0$, *there exists* $\tau_a \in \mathbb{R}$ *such that* $\nu_{a,\tau_a}(0) = \frac{1}{2}$.

Proof 1.14. *Define* $I(\tau) := \nu_{a,\tau}(0)$. *It follows from the classical regularity estimates and from the uniqueness of* $\nu_{a,\tau}$ *that I is a continuous function. Moreover, as* $\nu_{a,\tau}$ *is nonincreasing, one has*

$$\overline{v}(a+\tau) \leq I(\tau) \leq \overline{v}(-a+\tau),$$

where $\overline{v}$ *is defined in Proposition 1.7. As* $\overline{v}(\cdot+\tau) \to 0$ *as* $\tau \to +\infty$ *and* $\overline{v}(\cdot+\tau) \to 1$ *as* $\tau \to -\infty$ *locally uniformly on* $\mathbb{R}$, *one has* $I(-\infty) = 1$ *and* $I(+\infty) = 0$. *The conclusion follows.*

Existence of travelling fronts with speeds $s \in [s^*(\varepsilon), \varepsilon^{-1})$

We conclude by giving the proof of Theorem 1.1 as a combination of the above results.

Proof 1.15 (Proof of Theorem 1.1.). *Consider a sequence* $(a_n)_n$ *such that* $\lim_{n\to+\infty} a_n = +\infty$ *and define* $\nu_n(z) := \nu_{a_n,\tau_{a_n}}$ *for all* $z \in [-a_n, a_n]$. *This function is decreasing and satisfies* $\nu_n(0) = 1/2$, $0 \leq \nu_n \leq 1$ *and*

$$(\varepsilon^2 s^2 - 1)\nu_n'' + (\varepsilon^2 F'(\nu_n) - 1)s\nu_n' = F(\nu_n) \text{ in } (-a_n, a_n).$$

As in the proof of Proposition 1.4, the uniform boundedness of $(\nu_n)_n$ *together with* L^p *elliptic regularity estimates ensure that the sequence* $(\nu_n)_n$ *is uniformly bounded in* $W^{2,p}(K)$ *for all compact set* $K \in \mathbb{R}$ *and* $p \in (1,\infty)$*. It follows from Sobolev injections and the Ascoli theorem that the sequence* $(\nu_n)_n$ *converges in* $\mathcal{C}^0_{loc}(\mathbb{R})$ *as* $n \to +\infty$ *to a function* ν*, up to extraction. Then* ν *satisfies*

$$(\varepsilon^2 s^2 - 1)\nu'' + (\varepsilon^2 F'(\nu) - 1)s\nu' = F(\nu)\,.$$

Moreover it is nonincreasing, $0 \le \nu \le 1$ *and* $\nu(0) = 1/2$*.*

Define $\ell_\pm := \lim_{z\to\pm\infty} \nu(z)$*. Passing to the (weak) limit in the equation satisfied by* ν*, one gets* $F(\ell_\pm) = 0$*. As* $0 \le \ell_\pm \le 1$*, the hypotheses on* F *give* $\ell_\pm \in \{0,1\}$*. On the other hand, as* ν *is nonincreasing, one has*

$$\ell_+ \le \nu(0) = 1/2 \le \ell_-.$$

We conclude that $\ell_- = \nu(-\infty) = 1$ *and* $\ell_+ = \nu(+\infty) = 0$*.*

The following classical inequality satisfied by the travelling profile will be required later.

Lemma 1.16. *The travelling profile* ν *satisfies :* $\forall z \; \nu'(z) + \lambda\nu(z) \ge 0$*, where* λ *is the smallest positive root of* (1.9).

Proof 1.17. *We introduce* $\varphi(z) = -\frac{\nu'(z)}{\nu(z)}$*. It is nonnegative, and it satisfies the following first-order ODE with a source term*

$$\left(\varepsilon^2 s^2 - 1\right)\left(-\varphi'(z) + \varphi(z)^2\right) + \left(1 - \varepsilon^2 F'(\nu(z))\right) s\varphi(z) = \frac{F(\nu(z))}{\nu(z)}\,.$$

Since F *is concave,* φ *satisfies the differential inequality*

$$\left(1 - \varepsilon^2 s^2\right)\varphi'(z) \le \left(1 - \varepsilon^2 s^2\right)\varphi(z)^2 - \left(1 - \varepsilon^2 F'(0)\right) s\varphi(z) + F'(0)\,.$$

The right-hand-side is the characteristic polynomial of the linearized equation (1.9)*. Moreover the function* φ *verifies* $\lim_{z\to-\infty} \varphi(z) = 0$*. Hence a simple ODE argument shows that* $\forall z \; \varphi(z) \le \lambda$*.*

1.3.4 Proof of Theorem 1.1.(c) : Existence of weak travelling fronts of speed $s = \varepsilon^{-1}$ in the parabolic regime

The aim of this Section is to prove that in the parabolic regime $\varepsilon^2 F'(0) < 1$, there still exists travelling fronts in the limit case $s = \varepsilon^{-1}$ but in the weak sense.

Proposition 1.18. *Assume that* $\varepsilon^2 F'(0) < 1$*. Then there exists a weak travelling front of speed* $s = \varepsilon^{-1}$*.*

Proof 1.19. *Let* $s_n = \varepsilon^{-1} - 1/n$ *for all* n *large enough so that* $s_n \ge s^*(\varepsilon)$*. We know from the previous Section that we can associate with the speed* s_n *a smooth travelling front* ν_n *and that we can assume, up to translation, that* $\nu_n(0) = 1/2$*. Multiplying equation (1.7) by* ν_n' *and integrating by parts over* $\mathbb{R}$*, one gets*

$$\begin{aligned} s_n\left(1 - \varepsilon^2 F'(0)\right)\int_{\mathbb{R}} \nu_n'(z)^2 dz &\le s_n \int_{\mathbb{R}} \Big(1 - \varepsilon^2 F'\big(\nu_n(z)\big)\Big)\nu_n'(z)^2 dz \\ &= -\int_{\mathbb{R}} F\big(\nu_n(z)\big)\nu_n'(z) dz \\ &= -\int_0^1 F(u) du. \end{aligned}$$

Hence, as $\varepsilon^2 F'(0) < 1$*, the sequence* $(\nu_n')_n$ *is bounded in* $L^2(\mathbb{R})$ *and one can assume, up to extraction, that it admits a weak limit* V_∞ *in* $L^2(\mathbb{R})$*. It follows that the sequence* $(\nu_n)_n$ *converges locally uniformly to* $\nu_\infty(z) := \int_0^z V_\infty(z')dz' + 1/2$*. Passing to the limit in (1.7), we get that this function is a weak solution of*

$$-\left(1-\varepsilon^2 F'(\nu_\infty(z))\right) s\nu_\infty'(z) = F(\nu_\infty(z)), \quad z \in \mathbb{R},$$

which ends the proof.

1.3.5 Proof of Theorem 1.2.(b) : Existence of weak travelling fronts of speed $s = \varepsilon^{-1}$ in the hyperbolic regime

In this Section we investigate the existence of travelling fronts with critical speed $s = \varepsilon^{-1}$ in the hyperbolic regime $\varepsilon^2 F'(0) = 1$.

Proof 1.20 (Proof of Theorem 1.2.). *The function* $G(\rho) := \varepsilon^2 F(\rho) - \rho$ *is concave, and vanishes when* $\rho = 0$*. Furthermore,* $G(1) < 0$ *and* $G'(0) = \varepsilon^2 F'(0) - 1 \geq 0$*. We now distinguish between the two cases* $\varepsilon^2 F'(0) > 1$ *and* $\varepsilon^2 F'(0) = 1$*.*

1. ***First case :*** $\varepsilon^2 F'(0) > 1$. *As* G' *is decreasing, there exists a unique* $\theta_\varepsilon \in (0,1)$ *such that* G *vanishes.*
2. ***Second case :*** $\varepsilon^2 F'(0) = 1$. *The only root of* G *is* $\rho = 0$*. In this case we set* $\theta_\varepsilon = 0$*.*

For both cases, we have $G'(\rho) < 0$ *for all* $\rho > \theta_\varepsilon$ *since* G *is strictly concave and* $G(0) = G(\theta_\varepsilon) = 0$*. Hence,* $\varepsilon^2 F'(\rho) < 1$ *for all* $\rho > \theta_\varepsilon$*. Set* ν *the maximal solution of*

$$\begin{cases} \nu'(z) = \dfrac{\varepsilon F(\nu(z))}{\varepsilon^2 F'(\nu(z)) - 1}, \\ \nu(0) = \dfrac{1+\theta_\varepsilon}{2} > \theta_\varepsilon. \end{cases} \tag{1.15}$$

Let I *be the (maximal) interval of definition of* ν*, with* $0 \in I$*, and*

$$z_0 = \sup\{z \in I, \nu(z) > \theta_\varepsilon\}.$$

1- Conclusion of the argument in the first case : $\varepsilon^2 F'(0) > 1$**.**

Since $\theta_\varepsilon > 0$*, we have necessarily* $z_0 < +\infty$*. From* (1.15)*,* ν *is decreasing on* $(-\infty, z_0)$*. Thus, we have* $\nu(z) \to \theta_\varepsilon$ *as* $z \to z_0^-$*. Moreover, one easily gets* $\nu(-\infty) = 1$*.*

We set $\nu(z_0) = \theta_\varepsilon$ *and we extend* ν *by* 0 *over* (z_0, ∞)*. We observe that* ν *is a weak solution, in the sense of distributions, of*

$$\left(\varepsilon^2 F(\nu) - \nu\right)' = \varepsilon F(\nu) \text{ on } \mathbb{R}$$

since $\varepsilon^2 F(0) = 0$ *and* $\varepsilon^2 F(\theta_\varepsilon) = \theta_\varepsilon$*.*

Up to space shifting $z - z_0$*, we may assume that the discontinuity arises at* $z = 0$*.*

Example : the case $F(\rho) = \rho(1-\rho)$ and $\varepsilon > 1$. *The travelling profile solves*

$$\nu'(z) = \frac{\varepsilon\nu(z)(1-\nu(z))}{\varepsilon^2 - 1 - 2\varepsilon^2\nu(z)} ,$$

or equivalently

$$\nu(z)^{\varepsilon^2-1}\left(1-\nu(z)\right)^{\varepsilon^2+1} = ke^{\varepsilon z} .$$

The constant k is determined by the condition $\nu(0) = \theta_\varepsilon = 1 - \varepsilon^{-2}$. Finally the travelling profile $\nu(z)$ satisfies the following implicit relation :

$$\nu(z)^{\varepsilon^2-1}\left(1-\nu(z)\right)^{1+\varepsilon^2} = \left(1-\varepsilon^{-2}\right)^{\varepsilon^2-1}\left(\varepsilon^{-2}\right)^{\varepsilon^2+1} e^{\varepsilon z} = \left(\varepsilon^2-1\right)^{\varepsilon^2-1} e^{\varepsilon z + 2\varepsilon^2 \log \varepsilon^2} . \tag{1.16}$$

2- Conclusion of the argument in the second case : $\varepsilon^2 F'(0) = 1$.

The difference here is that $\theta_\varepsilon = 0$. To conclude the proof as previously, we just need to check that z_0 is finite. We argue by contradiction. Assume $z_0 = +\infty$. Linearizing the r.h.s. of (1.15) *near $\nu = 0$, we get*

$$\nu'(z) = \frac{F'(0)}{\varepsilon F''(0)} + o(\nu(z)) , \quad \text{as } z \to +\infty \tag{1.17}$$

We get a contradiction because $\varepsilon^{-1}F'(0)/F''(0) < 0$.

Finally, we create a continuous front with the same extension idea as for the first case.

Example : the case $F(\rho) = \rho(1-\rho)$ and $\varepsilon = 1$. *The travelling profile reads* (1.16) *:*

$$\nu(z) = \left(1 - e^{z/2}\right)_+ .$$

1.3.6 Proof of Theorem 1.1.(d) and Theorem 1.2.(c) : Existence of supersonic travelling fronts $s > \varepsilon^{-1}$

In this Section we investigate the existence of supersonic travelling fronts with speeds above the maximal speed of propagation $s > \varepsilon^{-1}$. These fronts are essentially driven by growth. The existence of such "unrealistic" fronts is motivated by the extreme case $\varepsilon \to +\infty$ for which we have formally $\partial_t \rho = F(\rho)$ (1.3). There exist travelling fronts of arbitrary speed which are solutions to $-s\nu' = F(\nu)$.

Proposition 1.21. *Given any speed $s > \varepsilon^{-1}$ there exists a smooth travelling front $\nu(x - st)$ with this speed.*

Proof 1.22. *We sketch the proof. We give below the key arguments derived from phase plane analysis. The same procedure as developped in Section 1.3.3 based on sub- and supersolutions could be reproduced based on the following ingredients.*

We learn from simple phase plane considerations associated to (1.7) *that the situation is reversed in comparison to the classical Fisher-KPP case (or $\varepsilon^2 F'(0) < 1$ and $s \in [s^*(\varepsilon), \varepsilon^{-1})$). Namely the point $(0,0)$ is a saddle point (instead of a stable node) whereas $(1,0)$ is an unstable node (instead of saddle point). This motivates "time reversal" : $V(z) = \nu(-z)$. Equation* (1.7) *becomes*

$$(\varepsilon^2 s^2 - 1)V''(z) - \left(\varepsilon^2 F'\left(V(z)\right) - 1\right) sV'(z) = F\left(V(z)\right) , \quad z \in \mathbb{R} .$$

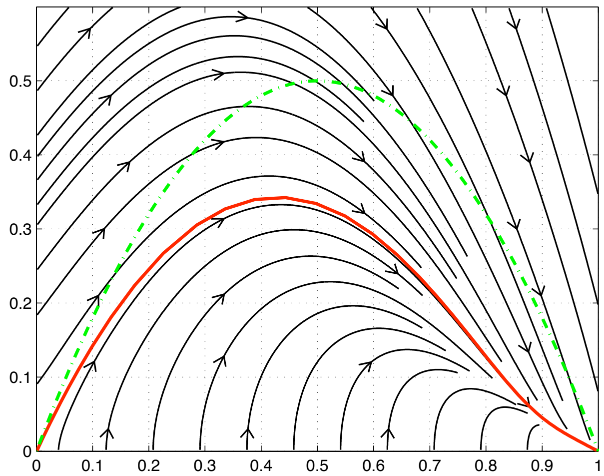


FIGURE 1.2 – Supersonic travelling front in the phase plane (V, V') for the nonlinearity $F(\rho) = \rho(1-\rho)$, and parameters $\varepsilon = \sqrt{2}$, $s = 1 > \varepsilon^{-1}$. Beware of the time reversal $\nu(z) = V(-z)$, which is the reason why $V' \geq 0$. The red line represents the travelling profile, and the green line represents the supersolution $kF(v)$.

We make the classical phase-plane transformation $V' = P$ [143, 95]. We end up with the implicit ODE with Dirichlet boundary conditions for P :

$$(\varepsilon^2 s^2 - 1)P'(v) - \left(\varepsilon^2 F'(v) - 1\right) s = \frac{F(v)}{P(v)}, \quad P(0) = P(1) = 0\,.$$

The unstable direction is given by $P(v) = \lambda v$ where λ is the positive root of

$$(\varepsilon^2 s^2 - 1)\lambda - \left(\varepsilon^2 F'(0) - 1\right) s = \frac{F'(0)}{\lambda}\,. \tag{1.18}$$

Since F is concave we deduce that $P(v) = \lambda v$ is a supersolution as in Proposition 1.7. In fact, denoting $Q(v) = P(v) - \lambda v$ we have

$$\begin{aligned}(\varepsilon^2 s^2 - 1)Q'(v) = (\varepsilon^2 s^2 - 1)(P'(v) - \lambda) &\leq s\varepsilon^2 \left(F'(v) - F'(0)\right) + \frac{F(v)}{P(v)} - \frac{F'(0)}{\lambda}, \\ &\leq F'(0)v \left(\frac{1}{P(v)} - \frac{1}{\lambda v}\right) \\ &\leq -\frac{F'(0)}{\lambda P(v)} Q(v)\,.\end{aligned}$$

Hence the trajectory leaving the saddle point $(0,0)$ in the phase plane (V, V') remains below the line $V' \leq \lambda V$.

On the other hand it is straightforward to check that $kF(v)$ is a supersolution where $k = \varepsilon^2 s/(\varepsilon^2 s^2 -$

1). *We denote* $R(v) = P(v) - kF(v)$. *We have* $ks > 1$ *and*

$$
\begin{aligned}
(\varepsilon^2 s^2 - 1)R'(v) &= (\varepsilon^2 s^2 - 1)(P'(v) - kF'(v)) \\
&= \varepsilon^2 s F'(v) - s + \frac{F(v)}{P(v)} - (\varepsilon^2 s^2 - 1)kF'(v) \\
&= -s + \frac{1}{k} - \frac{R(v)}{kP(v)} < -\frac{R(v)}{kP(v)} \, .
\end{aligned}
$$

We also show that initially (as $v \to 0$*) we have* $kF'(0) > \lambda$. *This proves that* $R(v) \leq 0$ *for all* $v \in (0,1)$. *Indeed, we plug* $kF'(0)$ *in place of* λ *into* (1.18) *and we get*

$$
(\varepsilon^2 s^2 - 1)kF'(0) - \left(\varepsilon^2 F'(0) - 1\right) s = s > \frac{1}{k} = \frac{F'(0)}{kF'(0)} \, .
$$

As a conclusion the trajectory leaving the saddle node at $(0,0)$ *is trapped in the set* $\{0 \leq v \leq 1 \, , \, 0 \leq p \leq kF(v)\}$ *(see Fig. 1.2). By the Poincaré-Bendixon Theorem it necessarily converges to the stable node at* $(1,0)$. *This heteroclinic trajectory is the travelling front in the supersonic case.*

1.4 Linear stability of travelling front solutions

In this Section we investigate the linear stability of the travelling front having minimal speed $s = s^*(\varepsilon)$ in both the parabolic and the hyperbolic regime. We seek stability in some weighted L^2 space. The important matter here is to identify the weight e^{ϕ}. The same weight shall be used crucially for the nonlinear stability analysis (Section 1.5).

We recall that the minimal speed is given by

$$
s^*(\varepsilon) = \begin{cases} \dfrac{2\sqrt{F'(0)}}{1 + \varepsilon^2 F'(0)} & \text{if } \varepsilon^2 F'(0) < 1 \\ \varepsilon^{-1} & \text{if } \varepsilon^2 F'(0) \geq 1 \end{cases}
$$

The profile of the wave has the following properties in the case $\varepsilon^2 F'(0) < 1$:

$$
\forall z \quad \nu(z) \geq 0 \, , \quad \partial_z \nu(z) \leq 0 \, , \quad \partial_z \nu(z) + \lambda \nu(z) \geq 0 \, ,
$$

where the decay exponent λ is

$$
\lambda = \frac{s(1 - \varepsilon^2 F'(0))}{2(1 - \varepsilon^2 s^2)} = \frac{1 + \varepsilon^2 F'(0)}{1 - \varepsilon^2 F'(0)} \, .
$$

We will use in this Section the formulation (1.3) of our system. The linearized system around the stationary profile ν in the moving frame $z = x - st$ reads

$$
\begin{cases} (\partial_t - s\partial_z)u + \partial_z \left(\dfrac{v}{\varepsilon} \right) = F'(\nu)u \\ \varepsilon(\partial_t - s\partial_z)v + \partial_z u = -\dfrac{v}{\varepsilon} \, . \end{cases} \tag{1.19}
$$

Proposition 1.23. *Let* $\varepsilon > 0$. *In the hyperbolic regime* $\varepsilon^2 F'(0) \geq 1$ *assume in addition that the initial perturbation has the same support as the wave. There exists a function* $\phi_\varepsilon(z)$ *such that the minimal* *speed travelling front is linearly stable in the weighted $L^2(e^{2\phi_\varepsilon(z)}dz)$ space. More precisely the following Lyapunov identity holds true for solutions of the linear system* (1.19),

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}}\left(|u|^2+|v|^2\right)e^{2\phi_\varepsilon(z)}\,dz\right)\le 0\,.$$

Proof 1.24. *We denote $\phi=\phi_\varepsilon$ for the sake of clarity. We multiply the first equation by $ue^{2\phi}$, and the second equation by $ve^{2\phi}$, where ϕ is to be determined. We get*

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}}|u|^2e^{2\phi(z)}\,dz\right)+\frac{s}{2}\int_{\mathbb{R}}|u|^2\partial_z e^{2\phi(z)}\,dz+\int_{\mathbb{R}}\partial_z\left(\frac{v}{\varepsilon}\right)ue^{2\phi(z)}\,dz = \int_{\mathbb{R}}F'(\nu)|u|^2e^{2\phi(z)}\,dz\,,$$

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}}|v|^2e^{2\phi(z)}\,dz\right)+\frac{s}{2}\int_{\mathbb{R}}|v|^2\partial_z e^{2\phi(z)}\,dz+\int_{\mathbb{R}}\partial_z\left(\frac{u}{\varepsilon}\right)ve^{2\phi(z)}\,dz = -\frac{1}{\varepsilon^2}\int_{\mathbb{R}}|v|^2e^{2\phi(z)}\,dz\,.$$

Summing the two estimates we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}}\left(|u|^2+|v|^2\right)e^{2\phi(z)}\,dz\right)+\int_{\mathbb{R}}\left(s\partial_z\phi(z)-F'(\nu)\right)|u|^2e^{2\phi(z)}\,dz + \int_{\mathbb{R}}\left(s\partial_z\phi(z)+\frac{1}{\varepsilon^2}\right)|v|^2e^{2\phi(z)}\,dz-\frac{2}{\varepsilon}\int_{\mathbb{R}}\left(\partial_z\phi(z)\right)uve^{2\phi(z)}\,dz=0\,.$$

We seek an energy dissipation estimate, see (1.21) *below. Therefore we require that the last quadratic form acting on (u,v) is nonnegative. This is guaranteed if $\partial_z\phi\ge 0$ and the following discriminant is nonpositive :*

$$\begin{aligned}\Delta(z)&=\frac{4}{\varepsilon^2}\left(\partial_z\phi(z)\right)^2-4\left(s\partial_z\phi(z)-F'(\nu)\right)\left(s\partial_z\phi(z)+\frac{1}{\varepsilon^2}\right)\\&=\frac{4}{\varepsilon^2}\left(\left(1-\varepsilon^2s^2\right)\left(\partial_z\phi(z)\right)^2-s\left(1-\varepsilon^2F'(\nu)\right)\partial_z\phi(z)+F'(\nu)\right)\,.\end{aligned}\tag{1.20}$$

The rest of the proof is devoted to finding such a weight $\phi(z)$ satisfying this sign condition. We distinguish between the parabolic and the hyperbolic regime.

1- The parabolic regime.

In the case $\varepsilon^2F'(0)<1$ we have $\varepsilon^2s^2<1$. Hence the optimal choice for $\partial_z\phi$ is :

$$\partial_z\phi(z)=\frac{s\left(1-\varepsilon^2F'(\nu)\right)}{2\left(1-\varepsilon^2s^2\right)}=\lambda\frac{1-\varepsilon^2F'(\nu)}{1-\varepsilon^2F'(0)}\ge 0\,.$$

Notice that $\partial_z\phi\to\lambda$ as $z\to+\infty$. We check that the discriminant is indeed nonpositive :

$$\begin{aligned}\varepsilon^2\Delta(z)&=-4\left(1-\varepsilon^2s^2\right)\left(\partial_z\phi(z)\right)^2+4F'(\nu)\\&=\frac{1}{(1-\varepsilon^2s^2)}\left(-s^2\left(1+\varepsilon^2F'(\nu)\right)^2+4F'(\nu)\right)\\&=\frac{1}{\left(1-\varepsilon^2F'(0)\right)^2}\left(-4F'(0)\left(1+\varepsilon^2F'(\nu)\right)^2+4F'(\nu)\left(1+\varepsilon^2F'(0)\right)^2\right)\\&=\frac{-4}{\left(1-\varepsilon^2F'(0)\right)^2}\left(F'(0)-F'(\nu)\right)\left(1-\varepsilon^4F'(0)F'(\nu)\right)\,.\end{aligned}$$

We have $\Delta(z) \leq 0$ since $\forall z$ $F'(\nu(z)) \leq F'(0)$ and $\varepsilon^2 F'(0) < 1$. Since the quadratic form is non-negative, we may control it by a sum of squares. This is the purpose of the next computation. We have

$$\left|\frac{2}{\varepsilon}\int_{\mathbb{R}} (\partial_z\phi(z))\, uv e^{2\phi(z)}\, dz\right| \leq \int_{\mathbb{R}} \left(s\partial_z\phi(z) - F'(\nu) - A(z)\right) |u|^2 e^{2\phi(z)}\, dz + \int_{\mathbb{R}} \left(s\partial_z\phi(z) + \frac{1}{\varepsilon^2} - A(z)\right) |v|^2 e^{2\phi(z)}\, dz\,,$$

where $A(z)$ is solution of

$$4\left(s\partial_z\phi(z) - F'(\nu) - A(z)\right)\left(s\partial_z\phi(z) + \frac{1}{\varepsilon^2} - A(z)\right) = \frac{4}{\varepsilon^2}\left(\partial_z\phi(z)\right)^2 .$$

A straightforward computation gives

$$\begin{aligned}
2A(z) &= \left(2s\partial_z\phi(z) - F'(\nu) + \frac{1}{\varepsilon^2}\right) - \left(\left(F'(\nu) + \frac{1}{\varepsilon^2}\right)^2 + \frac{4}{\varepsilon^2}\left(\partial_z\phi(z)\right)^2\right)^{1/2} \\
&= \frac{1-\varepsilon^2 F'(\nu)}{\varepsilon^2\left(1-\varepsilon^2 s^2\right)} - \frac{1}{\varepsilon^2\left(1-\varepsilon^2 s^2\right)}\left(\left(1-\varepsilon^2 s^2\right)^2\left(1+\varepsilon^2 F'(\nu)\right)^2 + \varepsilon^2 s^2\left(1-\varepsilon^2 F'(\nu)\right)^2\right)^{1/2} \\
&= \frac{1-\varepsilon^2 F'(\nu)}{\varepsilon^2\left(1-\varepsilon^2 s^2\right)}\left(1 - \left(\left(\frac{1-\varepsilon^2 F'(0)}{1+\varepsilon^2 F'(0)}\right)^4\left(\frac{1+\varepsilon^2 F'(\nu)}{1-\varepsilon^2 F'(\nu)}\right)^2 + \frac{4\varepsilon^2 F'(0)}{\left(1+\varepsilon^2 F'(0)\right)^2}\right)^{1/2}\right) \\
&= \frac{1-\varepsilon^2 F'(\nu)}{\varepsilon^2\left(1-\varepsilon^2 s^2\right)}\left(1 - \left(1 + \left(\frac{1-\varepsilon^2 F'(0)}{1+\varepsilon^2 F'(0)}\right)^4\left(\frac{1+\varepsilon^2 F'(\nu)}{1-\varepsilon^2 F'(\nu)}\right)^2 - \frac{\left(1-\varepsilon^2 F'(0)\right)^2}{\left(1+\varepsilon^2 F'(0)\right)^2}\right)^{1/2}\right) .
\end{aligned}$$

We clearly have $A(z) \geq 0$ since

$$\forall z \quad \frac{1+\varepsilon^2 F'(\nu)}{1-\varepsilon^2 F'(\nu)} \leq \frac{1+\varepsilon^2 F'(0)}{1-\varepsilon^2 F'(0)} .$$

Finally we obtain in the case $\varepsilon^2 F'(0) < 1$,

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}} \left(|u|^2 + |v|^2\right) e^{2\phi(z)}\, dz\right) + \int_{\mathbb{R}} A(z)\left(|u|^2 + |v|^2\right) e^{2\phi(z)}\, dz \leq 0 . \tag{1.21}$$

2- The hyperbolic regime.

We assume for simplicity that the support of the travelling profile is $\operatorname{Supp}\nu = (-\infty, 0]$. *In the hyperbolic regime we have $s = \varepsilon^{-1}$, so the discriminant equation* (1.20) *reduces to*

$$\Delta(z) = \frac{4}{\varepsilon^2}\left(-s\left(1-\varepsilon^2 F'(\nu)\right)\partial_z\phi(z) + F'(\nu)\right) .$$

We naturally choose

$$\partial_z\phi(z) = \frac{\varepsilon F'(\nu)}{1-\varepsilon^2 F'(\nu)} .$$

Within this choice for ϕ we get,

$$\frac{d}{dt}\left(\frac{1}{2}\int_{z\le 0}\left(|u|^2+|v|^2\right)e^{2\phi(z)}\,dz\right)+\int_{z\le 0}A(z)\left(\varepsilon^2F'(\nu(z))u-v\right)^2e^{2\phi(z)}\,dz=0\,,$$

where the additional weight in the dissipation writes :

$$A(z)=\frac{1}{\varepsilon^2\left(1-\varepsilon^2F'(\nu(z))\right)}\,.$$

In the case $\varepsilon^2F'(0)>1$ we have $1-\varepsilon^2F'(\nu(z))>0$ on $\operatorname{Supp}\nu$ (see Section 1.3.5). Notice that the monotonicity of ϕ may change on $\operatorname{Supp}\nu$ since $F'(\nu(z))$ may change sign. We observe that $A(z)$ is uniformly bounded from below on $\operatorname{Supp}\nu$.

In the transition case $\varepsilon^2F'(0)=1$, we have $\partial_z\phi(z)\to+\infty$ as $z\to 0^-$. We observe that $A(z)\to+\infty$ as $z\to 0^-$ too.

Example : the case $F(\rho)=\rho(1-\rho)$, and $\varepsilon=1$. We can easily compute from Section 1.3.5

$$\phi(z)=-\frac{z}{2}-\log\left(1-e^{z/2}\right)\,.$$

Remark 1.25 (Lack of coercivity)**. 1- The parabolic regime.** *We directly observe that $A(z)\to 0$ as $z\to+\infty$ in the Lyapunov identity* (1.21). *This corresponds to the lack of coercivity of the linear operator. It has been clearly identified for the classical Fisher-KPP equation [104, 105]. This lack of coercivity is a source of complication for the next question,* i.e. *nonlinear stability (see Section 1.5).*
2- The hyperbolic regime. *The situation is more degenerated here : the dissipation provides information about the relaxation of v towards $\varepsilon^2F'(\nu)u$ only.*

1.5 Nonlinear stability of travelling front solutions in the parabolic regime $\varepsilon^2 F'(0) < 1$

In this Section we investigate the stability of the travelling profile having minimal speed $s=s^*(\varepsilon)$ in the parabolic regime. We seek stability in the energy class. Energy methods have been successfully applied to reaction-diffusion equations [104, 105, 183, 106]. We follow the strategy developped in [105] for a simpler equation, namely the damped hyperbolic Fisher-KPP equation.

Before stating the theorem we give some useful notations. The perturbation is denoted by $u(t,z)=\rho(t,z)-\nu(z)$ where $z=x-st$ is the space variable in the moving frame. We also need some weighted perturbation $w=e^{\phi}u$, where ϕ is an explicit weight to be precised later (1.33).

Theorem 1.26. *For all $\varepsilon\in\left(0,1/\sqrt{F'(0)}\right)$ there exists a constant $c(\varepsilon)$ such that the following claim holds true : let u^0 be any compactly supported initial perturbation which satisfies*

$$\|u^0\|^2_{H^1(\mathbb{R})}+\|w^0\|^2_{H^1(\mathbb{R})}\le c(\varepsilon)\,,$$

then there exists $z_0\in\mathbb{R}$ such that

$$\sup_{t>0}\left(\|\partial_z u(t,\cdot)\|_2^2+\int_{z<z_0}|u(t,z)|^2\,dz+\|w(t,\cdot)\|^2_{H^1}\right)\le c(\varepsilon)\,,$$

remains uniformly small for all time $t > 0$, and the perturbation is globally decaying in the following sense :

$$\left(\|\partial_z u\|_2^2 + \int_{z<z_0} |u|^2 \, dz + \|\partial_z w\|_2^2 + \int_{z>z_0} e^{-\phi(z)} |w|^2 \, dz \right) \in L^2(0, +\infty)\,.$$

Remark 1.27. *1. The additional weight $e^{-\phi(z)}$ in the last contribution (weighted L^2 space) is specific to the lack of coercivity in the energy estimates.*

2. *The constant $c(\varepsilon)$ that we obtain degenerates as $\varepsilon \to 1/\sqrt{F'(0)}$, due to the transition from a parabolic to an hyperbolic regime.*
3. *We restrict ourselves to compactly supported initial perturbations u^0 to justify all integration by parts. Indeed the solution $u(t,z)$ remains compactly supported for all $t > 0$ because of the finite speed of propagation (see the kinetic formulation* (1.8) *and [91, Chapter 12]). The result would be the same if we were assuming that u^0 decays sufficiently fast at infinity.*

Proof 1.28. *We proceed in several steps.*

1- Derivation of the energy estimates. *The equation satisfied by the perturbation u writes*

$$\varepsilon^2 \left(\partial_{tt} u - 2s\partial_{tz} u + s^2 \partial_{zz} u \right) + \left(1 - \varepsilon^2 F'(v+u) \right) \left(\partial_t u - s\partial_z u \right) - \partial_{zz} u \\ + \varepsilon^2 (F'(v+u) - F'(v)) s\partial_z v = F(v+u) - F(v)\,. \quad (1.22)$$

We write the nonlinearities as follows :

$$F'(v+u) = F'(v) + K_1(z;u)u\,,$$
$$F'(v+u) - F'(v) = F''(v)u + K_2(z;u)u^2\,,$$
$$F(v+u) - F(v) = F'(v)u + K_3(z;u)u^2\,.$$

where the functions K_i are uniformly bounded in $L^\infty(\mathbb{R})$. More precisely we have

$$K_1(z;u) = \int_0^1 F''(v+tu)dt\,,$$
$$K_2(z;u) = \int_0^1 (1-t)F'''(v+tu)dt\,,$$
$$K_3(z;u) = \int_0^1 (1-t)F''(v+tu)dt\,.$$

Thus we can decompose equation (1.22) *into linear and nonlinear contributions :*

$$\varepsilon^2 \left(\partial_{tt} u - 2s\partial_{tz} u + s^2 \partial_{zz} u \right) \\ + \left(1 - \varepsilon^2 F'(v) \right) \left(\partial_t u - s\partial_z u \right) - \partial_{zz} u + \left(s\varepsilon^2 F''(v)\partial_z v - F'(v) \right) u \\ = \varepsilon^2 K_1(z;u) u \left(\partial_t u - s\partial_z u \right) + \left(K_3(z;u) - s\varepsilon^2 K_2(z;u)\partial_z v \right) u^2\,. \quad (1.23)$$

Testing equation (1.23) *against $\partial_t u - s\partial_z u$ yields our first energy estimate (hyperbolic energy) :*

$$\frac{d}{dt} \left\{ \frac{\varepsilon^2}{2} \int_{\mathbb{R}} |\partial_t u - s\partial_z u|^2 \, dz + \frac{1}{2} \int_{\mathbb{R}} |\partial_z u|^2 \, dz + \frac{1}{2} \int_{\mathbb{R}} \left(s\varepsilon^2 F''(v)\partial_z v - F'(v) \right) |u|^2 \, dz \right\} \\ + \int_{\mathbb{R}} \left(1 - \varepsilon^2 F'(v) \right) |\partial_t u - s\partial_z u|^2 + \frac{s}{2} \int_{\mathbb{R}} \partial_z \left(s\varepsilon^2 F''(v)\partial_z v - F'(v) \right) |u|^2 \, dz \\ = \varepsilon^2 \int_{\mathbb{R}} K_1(z;u) u \, |\partial_t u - s\partial_z u|^2 \, dz + \int_{\mathbb{R}} \left(K_3(z;u) - se^2 K_2(z;u)\partial_z v \right) u^2 \left(\partial_t u - s\partial_z u \right) \, dz\,. \quad (1.24)$$

We are lacking coercivity with respect to H^1 norm in the energy dissipation. Testing equation (1.23) *against u yields our second energy estimate (parabolic energy) :*

$$\frac{d}{dt}\left\{\varepsilon^2 \int_{\mathbb{R}} u\left(\partial_t u - s\partial_z u\right)\,dz + \frac{1}{2}\int_{\mathbb{R}}\left(1-\varepsilon^2 F'(\nu)\right)|u|^2\,dz\right\}$$
$$- \varepsilon^2 \int_{\mathbb{R}} |\partial_t u - s\partial_z u|^2\,dz + \int_{\mathbb{R}} |\partial_z u|^2\,dz + \int_{\mathbb{R}} \left(\frac{s\varepsilon^2}{2}F''(\nu)\partial_z\nu - F'(\nu)\right)|u|^2\,dz$$
$$= \varepsilon^2 \int_{\mathbb{R}} K_1(z;u)u^2\left(\partial_t u - s\partial_z u\right)\,dz + \int_{\mathbb{R}}\left(K_3(z;u) - se^2 K_2(z;u)\partial_z\nu\right)u^3\,dz\,. \quad (1.25)$$

We introduce the following notations for the two energy contributions and the respective quadratic dissipations (1.24),(1.25) *:*

$$E_1^u(t) = \frac{\varepsilon^2}{2}\int_{\mathbb{R}} |\partial_t u - s\partial_z u|^2\,dz + \frac{1}{2}\int_{\mathbb{R}}|\partial_z u|^2\,dz + \frac{1}{2}\int_{\mathbb{R}}\left(s\varepsilon^2 F''(\nu)\partial_z\nu - F'(\nu)\right)|u|^2\,dz\,,$$
$$E_2^u(t) = \varepsilon^2 \int_{\mathbb{R}} u\left(\partial_t u - s\partial_z u\right)\,dz + \frac{1}{2}\int_{\mathbb{R}}\left(1-\varepsilon^2 F'(\nu)\right)|u|^2\,dz\,,$$
$$Q_1^u(t) = \int_{\mathbb{R}}\left(1-\varepsilon^2 F'(\nu)\right)|\partial_t u - s\partial_z u|^2 + \frac{s}{2}\int_{\mathbb{R}}\partial_z\left(s\varepsilon^2 F''(\nu)\partial_z\nu - F'(\nu)\right)|u|^2\,dz\,,$$
$$Q_2^u(t) = -\varepsilon^2\int_{\mathbb{R}}|\partial_t u - s\partial_z|^2\,dz + \int_{\mathbb{R}}|\partial_z u|^2\,dz + \int_{\mathbb{R}}\left(\frac{s\varepsilon^2}{2}F''(\nu)\partial_z\nu - F'(\nu)\right)|u|^2\,dz\,.$$

The delicate issue is to control the zeroth-order terms. In particular we define the weights

$$A_1(z) = s\varepsilon^2 F''(\nu)\partial_z\nu - F'(\nu)\,,$$
$$A_2(z) = \frac{s\varepsilon^2}{2}F''(\nu)\partial_z\nu - F'(\nu)\,.$$

They change sign over $\mathbb{R}$. More precisely we have

$$\lim_{z\to-\infty} A_1(z) = \lim_{z\to-\infty} A_2(z) = -F'(1) > 0\,,$$
$$\lim_{z\to+\infty} A_1(z) = \lim_{z\to+\infty} A_2(z) = -F'(0) < 0\,.$$

To circumvent this issue we introduce $w(t,z) = e^{\phi(z)}u(t,z)$ as in [105] and the previous Section 1.4, where $\phi(z)$ is a weight to be determined later (1.32)*. The new function $w(t,z)$ satisfies the following equation :*

$$\varepsilon^2\partial_{tt}w - 2\varepsilon^2 s\partial_{tz}w + \left(2\varepsilon^2 s\partial_z\phi + 1 - \varepsilon^2 F'(\nu)\right)\partial_t w$$
$$+ \left(-s\left(1-\varepsilon^2 F'(\nu)\right) - 2(\varepsilon^2 s^2 - 1)\partial_z\phi\right)\partial_z w$$
$$+ (\varepsilon^2 s^2 - 1)\partial_{zz}w + \left(s\left(1-\varepsilon^2 F'(\nu)\right)\partial_z\phi\right.$$
$$+(\varepsilon^2 s^2 - 1)\left(-\partial_{zz}\phi + |\partial_z\phi|^2\right)$$
$$\left. + \varepsilon^2 s F''(\nu)\partial_z\nu - F'(\nu)\right) w = \varepsilon^2 K_1(z;u)u\left(\partial_t w - s\partial_z w\right)$$
$$+ \left(K_3(z;u) - \varepsilon^2 s K_2(z;u)\partial_z\nu + \varepsilon^2 s K_1(z;u)\partial_z\phi\right)uw\,. \quad (1.26)$$

We denote the prefactors of $\partial_t w$, $\partial_z w$ and w as A_3, A_4 and A_5 respectively :

$$A_3(z) = 2\varepsilon^2 s\partial_z\phi + 1 - \varepsilon^2 F'(\nu)\,,$$
$$A_4(z) = -s\left(1-\varepsilon^2 F'(\nu)\right) - 2(\varepsilon^2 s^2 - 1)\partial_z\phi\,, \quad (1.27)$$
$$A_5(z) = s\left(1-\varepsilon^2 F'(\nu)\right)\partial_z\phi + (\varepsilon^2 s^2 - 1)\left(-\partial_{zz}\phi + |\partial_z\phi|^2\right) + \varepsilon^2 s F''(\nu)\partial_z\nu - F'(\nu)\,.$$

Testing (1.26) *against* $\partial_t w$, *we obtain our third energy estimate :*

$$\begin{aligned}\frac{d}{dt}\left\{\frac{\varepsilon^2}{2}\int_{\mathbb{R}}|\partial_t w|^2\,dz + \frac{1-\varepsilon^2 s^2}{2}\int_{\mathbb{R}}|\partial_z w|^2\,dz + \frac{1}{2}\int_{\mathbb{R}}A_5(z)|w|^2\,dz\right\}&\\ + \int_{\mathbb{R}}A_3(z)|\partial_t w|^2\,dz + \int_{\mathbb{R}}A_4(z)\partial_t w\partial_z w\,dz&\\ = \varepsilon^2\int_{\mathbb{R}}K_1(z;u)u\left(|\partial_t w|^2 - s\partial_t w\partial_z w\right)\,dz&\\ + \int_{\mathbb{R}}\left(K_3(z;u) - \varepsilon^2 sK_2(z;u)\partial_z\nu + \varepsilon^2 sK_1(z;u)\partial_z\phi\right)uw\partial_t w\,dz\,,&\end{aligned}$$

Testing (1.26) *against* w *we obtain our last energy estimate :*

$$\begin{aligned}\frac{d}{dt}\left\{\varepsilon^2\int_{\mathbb{R}}w\partial_t w\,dz + \frac{1}{2}\int_{\mathbb{R}}A_3(z)|w|^2\,dz\right\}&\\ -\varepsilon^2\int_{\mathbb{R}}|\partial_t w|^2\,dz + (1-\varepsilon^2 s^2)\int_{\mathbb{R}}|\partial_z w|^2\,dz&\\ + 2s\varepsilon^2\int_{\mathbb{R}}\partial_t w\partial_z w\,dz + \int_{\mathbb{R}}\left(A_5(z) - \frac{\partial_z A_4(z)}{2}\right)|w|^2\,dz&\\ = \int_{\mathbb{R}}\left(K_3(z;u) - \varepsilon^2 sK_2(z;u)\partial_z\nu + \varepsilon^2 sK_1(z;u)\partial_z\phi\right)uw^2\,dz&\\ + \int_{\mathbb{R}}\varepsilon^2 K_1(z;u)uw\left(\partial_t w - s\partial_z w\right)\,dz\,,&\end{aligned}$$

We introduce again useful notations for the two energy contributions and the associated quadratic dissipations :

$$E_1^w(t) = \frac{\varepsilon^2}{2}\int_{\mathbb{R}}|\partial_t w|^2\,dz + \frac{1-\varepsilon^2 s^2}{2}\int_{\mathbb{R}}|\partial_z w|^2\,dz + \frac{1}{2}\int_{\mathbb{R}}A_5(z)|w|^2\,dz\,, \tag{1.28}$$

$$E_2^w(t) = \varepsilon^2\int_{\mathbb{R}}w\partial_t w\,dz + \frac{1}{2}\int_{\mathbb{R}}A_3(z)|w|^2\,dz\,, \tag{1.29}$$

$$Q_1^w(t) = \int_{\mathbb{R}}A_3(z)|\partial_t w|^2\,dz + \int_{\mathbb{R}}A_4(z)\partial_t w\partial_z w\,dz\,.$$

$$Q_2^w(t) = -\varepsilon^2\int_{\mathbb{R}}|\partial_t w|^2\,dz + (1-\varepsilon^2 s^2)\int_{\mathbb{R}}|\partial_z w|^2\,dz + 2\varepsilon^2 s\int_{\mathbb{R}}\partial_t w\partial_z w\,dz \tag{1.30}$$

$$+ \int_{\mathbb{R}}\left(A_5(z) - \frac{\partial_z A_4(z)}{2}\right)|w|^2\,dz\,. \tag{1.31}$$

To determine $\phi(z)$ *we examine* (1.28)–(1.29). *We first require the natural condition*

$$\partial_z\phi(z) \geq 0.$$

This clearly ensures $A_3(z) \geq 1 - \varepsilon^2 F'(0)$. *We examinate the condition* $A_5(z) - \frac{1}{2}\partial_z A_4(z) \geq 0$ (1.31) *in order to fully determine the weight* $\phi(z)$ *:*

$$\begin{aligned}A_5(z) - \frac{\partial_z A_4(z)}{2} &= s\left(1-\varepsilon^2 F'(\nu)\right)\partial_z\phi + (\varepsilon^2 s^2 - 1)\left(-\partial_{zz}\phi + |\partial_z\phi|^2\right) + \varepsilon^2 sF''(\nu)\partial_z\nu\\ &\quad - F'(\nu) - \frac{1}{2}\varepsilon^2 sF''(\nu)\partial_z\nu + (\varepsilon^2 s^2 - 1)\partial_{zz}\phi\\ &= (\varepsilon^2 s^2 - 1)\,|\partial_z\phi|^2 + s\left(1-\varepsilon^2 F'(\nu)\right)\partial_z\phi + \frac{1}{2}\varepsilon^2 sF''(\nu)\partial_z\nu - F'(\nu)\,.\end{aligned}$$

This is a second-order equation in the variable $\partial_z \phi$. Maximization of this quantity is achieved when

$$\partial_z \phi = \frac{s\left(1-\varepsilon^2 F'(\nu)\right)}{2(1-\varepsilon^2 s^2)} \;\geq\; 0\,. \tag{1.32}$$

We notice that this is equivalent to setting $A_4(z) = 0$ (1.27). Then we obtain

$$\begin{aligned}
A_5(z) &= \frac{s^2\left(1-\varepsilon^2 F'(\nu)\right)^2}{4(1-\varepsilon^2 s^2)} + \frac{1}{2}\varepsilon^2 s F''(\nu)\partial_z \nu - F'(\nu) \\
&= \frac{1}{4(1-\varepsilon^2 s^2)}\left(s^2\left(1+\varepsilon^2 F'(\nu)\right)^2 - 4F'(\nu)\right) + \frac{1}{2}\varepsilon^2 s F''(\nu)\partial_z \nu \\
&= \frac{1}{4\left(1-\varepsilon^2 F'(0)\right)^2}\left(4F'(0)\left(1+\varepsilon^2 F'(\nu)\right)^2 - 4F'(\nu)\left(1+\varepsilon^2 F'(0)\right)^2\right) + \frac{1}{2}\varepsilon^2 s F''(\nu)\partial_z \nu \\
&= \frac{1}{\left(1-\varepsilon^2 F'(0)\right)^2}\left(F'(0)-F'(\nu)\right)\left(1-\varepsilon^4 F'(0)F'(\nu)\right) + \frac{1}{2}\varepsilon^2 s F''(\nu)\partial_z \nu\,.
\end{aligned}$$

We check that $A_5(z) \geq 0$ since $\forall z\; F'(\nu(z)) \leq F'(0)$, $\varepsilon^2 F'(0) < 1$, $\forall z\; F''(\nu(z)) \leq 0$ and $\forall z\; \partial_z\, \nu(z) \leq 0$.

We recall that the exponential decay of ν at $+\infty$ is given by the eigenvalue $\lambda > 0$, where

$$\lambda = \frac{s(1-\varepsilon^2 F'(0))}{2(1-\varepsilon^2 s^2)}\,.$$

Therefore we can rewrite

$$\partial_z \phi = \lambda \frac{1-\varepsilon^2 F'(\nu)}{1-\varepsilon^2 F'(0)}\,. \tag{1.33}$$

Remark 1.29. *As far as we are concerned with linear stability, the energies E_1^w and E_2^w contain enough information. However proving nonlinear stability requires an additional control of u in L^∞ which can be obtained using E_1^u and E_2^u [105].*

2- Combination of the energy estimates. *We first examinate the energies E_1^u and E_2^u. We clearly have*

$$\begin{aligned}
E_2^u(t) &\geq -\frac{\varepsilon^4}{1-\varepsilon^2 F'(0)}\left\|\partial_t u - s\partial_z u\right\|_2^2 - \frac{1-\varepsilon^2 F'(0)}{4}\|u\|_2^2 + \frac{1-\varepsilon^2 F'(0)}{2}\|u\|_2^2 \\
&\geq -\frac{\varepsilon^4}{1-\varepsilon^2 F'(0)}\left\|\partial_t u - s\partial_z u\right\|_2^2 + \frac{1-\varepsilon^2 F'(0)}{4}\|u\|_2^2
\end{aligned}$$

We set

$$\delta = \frac{1-\varepsilon^2 F'(0)}{2\varepsilon^2}\,.$$

We have on the one hand

$$E_1^u(t) + \delta E_2^u(t) \geq \frac{1}{2}\left\|\partial_z u\right\|_2^2 dz + \int_{\mathbb{R}} A_6(z)|u|^2\, dz\,,$$

where $A_6(z)$ is defined as

$$A_6(z) = \frac{1}{2}A_1(z) + \delta\frac{1-\varepsilon^2 F'(0)}{4}\,.$$

We have on the other hand,

$$Q_1^u(t) + \delta Q_2^u(t) \geq \frac{1-\varepsilon^2 F'(0)}{2} \|\partial_t u - s\partial_z u\|_2^2 + \delta \|\partial_z u\|_2^2 + \int_{\mathbb{R}} A_7(z)|u|^2\, dz\,,$$

where $A_7(z)$ *is defined as*

$$A_7(z) = \frac{s}{2}\partial_z \left(s\varepsilon^2 F''(v)\partial_z v - F'(v)\right) + \delta A_2(z)\,.$$

We have both $\lim_{z\to-\infty} A_6(z) > 0$ *and* $\lim_{z\to-\infty} A_7(z) > 0$*. Accordingly there exists* $\alpha > 0$ *and* $z_0 \in \mathbb{R}$ *such that*

$$\forall z < z_0 \quad \min(A_6(z), A_7(z)) > \alpha\,.$$

In order to control the zeroth-order terms over $(z_0, +\infty)$ *we shall use the last two energy estimates. First we observe that* $\forall z > z_0$ $|u(z)| = |e^{-\phi(z)}w(z)| \leq e^{-\phi(z_0)}|w(z)|$ *since* ϕ *is increasing. We set* $\phi(z_0) = 0$ *without loss of generality. This determines completely* ϕ *together with the condition* (1.32)*. We have*

$$E_1^u(t) + \delta E_2^u(t) \geq \frac{1}{2}\|\partial_z u\|_2^2\, dz + \alpha \int_{z<z_0} |u|^2\, dz - \left\| A_6 e^{-2\phi}\mathbf{1}_{z>z_0}\right\|_\infty \int_{z>z_0} |w|^2\, dz\,,$$

$$Q_1^u(t) + \delta Q_2^u(t) \geq \frac{1-\varepsilon^2F'(0)}{2}\|\partial_t u - s\partial_z u\|_2^2 + \delta\|\partial_z u\|_2^2 + \alpha \int_{z<z_0} |u|^2\, dz$$
$$- \left\| \frac{A_7 e^{-2\phi}}{A_5}\mathbf{1}_{z>z_0}\right\|_\infty \int_{z>z_0} A_5(z)|w|^2\, dz\,.$$

Lemma 1.30. *We have* $\dfrac{A_7 e^{-2\phi}}{A_5} \in L^\infty(z_0, +\infty)$ *and* $\dfrac{e^{-\phi}}{A_5} \in L^\infty(z_0,+\infty)$*.*

Proof 1.31. *The first claim is clearly a consequence of the second claim since* $A_7 e^{-\phi} \in L^\infty(z_0, +\infty)$*. First we have*

$$F'(0) - F'(v) \geq \left(\inf_{[0,1]} \left(-F''\right)\right) v \;= \alpha v\;,$$

where $\alpha > 0$ *is the coercivity constant of* $-F$ (1.2)*. As a consequence,* $A_5 \geq \left(\frac{(1+\varepsilon^2F'(0))\alpha}{1-\varepsilon^2F'(0)}\right) v$*. Second we recall* $\partial_z v + \lambda v \geq 0$ *(Lemma 1.16), so that* $\forall z > z_0,\ v(z) \geq v(z_0)e^{-\lambda(z-z_0)}$*. Finally we have* $\forall z,\ \partial_z\phi \geq \lambda$ (1.33)*, thus*

$$\forall z > z_0, \quad e^{-\phi(z)} \leq e^{-\phi(z_0)}e^{-\lambda(z-z_0)} \leq \frac{e^{-\phi(z_0)}}{v(z_0)} v(z)$$
$$\leq \frac{e^{-\phi(z_0)}}{v(z_0)}\left(\frac{1-\varepsilon^2F'(0)}{(1+\varepsilon^2F'(0))\,\alpha}\right) A_5(z)\,.$$

We now focus on the second series of energy estimates. We clearly have

$$E_2^w(t) \geq -\frac{\varepsilon^4}{1-\varepsilon^2F'(0)}\|\partial_t w\|_2^2 - \frac{1-\varepsilon^2F'(0)}{4}\|w\|_2^2 + \frac{1-\varepsilon^2F'(0)}{2}\|w\|_2^2$$
$$\geq -\frac{\varepsilon^4}{1-\varepsilon^2F'(0)}\|\partial_t w\|_2^2 + \frac{1-\varepsilon^2F'(0)}{4}\|w\|_2^2\,,$$

and

$$Q_2^w(t) \geq -\varepsilon^2 \|\partial_t w\|_2^2 + (1-\varepsilon^2 s^2)\|\partial_z w\|_2^2 - \frac{2\varepsilon^4 s^2}{1-\varepsilon^2 s^2}\|\partial_t w\|_2^2 - \frac{1-\varepsilon^2 s^2}{2}\|\partial_z w\|_2^2 + \int_{\mathbb{R}} A_5(z)|w|^2\, dz$$
$$\geq -\varepsilon^2 \frac{1+\varepsilon^2 s^2}{1-\varepsilon^2 s^2}\|\partial_t w\|_2^2 + \frac{1-\varepsilon^2 s^2}{2}\|\partial_z w\|_2^2 + \left\| \frac{e^{-\phi}}{A_5}\mathbf{1}_{z>z_0}\right\|_\infty^{-1} \int_{z>z_0} e^{-\phi}|w|^2\, dz\,.$$

We set

$$\delta' = \frac{1-\varepsilon^2 F'(0)}{2\varepsilon^2}\cdot\frac{1-\varepsilon^2 s^2}{1+\varepsilon^2 s^2} < \delta\,.$$

We have on the one hand

$$E_1^w(t) + \delta' E_2^w(t) \geq \frac{1-\varepsilon^2 s^2}{2}\, \|\partial_z w\|_2^2\, dz + \int_{\mathbb{R}} A_8(z)|w|^2\, dz\,,$$

where $A_8(z)$ is defined as

$$A_8(z) = \frac{1}{2}A_5(z) + \delta'\frac{1-\varepsilon^2 F'(0)}{4} \geq \delta'\frac{1-\varepsilon^2 F'(0)}{4}\,.$$

We have on the other hand,

$$Q_1^w(t) + \delta' Q_2^w(t) \geq \frac{1-\varepsilon^2 F'(0)}{2}\, \|\partial_t w\|_2^2 + \delta'\frac{1-\varepsilon^2 s^2}{2}\|\partial_z w\|_2^2 + \delta' \left\| \frac{e^{-\phi}}{A_5}\mathbf{1}_{z>z_0}\right\|_\infty^{-1} \int_{z>z_0} h|w|^2\, dz\,.$$

Combining all these estimates, we define $E(t) = E_1^w(t) + \delta' E_2^w(t) + \delta''\left(E_1^u(t) + \delta E_2^u(t)\right)$ and $Q(t) = Q_1^w(t) + \delta' Q_2^w(t) + \delta''\left(Q_1^u(t) + \delta Q_2^u(t)\right)$, where $\delta'' > 0$ is defined such as the following condition holds true

$$\delta'' < \delta' \min\left(\frac{1-\varepsilon^2 F'(0)}{4}\left\| A_6 e^{-2\phi}\mathbf{1}_{z>z_0}\right\|_\infty^{-1}, \left\| \frac{A_7 e^{-2\phi}}{A_5}\mathbf{1}_{z>z_0}\right\|_\infty^{-1}\right)\,.$$

We finally obtain our main estimate,

$$\frac{d}{dt}E(t) + Q(t) \leq \mathcal{O}\left(\int_{\mathbb{R}} |u||\partial_t u - s\partial_z u|^2\, dz + \int_{\mathbb{R}} |u|^3\, dz\right) + \mathcal{O}\left(\int_{\mathbb{R}} e^{-\phi}|w||\partial_t w|^2\, dz + \int_{\mathbb{R}} e^{-\phi}|w||\partial_z w|^2\, dz + \int_{\mathbb{R}} e^{-\phi}|w|^3\, dz\right)\,, \quad (1.34)$$

where

$$E(t) \geq \mathcal{O}\left(\|\partial_z u\|_2^2 + \int_{z<z_0} |u|^2\, dz + \|\partial_z w\|_2^2 + \|w\|_2^2\right)\,,$$
$$Q(t) \geq \mathcal{O}\left(\|\partial_t - s\partial_z u\|_2^2 + \|\partial_z u\|_2^2 + \int_{z<z_0} |u|^2\, dz + \|\partial_t w\|_2^2 + \|\partial_z w\|_2^2 + \int_{z>z_0} e^{-\phi}|w|^2\, dz\right)\,.$$

3- Control of the nonlinear contributions. *Our goal is to control the size of the perturbation* u *in* L^∞*. For this purpose we use the embeddings of* $H^1(\mathbb{R})$ *into* $L^\infty(\mathbb{R})$ *and* $L^4(\mathbb{R})$ *:*

$$\|u\|_\infty \le C\|u\|_2^{1/2}\|\partial_z u\|_2^{1/2}\,,$$
$$\|u\|_4 \le C\|u\|_2^{3/4}\|\partial_z u\|_2^{1/4}\,.$$

We examinate successively the nonlinear contributions. We recall $u = e^{-\phi}w$*. First we have*

$$\begin{aligned}\int_{\mathbb{R}} |u||\partial_t u - s\partial_z u|^2\,dz &\le \|u\|_\infty\|\partial_t - s\partial_z u\|_2^2\\ &\le \mathcal{O}\left(E(t)^{1/2}Q(t)\right)\,,\end{aligned}$$

and similar estimates can be derived for all the contributions in the r.h.s. of (6.1) *except for the last one. Second we have*

$$\begin{aligned}\int_{\mathbb{R}} e^{-\phi}|w|^3\,dz &= \int_{z<z_0} e^{-\phi}|w|^3\,dz + \int_{z>z_0} e^{-\phi}|w|^3\,dz\\ &\le \|u\|_{L^\infty(-\infty,z_0)}\|w\|^2_{L^2(-\infty,z_0)} + \|e^{-\phi/2}w\|^2_{L^4(z_0,+\infty)}\|w\|_{L^2(z_0,+\infty)}\\ &\le C\|u\|^{1/2}_{L^2(-\infty,z_0)}\|\partial_z u\|^{1/2}_{L^2(-\infty,z_0)}\|w\|^2_{L^2(-\infty,z_0)}\\ &\quad + C\|e^{-\phi/2}w\|^{3/2}_{L^2(z_0,+\infty)}\left\|\partial_z\left(e^{-\phi/2}w\right)\right\|^{1/2}_{L^2(z_0,+\infty)}\|w\|_{L^2(z_0,+\infty)}\,.\end{aligned}$$

We have

$$\|w\|^2_{L^2(-\infty,z_0)} \le \|u\|^2_{L^2(-\infty,z_0)}\,,$$
$$\|e^{-\phi/2}w\|^2_{L^2(z_0,+\infty)} = \int_{z>z_0} e^{-\phi}|w|^2\,dz\,,$$

and

$$\begin{aligned}\left\|\partial_z\left(e^{-\phi/2}w\right)\right\|^2_{L^2(z_0,+\infty)} &\le 2\int_{z>z_0}\left(e^{-\phi}|\partial_z w|^2 + \frac{1}{4}|\partial_z\phi|^2 e^{-\phi}|w|^2\right)dz\\ &\le 2\int_{z>z_0}|\partial_z w|^2\,dz + C\int_{z>z_0} e^{-\phi}|w|^2\,dz\,.\end{aligned}$$

Consequently we obtain

$$\int_{\mathbb{R}} e^{-\phi}|w|^3\,dz \le \mathcal{O}\left(E(t)^{1/2}Q(t)\right)\,.$$

Finally we get

$$\frac{d}{dt}E(t) + Q(t) \le \mathcal{O}\left(E(t)^{1/2}Q(t)\right)\,.$$

This estimate ensures that the energy is nonincreasing provided that it is initially small enough. Indeed there exists a constant C *such that* $\frac{d}{dt}E(t) + Q(t) \le CE(t)^{1/2}Q(t)$*. We set* $c = C^{-2}/2$*. If initially* $E^0 \le c$ *then the previous differential inequality guarantees that* $E(t)$ *is decaying and remains below the level* c*. Therefore* $E(t)$ *is positive decaying, and the dissipation* $Q(t)$ *is integrable. This concludes the proof of Theorem 1.26.*

Chapitre 2

Equations cinétiques de transport-réaction : Le cas d'un continuum de vitesses

Dans cet article en collaboration avec Vincent Calvez et Grégoire Nadin, nous étudions l'existence et la stabilité d'ondes progressives solutions d'un modèle cinétique de transport-réaction. Le modèle décrit des particules qui se déplacent en changeant de direction via un processus de saut en vitesse ("velocity-jump process") et qui prolifèrent via un terme de réaction de type monostable. Le caractère borné ou non se révèle être une condition nécessaire et suffisante pour l'existence d'ondes progressives positives. La vitesse minimale d'existence de ces ondes est obtenue à partir d'une relation de dispersion explicite. Nous construisons les ondes en utilisant une technique de sur- et sous- solutions et prouvons qu'elles sont stables (dans un sens faible) dans un espace L^2 à poids. Dans le cas d'un espace de vitesses non-borné, nous prouvons que la propagation est sur-linéaire. Il apparait que la vitesse de propagation dépend fortement de la décroissance à l'infini de la distribution stationnaire. Dans le cas d'une distribution Gaussienne, nous prouvons que le front se propage comme $x \sim t^{\frac{3}{2}}$.

Contents

2.1 Introduction

We address the issue of front propagation in a reaction-transport equation of kinetic type,

$$\begin{cases} \partial_t g + v\partial_x g = \big(M(v)\rho_g - g\big) + r\rho_g\,(M(v) - g)\,, & (t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V\,, \\ g(0,x,v) = g^0(x,v)\,, & (x,v) \in \mathbb{R} \times V\,. \end{cases} \tag{2.1}$$

Here, the density $g(t,x,v)$ describes a population of individuals in a continuum setting, and $\rho_g(t,x) = \int_V g(t,x,v)\,dv$ is the macroscopic density. The subset $V \subset \mathbb{R}$ is the set of all possible velocities. Individuals move following a velocity-jump process : they run with speed $v \in V$, and change velocity at rate 1. They instantaneously choose a new velocity with the probability distribution M. Unless otherwise stated, we assume in this paper that V is symmetric and M satisfies the following properties : $M \in L^1(V) \cap \mathcal{C}^0(V)$, and

$$\int_V M(v)dv = 1\,, \quad \int_V vM(v)dv = 0\,, \quad \int_V v^2 M(v)dv = D < +\infty\,. \tag{2.2}$$

In addition, individuals are able to reproduce, with rate $r > 0$. New individuals start with a velocity chosen at random with the same probability distribution M. We could have chosen a different distribution without changing the main results, but we keep the same for the sake of clarity. Finally, we include a quadratic saturation term, which accounts for local competition between individuals, regardless of their speed.

The main motivation for this work comes from the study of pulse waves in bacterial colonies of *Escherichia coli* [3, 139, 186, 185]. Kinetic models have been proposed to describe the run-and-tumble motion of individual bacteria at the mesoscopic scale [5, 175]. Several works have been dedicated to derive macroscopic equations from those kinetic models in the diffusion limit [174, 85, 55, 186]. Recently it has been shown that for some set of experiments, the diffusion approximation is not valid, so one has to stick to the kinetic description at the mesoscopic scale to closely compare with data [185].

There is one major difference between this motivation and model (2.1). Pulse waves in bacterial colonies of *E. coli* are mainly driven by chemotaxis which generates macroscopic fluxes. Growth of the population can be merely ignored in such models. In model (2.1) however, growth and dispersion are the main reasons for front propagation, and there is no macroscopic flux due to the velocity-jump process since the distribution M satisfies $\int_V vM(v)dv = 0$. For the sake of applications, we also refer to the growth and branching of the plant pathogen *Phytophthora* by mean of a reaction-transport equation similar to (2.1) [125].

There is a strong link between (2.1) and the classical Fisher-KPP equation [97, 143]. In case of a suitable balance between scattering and growth (more scattering than growth), we can perform the parabolic rescaling $(r, t, x) \mapsto \left(\varepsilon^2 r, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)$ in (2.1),

$$\varepsilon^2 \partial_t g_\varepsilon + \varepsilon v \partial_x g_\varepsilon = \left(M(v)\rho_{g_\varepsilon} - g_\varepsilon\right) + \varepsilon^2 r \rho_{g_\varepsilon} \left(M(v) - g_\varepsilon\right) . \tag{2.3}$$

The diffusion limit yields $g_\varepsilon \to M(v)\rho_0$, where ρ_0 is solution to the Fisher-KPP equation (see [69] for example),

$$\partial_t \rho_0 - D\partial_{xx}\rho_0 = r\rho_0 \left(1 - \rho_0\right) . \tag{2.4}$$

We recall that for nonincreasing initial data decaying sufficiently fast at $x = +\infty$, the solution of (7.2) behaves asymptotically as a travelling front moving at the minimal speed $c^* = 2\sqrt{rD}$ [143, 10]. In addition, this front is stable in some weighted L^2 space [140, 104]. Therefore it is natural to address the same questions for (2.1). We give below the definition of a travelling wave for equation (2.1).

Definition 2.1. *A function $g(t, x, v)$ is a smooth travelling wave solution of speed $c \in \mathbb{R}_+$ of equation* (2.1) *if it can be written $g(t, x, v) = f(x - ct, v)$, where the profile $f \in \mathcal{C}^2(\mathbb{R} \times V)$ satisfies*

$$\forall (z, v) \in \mathbb{R} \times V\,, \quad 0 \leq f(z, v) \leq M(v)\,, \quad \lim_{z \to -\infty} f(z, v) = M(v)\,, \quad \lim_{z \to +\infty} f(z, v) = 0\,. \tag{2.5}$$

In fact, f is a solution of the stationary equation in the moving frame $z = x - ct$, for some $c \geq 0$,

$$(v - c)\partial_z f = \left(M(v)\rho_f - f\right) + r\rho_f \left(M(v) - f\right) , \quad (z, v) \in \mathbb{R} \times V\,. \tag{2.6}$$

The existence of travelling waves in reaction-transport equations has been adressed by Schwetlick [188, 189] for a similar class of equations. First, the set V is bounded and M is the uniform distribution over V. Second, the nonlinearity can be chosen more generally (either monostable as here, or bistable), but it depends only on the macroscopic density ρ_g [188, Eq. (4)]. For the monostable case, using a quite general method he has proved the existence of travelling waves of speed c for any $c \in [c^*, \sup V)$, a result very similar to the Fisher-KPP equation. We emphasize that, although the equations differ between his work and ours, they coincide in the linearized regime of low density $g \ll 1$. On the contrary to Schwetlick, we do not consider a general nonlinearity and we restrict to the logistic case, but we consider general velocity kernels $M(v)$.

More recently, the rescaled equation (2.3) has been investigated by Cuesta, Hittmeir and Schmeiser [69] in the parabolic regime $\varepsilon \ll 1$. Using a micro-macro decomposition, they construct possibly oscillatory travelling waves of speed $c \geq 2\sqrt{rD}$ for ε small enough (depending on c). In addition, when the set of admissible speeds V is bounded, $c > 2\sqrt{rD}$, and ε is small enough, they prove that the travelling wave constructed in this way is indeed nonnegative.

Lastly, when M is the measure $M = \frac{1}{2}(\delta_{-\nu} + \delta_{\nu})$ for some $\nu > 0$, equation (2.1) is analogous to the reaction-telegraph equation for the macroscopic density ρ_g (up to a slight change in the nonlinearity however). This equation has been the subject of a large number of studies [83, 120, 131, 105, 165, 92, 93, 99, 173, 155]. Recently, the authors proved the existence of a minimal speed c^* such that travelling waves exist for all speed $c \geq c^*$ [36]. Moreover these waves are stable in some L^2 weighted space, with a weight which differs from the classical exponential weight arising in the stability theory of the Fisher-KPP equation, see *e.g.* [140]. As the reaction-telegraph equation involves both parabolic and hyperbolic contributions, the smoothness of the wave depends on the balance between these contributions. In fact there is a transition between a parabolic (smooth waves) and a hyperbolic regime (discontinuous waves), see Remark 2.3 below. The authors also prove the existence of supersonic waves, having speed $c > \nu$ (see Remark 2.4).

The aim of the present paper is to investigate the existence and stability of travelling waves for equation (2.1) for arbitrary kernels M satisfying (2.2). For the existence part, we shall use the method of sub- and supersolutions, which do not rely on a perturbation argument. The stability part relies on the derivation of a suitable weight from which we can build a Lyapunov functional for the linearized version of (2.1). The crucial assumption for the existence of travelling waves is the boundedness of V. We prove in fact that under the condition $(\forall v \in \mathbb{R})\ M(v) > 0$, there cannot exist a positive travelling wave. We finally investigate the spreading rate when M is a Gaussian distribution.

In the last stage of writting of this paper, we realized that similar issues were formally addressed by Méndez et al. for a slightly different equation admitting the same linearization near the front edge [166]. Our results are in agreement with their predictions.

Existence of travelling waves when the velocity set is bounded.

Theorem 2.2. *Assume that the set V is compact, and that $M \in \mathcal{C}^0(V)$ satisfies* (2.2). *Let $v_{\max} = \sup V$. There exists a speed $c^* \in (0, v_{\max})$ such that for all $c \in [c^*, v_{\max})$, there exists a travelling wave $f(x - ct, v)$ solution of* (6.2) *with speed c. The travelling wave is nonincreasing with respect to the space variable : $\partial_z f \leq 0$. Moreover, if $\inf_V M > 0$ then there exists no positive travelling wave of speed $c \in [0, c^*)$.*

The minimal speed c^* is given through the following implicit dispersion relation. First, we observe that, for each $\lambda > 0$, there is a unique $c(\lambda) \in (v_{\max} - \lambda^{-1}, v_{\max})$ such that

$$(1 + r) \int_V \frac{M(v)}{1 + \lambda(c(\lambda) - v)}\, dv = 1\,. \tag{2.7}$$

Then we have the formula

$$c^* = \inf_{\lambda > 0} c(\lambda)\,.$$

Remark 2.3. *In the special case of two possible velocities only [36], corresponding to* $M(v) = \frac{1}{2}(\delta_{-v_{\max}} + \delta_{v_{\max}})$*, two regimes have to be distinguished, namely* $r < 1$ *and* $r \geq 1$*. In the case* $r \geq 1$ *the travelling wave with minimal speed vanishes on a half-line. There, the speed of the wave is not characterized by the linearized problem for* $f \ll 1$*. Note that this case is not contained in the statement of Theorem 2.2 since it is assumed that* $M \in \mathcal{C}^0(V)$*. This makes a clear difference between the case of a measure* M *which is absolutely continuous with respect to the Lebesgue measure, and the case of a measure with atoms.*

Remark 2.4. *We expect that travelling waves exist for any* $c \geq c^*$*, although this seems to contradict the finite speed of propagation when* $c > v_{\max}$*. In fact supersonic waves corresponding to* $c > v_{\max}$ *should be driven by growth mainly, as it is the case in a simplified model with only two speeds [36]. A simple argument to support the existence of such waves consists in eliminating the transport part, and seeking waves driven by growth only,* $-c\partial_z f = M(v)\rho_f - f + r\rho_f(M - f)$*. Integrating with respect to* v *yields a logistic equation for* ρ_f*,* $-c\partial_z\rho_f = r\rho_f(1-\rho_f)$*, which as a solution connecting* 1 *and* 0 *for any positive* c*. However these waves are quite artificial and we do not address this issue further.*

We now define $c^* = c^*(M)$ and investigate the dependence of the minimal speed upon M. In the following Proposition, we give some general bounds on the minimal speed.

Proposition 2.5. *Under the same conditions as Theorem 2.2, assume in addition that* M *is symmetric. Then, the minimal speed satisfies the following properties,*

a- [Scaling] *For* $\sigma > 0$*, define* $M_\sigma(v) = \sigma^{-1}M(\sigma^{-1}v)$*, and rescale the velocity set accordingly* ($V_\sigma := \sigma V$)*, then*

$$c^*(M_\sigma) = \sigma c^*(M).$$

b- [Rearrangement] *Denote by* $M^\star$ *the Schwarz decreasing rearrangement of the function* M *[146] and* $M_\star = -(-M)^\star$ *the Schwarz increasing rearrangement of the density distribution* M*, then*

$$c^*(M^\star) \leq c^*(M) \leq c^*(M_\star).$$

c- [Comparison] *If* $r < 1$ *then*

$$\frac{2\sqrt{rD}}{1+r} \leq c^*(M) \leq \frac{2\sqrt{r}}{1+r}v_{\max}\,,$$

whereas, if $r \geq 1$ *then*

$$\sqrt{D} \leq c^*(M) \leq v_{\max}\,,$$

d- [Diffusion limit] *In the diffusion limit* $(r,t,x) \mapsto \left(\varepsilon^2 r, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)$*, the dispersion relation for the rescaled equation* (2.3) *reads*

$$(1+\varepsilon^2 r)\int_V \frac{M(v)}{1+\varepsilon^2\lambda(c - v/\varepsilon)}\,dv = 1. \tag{2.8}$$

We recover the KPP speed of the wave in the diffusive limit,

$$\lim_{\varepsilon\to 0} c^*_\varepsilon = 2\sqrt{rD}$$

Spreading of the front.

In the case where V is compact, we prove that for suitable initial data $g(0,x,v)$, the front spreads asymptotically with speed c^*, in a weak sense.

Proposition 2.6. *Under the same conditions as Theorem 2.2, assume in addition that* $\inf_V M > 0$. *Let* $g^0 \in L^\infty(\mathbb{R}\times V)$ *such that* $0 \le g^0(x,v) \le M(v)$ *for all* $(x,v)\in \mathbb{R}\times V$. *Let g be the solution of the Cauchy problem* (2.1). *Then*

1. *if there exists x_R such that $g^0(x,v) = 0$ for all $x \ge x_R$ and $v\in V$, then for all $c > c^*$,*
$$(\forall v \in V) \quad \lim_{t\to+\infty}\left(\sup_{x\ge ct} g(t,x,v)\right) = 0\,,$$
2. *if there exists x_L and $\gamma\in(0,1)$ such that $g^0(x,v)\ge \gamma M(v)$ for all $x\le x_L$ and $v\in V$, then for all $c < c^*$,*
$$(\forall v \in V) \quad \lim_{t\to+\infty}\left(\sup_{x\le ct} |M(v) - g(t,x,v)|\right) = 0\,,$$

where c^* is the minimal speed of existence of travelling waves given by Theorem 2.2.

Stability of the travelling waves.

We also establish linear and nonlinear stability of the travelling wave of speed $c\in[c^*, v_{\max})$ in some weighted L^2 space. The key point is to derive a suitable weight which enables to build a Lyapunov functional for the linear problem. The weight $\phi(z,v)$ is constructed in a systematic way, following [36]. However, we believe it is not optimal, as opposed to [36], for some technical reason (see Remark 2.30).

Let f be a travelling wave (6.2) of speed $c\in[c^*, v_{\max})$, and let $u = g - f$ (resp. $u^0 = g^0 - f$) be the perturbation of f in the moving frame. Neglecting the nonlinear contributions, we are led to investigate the linear equation

$$\partial_t u + (v-c)\partial_z u + \left(1 + r\rho_f\right) u = \left((1+r)M - rf\right)\rho_u\,. \tag{2.9}$$

Theorem 2.7 (Linear stability)**.** *There exists a weight $\phi(z,v)$ such that the travelling front of speed $c\in[c^*, v_{\max})$ is linearly stable in the weighted space $L^2\left(e^{-2\phi(z,v)}dzdv\right)$ in the following sense : if $u^0\in L^2\left(e^{-2\phi(z,v)}dzdv\right)$, then*

$$(\forall t\in\mathbb{R}^+) \quad \|u(t)\|_{L^2(e^{-2\phi})} \le \|u^0\|_{L^2(e^{-2\phi})}.$$

Moreover, the perturbation is globally decaying as the dissipation is integrable in time :

$$(\forall z_0\in\mathbb{R})$$
$$\int_{\{z<z_0\}\times V} |u(t,z,v)|^2 e^{-2\phi(z,v)}dzdv + \int_{\{z>z_0\}\times V} \rho_f(z)\,|u(t,z,v)|^2 e^{-2\phi(z,v)}dzdv \in L^1\left(\mathbb{R}^+\right).$$

The proposition will appear as a corollary of the following Lyapunov identity, which holds true for any solution u of the linear equation (2.9),

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}\times V}|u|^2 e^{-2\phi(z,v)}dzdv\right) + \int_{\mathbb{R}\times V}\frac{r}{2}\left(\rho_f + \frac{f}{M(v)+r\left(M(v)-f\right)}\right)|u|^2 e^{-2\phi(z,v)}dzdv \leq 0\,. \quad (2.10)$$

The weight ϕ is given in Definition 2.27. It is equivalent to $-z$ as $z \to +\infty$, uniformly with respect to v.

The weighted energy estimate 2.10 does not provide any exponential decay, because of the presence of $\rho_f(z)$ in the dissipation. This is a general concern for reaction-diffusion equations, see [140] and references therein. However, in [69] the authors prove such an exponential decay in the case of supercritical speeds $c > 2\sqrt{rD}$, and ε small enough (diffusive regime). We do not follow this argument further in this work.

Then we adapt the method of [69], using a comparison argument together with the explicit formula of the dissipation (2.10), in order to prove a nonlinear stability result.

Corollary 2.8 (Nonlinear stability). *Under the same conditions as Theorem 2.7, assume in addition that there exists $\gamma \in \left(\frac{1}{2}, 1\right]$ such that*

$$(\forall\,(x,v) \in \mathbb{R}\times V) \quad g^0(x,v) \geq \gamma f(x,v)\,. \quad (2.11)$$

Then the same conclusion as in Theorem 2.7 holds true.

We expect that nonlinear stability holds true for any $\gamma \in (0,1]$. However this would require to redefine the weight ϕ, since we believe it is not the optimal one, see Remark 2.30 below.

Superlinear propagation when velocity is unbounded.

Boundedness of V is a crucial hypothesis in order to build the travelling waves. We believe that it is a necessary and sufficient condition. We make a first step to support this conjecture by investigating the case $V = \mathbb{R}$. We first prove infinite speed of spreading of the front under the natural assumption $(\forall v \in \mathbb{R})\ M(v) > 0$. As a corollary there cannot exist travelling wave in the sense of Definition 2.1. Note that there exist travelling waves with less restrictive conditions than Definition 2.1, at least in the diffusive regime [69]. These fronts are expected not to verify the nonnegativity condition, as $x \to +\infty$. We believe that such oscillating fronts do exist far from the diffusive regime. In the case where $V = \mathbb{R}$ and M is a Gaussian distribution, we have plotted the dispersion relation (2.7) in the complex plane $\lambda \in \mathbb{C}$, for an arbitrary given $c > 0$. We have observed that it selects two complex conjugate roots, supporting the fact that damped oscillating fronts should exist (results not shown).

Proposition 2.9. *Assume that $M(v) > 0$ for all $v \in \mathbb{R}$. Let $g^0 \in L^\infty(\mathbb{R}\times V)$ such that $0 \leq g^0(x,v) \leq M(v)$ for all $(x,v) \in \mathbb{R}\times V$ and there exists x_L and $\gamma \in (0,1)$ such that $g^0(x,v) \geq \gamma M(v)$ for all $x \leq x_L$ and $v \in V$. Let g be the solution of the Cauchy problem* (2.1). *Then for all $c > 0$,*

$$(\forall v \in V) \quad \lim_{t\to+\infty}\left(\sup_{x\leq ct}|M(v) - g(t,x,v)|\right) = 0\,.$$

We can immediately deduce from this result the non-existence of travelling waves when $V = \mathbb{R}$, by taking such a travelling wave as an initial datum g^0 in order to reach a contradiction.

Corollary 2.10. *Assume that $M(v) > 0$ for all $v \in \mathbb{R}$. Then equation* (2.1) *does not admit any travelling wave solution.*

Accelerating fronts for a Gaussian distribution.

Accelerating fronts in reaction-diffusion equations have raised a lot of interest in the recent years. They occur for the Fisher-KPP equation (7.2) when the initial datum decays more slowly than any exponential [123]. They also appear when the diffusion operator is replaced by a nonlocal dispersal operator with fat tails [144, 153, 107], or by a nonlocal fractional diffusion operator [46, 45]. Recently, accelerating fronts have been conjectured to occur in a reaction-diffusion-mutation model which generalizes the Fisher-KPP equation to a population structured with respect to the diffusion coefficient [34].

Here, we investigate the case of a Gaussian distribution M. The spreading rate $\langle x \rangle = \mathcal{O}(t^{3/2})$ is expected in this case (heuristics, and see [166]). We prove that spreading occurs with this rate. For this purpose, we build suitable sub- and supersolution which spread with this rate.

We split our results into two parts, respectively the upper bound and the lower bound of the spreading rate. The reason is that the constructions are quite different. The construction of the supersolution relies on a first guess inspired from [107], plus convolution tricks which are made easier in the gaussian case. On the other hand, the construction of the subsolution is based on a better comprehension of the growth-dispersion process. Again, some technical estimates are facilitated in the gaussian case. We believe that these results can be generalized to a large class of distributions M, at the expense of clarity.

Theorem 2.11. *Let $M(v) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{v^2}{2\sigma^2}\right)$, defined for $v \in \mathbb{R}$. Let $g^0 \in L^\infty(\mathbb{R} \times V)$ such that $0 \le g^0(x,v) \le M(v)$ for all $(x,v) \in \mathbb{R} \times V$. We have the two following, independent, items,*

1. *Assume that there exist $1 \le b \le a$ such that*

$$(\forall (x,v) \in \mathbb{R} \times V) \quad g^0(x,v) \le \frac{1}{b} M\left(\frac{x}{b}\right) M(v) e^{ra} .$$

Let g be the solution of the Cauchy problem (2.1)*. Then for all $\varepsilon > 0$, one has*

$$\lim_{t \to +\infty} \left(\sup_{|x| \ge (1+\varepsilon)\sigma\sqrt{2r} t^{3/2}} \rho_g(t,x) \right) = 0 .$$

2. *Assume that there exists $\gamma \in (0,1)$, and $x_L \in \mathbb{R}$ such that*

$$(\forall (x,v) \in \mathbb{R} \times V) \quad g^0(x,v) \ge \gamma M(v) \mathbf{1}_{x < x_L},$$

Let g be the solution of the Cauchy problem (2.1)*. Then for all $\varepsilon > 0$, one has*

$$\lim_{t \to +\infty} \left(\inf_{x \le (1-\varepsilon)\sigma(\frac{r}{r+2} t)^{3/2}} \rho_g(t,x) \right) \ge 1 - \gamma .$$

Remark 2.12 (Front propagation and diffusive limit)**.** *There is some subtlety hidden behind this phenomenon of infinite speed of spreading. In fact the diffusion limit of the scattering equation (namely* $r = 0$*) towards the heat equation makes no difference between bounded or unbounded velocity sets, as soon as the variance D is finite (see [71] and the references therein). However very low densities behave quite differently, which can be measured in the setting of large deviations or WKB limit. This can be observed even in the case of a bounded velocity set. In [33] the large deviation limit of the scattering equation is performed. It differs from the classical eikonal equation obtained from the heat equation. The case of unbounded velocities is even more complicated [31]. To conclude, let us emphasize that low densities are the one that drive the front here (pulled front). So the diffusion limit is irrelevant in the case of unbounded velocities, since very low density of particles having very large speed makes a big contribution.*

2.2 Preliminary results

We first recall some useful results concerning the Cauchy problem associated with (2.1) : well-posedness and a strong maximum principle. These statements extend some results given in [69]. They do not rely on the boundedness of V.

Proposition 2.13 (Global existence : Theorem 4 in [69])**.** *Let* g^0 *a measurable function such that* $0 \leq g^0(x,v) \leq M(v)$ *for all* $(x,v) \in \mathbb{R} \times V$*. Then the Cauchy problem* (2.1) *has a unique solution* $g \in \mathcal{C}_b^0(\mathbb{R}_+ \times \mathbb{R} \times V)$ *in the sense of distributions, satisfying*

$$(\forall (t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V) \quad 0 \leq g(t,x,v) \leq M(v)\,.$$

The next result refines the comparison principle of [69] in order to extend it to sub and supersolutions in the sense of distributions and to state a strong maximum principle. Its proof is given in Appendix.

Proposition 2.14 (Comparison principle)**.** *Assume that* $u_1, u_2 \in \mathcal{C}(\mathbb{R}_+, L^\infty(\mathbb{R} \times V))$ *are respectively a super- and a subsolution of* (2.1), i.e.

$$\partial_t g_1 + v\partial_x g_1 \geq \left(M(v)\rho_{g_1} - g_1\right) + r\rho_{g_1}\left(M(v) - g_1\right)\,,$$
$$\partial_t g_2 + v\partial_x g_2 \leq \left(M(v)\rho_{g_2} - g_2\right) + r\rho_{g_2}\left(M(v) - g_2\right)\,,$$

in the sense of distributions. Assume in addition that g_2 *satisfies* $g_2(t,x,v) \leq M(v)$ *for all* $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V$*. Then* $g_2(t,x,v) \leq g_1(t,x,v)$ *for all* $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V$*.*

Assume in addition that V is an interval, and that $\inf_V M > 0$*. If there exists* (x_0, v_0) *such that* $g_2(0,x_0,v_0) > g_1(0,x_0,v_0)$*, then one has* $g_1(t,x,v) > g_2(t,x,v)$ *for all* $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V$ *such that* $|x - x_0| < v_{\max} t$*.*

Remark 2.15. *If* $V = \mathbb{R}$*, then this statement reads as in the parabolic framework : if* $g_2 \geq g_1$ *and* $g_2 \not\equiv g_1$ *at* $t = 0$*, then* $g_2 > g_1$ *for all* $t > 0$*. In the case* $V = [-v_{\max}, v_{\max}]$ *we have to take into account finite speed of propagation, obviously.*

2.3 Existence and construction of travelling wave solutions

We assume throughout this Section that $V = \operatorname{Supp} M$ is compact. We construct the travelling waves for $c \in [c^*, v_{\max})$. The proof is divided into several steps. It is based on a sub and supersolutions method.

2.3.1 The linearized problem.

The aim of this first step is to solve the linearized equation of (6.2) at $+\infty$, in the regime of low density $f \ll 1$. Such an achievement gives information about the speed and the space decreasing rate of a travelling wave solution of the nonlinear problem, as for the Fisher-KPP equation. The linearization of (6.2) at $f = 0$ writes

$$(v - c)\partial_x f = \big(M(v)\rho_f - f\big) + rM(v)\rho_f\,, \tag{2.12}$$

We seek a solution having exponential decay at $+\infty$. More specifically we separate the variables in our ansatz : $f(x, v) = e^{-\lambda x}F(v)$, with $\int_V F(v)dv = 1$. The next Proposition gathers the results concerning the linear problem.

Proposition 2.16 (Existence of a minimal speed for the linearized equation)**.** *There exists a minimal speed c^* such that for all $c \in [c^*, v_{\max})$, there exists $\lambda > 0$ such that $f_\lambda(x, v) = e^{-\lambda x}F_\lambda(v)$ is a nonnegative solution of* (2.12)*. The profile F_λ is explicitely given by*

$$F_\lambda(v) = \frac{(1 + r)M(v)}{1 + \lambda(c - v)} \geq 0\,.$$

The admissible (λ, c) are solutions of the following dispersion relation,

$$\int_V \frac{(1 + r)M(v)}{1 + \lambda(c - v)}\,dv = 1\,. \tag{2.13}$$

Moreover, among all possible λ for a given c, the minimal one λ_c is well defined and isolated.

Remark 2.17. *Here appears the crucial assumption on the boundedness of V. If this condition is not fulfilled, it is never possible to ensure that the profile F_λ is nonnegative since the denominator is linear with respect to v.*

Proof of Proposition 2.16. # Step 1. Plugging the ansatz $f_\lambda(x, v) = e^{-\lambda x}F_\lambda(v)$ into (2.12) yields

$$(c - v)\lambda F_\lambda(v) = (M(v) - F_\lambda(v)) + rM(v)\,. \tag{2.14}$$

The profile is given by

$$F_\lambda(v) = \frac{(1 + r)M(v)}{1 + \lambda(c - v)}.$$

The dispersion relation reads $\int_V F_\lambda(v)dv = 1$, or equivalently (2.13). Moreover, we require the profile F_λ to be nonnegative, which gives the condition $1 + \lambda(c - v) > 0$ for all $v \in V$, which implies $\lambda < \frac{1}{v_{\max} - c}$.

From now on, we focus on the existence of solutions (λ, c) of (2.13), with $c \in [0, v_{\max})$ and $\lambda \in \left[0, \frac{1}{v_{\max} - c}\right)$. Let us denote

$$I(\lambda; c) = \int_V \frac{(1 + r)M(v)}{1 + \lambda(c - v)}\,dv. \tag{2.15}$$

so that we look for solutions of $I(\lambda; c) = 1$.

 # Step 2. Technically speaking, for all $c \in [0, v_{\max})$, the function $\lambda \mapsto I(\lambda; c)$ is analytic over $\left[0, \frac{1}{v_{\max}-c}\right)$ Indeed, as $v \mapsto v^n M(v)$ is integrable for all n, it is clear that

$$I(\lambda; c) = \sum_{n\geq 0} (1+r)\lambda^n \int_V M(v)(v-c)^n dv$$

is the analytic development of I for $\lambda \in \left[0, \frac{1}{v_{\max}-c}\right)$. Next we observe that $c \mapsto I(\lambda; c)$ is decreasing for all $\lambda \in \left(0, \frac{1}{v_{\max}-c}\right)$, and that $\lambda \mapsto I(\lambda; c)$ is strictly convex. Moreover, the function I satisfies the following properties :

$$I(0; c) = 1 + r > 1\,,$$
$$I(\lambda; 0) = (1+r)\int_V \frac{M(v)}{1-\lambda v}\, dv > 1, \quad \text{for all } \lambda \in \left[0, \frac{1}{v_{\max}}\right)$$
$$I(\lambda; v_{\max}) = (1+r)\int_V \frac{M(v)}{1+\lambda(v_{\max}-v)}\, dv \xrightarrow[\lambda\to+\infty]{} 0\,.$$

The last property relies on the Lebesgue's dominated convergence theorem since $M \in L^1(V)$.

 # Step 3. Assume first that $\frac{M(v)}{v_{\max}-v} \notin L^1(V)$. Then Fatou's lemma gives

$$\liminf_{\lambda \nearrow \frac{1}{v_{\max}-c}} I(\lambda; c) = \liminf_{\lambda \nearrow \frac{1}{v_{\max}-c}} \int_V \frac{M(v)}{1+\lambda(c-v)} dv$$
$$\geq \int_V \liminf_{\lambda \nearrow \frac{1}{v_{\max}-c}} \frac{M(v)}{1+\lambda(c-v)} dv = \int_V \frac{M(v)}{1-\frac{v-c}{v_{\max}-c}} dv = +\infty.$$

As a consequence, $\theta(c) = \min\left\{ I(\lambda; c) : \lambda \in \left[0, \frac{1}{v_{\max}-c}\right)\right\}$ is well defined and finite for all $c \in [0, v_{\max})$. It follows from the earlier properties that $\theta(0) > 1$ and $\theta(v_{\max}) = 0$. Moreover, the regularity and monotonicity properties of I guarantee that θ is continuous and decreasing. Hence, there exists c^* such that $\theta(c^*) = 1$ and there exists λ_{c^*} such that $I(\lambda_{c^*}; c^*) = 1$.

Next, for all $c \in (c^*, v_{\max})$, as $c \mapsto I(\lambda; c)$ is decreasing, one has $I(\lambda_{c^*}; c) < 1$ for all $c > c^*$. Thus, as $I(0; c) > 1$, there exists λ such that $I(\lambda; c) = 1$ for all $c > c^*$.

Second, consider a general $M \in \mathcal{C}^0(V)$ possibly vanishing at $v = v_{\max}$. To recover the first step, we define for $n \in \mathbb{N}^*$ a new distribution $M_n = \frac{M+1/n}{1+|V|/n}$ over V (and 0 outside of V), where $|V|$ is the measure of V. Then $\frac{M_n(v)}{v_{\max}-v} \notin L^1(V)$ since $M_n(v_{\max}) \geq \frac{1/n}{1+|V|/n} > 0$, and thus the earlier step yields that there exists a sequence c_n^* of minimal speeds associated with $(M_n)_n$. We also associate I_n with M_n through (2.15). We define

$$c^* = \limsup_{n\to\infty} c_n^*,$$

and we now show that it is the minimal speed.

– Take $c < c^*$. Then for all $\lambda \in \left(0, \frac{1}{v_{\max}-c}\right)$ and for some arbitrarily large n so that $\lambda \in \left(0, \frac{1}{v_{\max}-c_n^*}\right)$, one has

$$I_n(\lambda; c) = I_n(\lambda; c_n^*) - \int_c^{c_n^*} \partial_c I_n(\lambda, c')dc'$$
$$\geq 1 - \int_c^{c_n^*} \partial_c I_n(\lambda, c')dc' \geq 1 + \frac{(1+r)\lambda}{(1+\lambda(c_n^*+v_{\max}))^2}(c_n^* - c).$$

Because $I_n(\lambda;c) \underset{n\to+\infty}{\to} I(\lambda;c)$ as $n \to +\infty$, we get

$$I(\lambda;c) \geq 1 + \frac{(1+r)\lambda}{1+\lambda(c^* + v_{\max})}(c^* - c) > 1.$$

Thus $I(\lambda;c) = 1$ has no solution for $\lambda \in \left(0, \frac{1}{v_{\max}-c}\right)$ if $c < c^*$.

– Assume that $c > c^*$. Then one has $c > c_n^*$ when n is large enough and thus for all n sufficiently large, there exists $\lambda_n \in \left(0, \frac{1}{v_{\max}-c}\right)$ such that $I_n(\lambda_n;c) = 1$. Up to extraction, one may assume that $(\lambda_n)_n$ converges to some $\lambda_\infty \in \left[0, \frac{1}{v_{\max}-c}\right]$. Fatou's lemma yields $I(\lambda_\infty;c) \leq 1$. Hence, there exists a solution $\lambda \in \left[0, \frac{1}{v_{\max}-c}\right]$ of $I(\lambda;c) = 1$ and obviously $\lambda \neq 0$ since $I(0;c) > 1$.
– Lastly, if $c = c^*$, we know that for all $k \in \mathbb{N}^*$, there exists $\lambda_k \in \left(0, \frac{1}{v_{\max}-(c^*+1/k))}\right]$ such that $I(\lambda_k; c^* + 1/k) = 1$. Assuming that $\lambda_k \to \lambda \in \left[0, \frac{1}{v_{\max}-c}\right]$ as $k \to +\infty$, we get $I(\lambda;c^*) = 1$.

□

Lemma 2.18 (Spatial decay rate). *For all $c \in [c^*, v_{\max})$, the quantity*

$$\lambda_c = \min\{f > 0 : I(f;c) = 1\}.$$

is well-defined. Moreover, for all $c \in (c^*, v_{\max})$, if $\gamma > 0$ is small enough, then $I(\lambda_c + \gamma; c) < 1$.

Proof of Lemma 2.18. We know from the definition of c^* that for all $c \in [c^*, v_{\max})$, the set $\Lambda_c = \{f > 0 : I(f;c) = 1\}$ is not empty. Thus, we can take a minimizing sequence λ_n which converges towards the infimum of Λ_c. As this sequence is bounded, one can assume, up to extraction, that $\lambda_n \to \lambda_c \geq 0$. Then Lebesgue's dominated convergence theorem gives $I(\lambda_c;c) = 1$. Hence $\lambda_c = \min \Lambda_c$.

Next, we have already noticed in the proof of Proposition 2.16 that $I(\lambda_{c^*}, c) < 1$ for all $c > c^*$. As $I(0,c) = 1 + r > 1$, the definition of λ_c yields $\lambda_c < \lambda_{c^*}$. The conclusion follows from the strict convexity of the function $\lambda \mapsto I(\lambda;c)$. □

2.3.2 Construction of sub and supersolutions when $c \in (c^*, v_{\max})$.

In this step we construct sub and supersolutions for (2.1). We fix $c \in (c^*, v_{\max})$ and we denote $\lambda = \lambda_c$ for legibility.

Lemma 2.19 (Supersolution). *Let*

$$\overline{f}(x,v) = \min\left\{M(v), e^{-\lambda x} F_\lambda(v)\right\}.$$

Then $\overline{f}$ is a supersolution of (6.2)*, that is, it satisfies in the sense of distributions :*

$$(v-c)\partial_x \overline{f} \geq \left(M(v)\rho_{\overline{f}} - \overline{f}\right) + r\rho_{\overline{f}}\left(M(v) - \overline{f}\right), \quad (x,v) \in \mathbb{R} \times V. \tag{2.16}$$

Lemma 2.20 (Subsolution)**.** *There exist $A > 0$ and $\gamma > 0$ such that if*

$$\underline{f}(x,v) = \max\left\{0, e^{-\lambda x}F_\lambda(v) - Ae^{-(\lambda+\gamma)x}F_{\lambda+\gamma}(v)\right\},$$

then $\underline{f}$ is a subsolution of (6.2)*, that is satisfies in the sense of distributions :*

$$(v-c)\partial_x \underline{f} \leq \left(M(v)\rho_{\underline{f}} - \underline{f}\right) + r\rho_{\underline{f}}\left(M(v) - \underline{f}\right), \quad (x,v) \in \mathbb{R}\times V. \tag{2.17}$$

Proof of Lemma 2.19. First, $(x,v) \mapsto e^{-\lambda x}F_\lambda(v)$ and $(x,v) \mapsto M(v)$ both clearly satisfy (2.16) since $\overline{f} \geq 0$. Next, as $\overline{f}$ is continuous, it immediately follows from the jump formula that, as a minimum of two supersolutions, it is a supersolution of (2.16) in the sense of distributions. □

Proof of Lemma 2.20. The same arguments as in the proof of Lemma 2.19 yield that it is enough to prove that (2.17) is satisfied by $\underline{f}$ over the open set $\{\underline{f} > 0\}$. As $c > c^*$, Proposition 2.16 gives $\gamma \in (0,\lambda)$ small enough such that $I(\lambda+\gamma; c) < 1$ and $F_{\lambda+\gamma}(v) > 0$. We compute the linear part :

$$(v-c)\partial_x \underline{f} - \left(M(v)\rho_{\underline{f}} - \underline{f}\right) - r\rho_{\underline{f}}M(v) = A\left(I(\lambda+\gamma,c) - 1\right)(1+r)e^{-(\lambda+\gamma)x}M(v).$$

To prove the Lemma, we now have to choose a relevant A such that

$$r\underline{f}\rho_{\underline{f}} \leq A(1+r)M(v)\left(1 - I(\lambda+\gamma,c)\right)e^{-(\lambda+\gamma)x}. \tag{2.18}$$

holds for all $(x,v) \in \mathbb{R}\times V$. As $\underline{f}(x,v) \leq e^{-\lambda x}F_\lambda(v)$ for all $(x,v) \in \mathbb{R}\times V$, one has $\rho_{\underline{f}}(x) \leq e^{-\lambda x}$ and thus it is enough to choose A such that

$$\begin{aligned} re^{-2\lambda x}F_\lambda(v) &\leq A(1+r)M(v)\left(1 - I(\lambda+\gamma,c)\right)e^{-(\lambda+\gamma)x}, \\ \frac{re^{-(\lambda-\gamma)x}}{1 - I(\lambda+\gamma,c)}\left(\frac{1}{1+\lambda(c-v)}\right) &\leq A. \end{aligned} \tag{2.19}$$

On the other hand for all $(x,v) \in \mathbb{R}\times V$ such that $\underline{f}(x,v) > 0$, we have $F_\lambda(v) > Ae^{-\gamma x}F_{\lambda+\gamma}(v)$, meaning that

$$e^{-\gamma x} < \frac{1}{A}\left(\frac{1+(\lambda+\gamma)(c-v)}{1+\lambda(c-v)}\right).$$

Plugging this estimate into (2.19), it is enough to choose A such that

$$\begin{gathered} \left(\frac{1}{A}\left(\frac{1+(\lambda+\gamma)(c-v)}{1+\lambda(c-v)}\right)\right)^{\frac{\lambda-\gamma}{\gamma}}\frac{r}{1 - I(\lambda+\gamma,c)}\left(\frac{1}{1+\lambda(c-v)}\right) \leq A \\ \sup_{v\in V}\left\{\left(\frac{1+(\lambda+\gamma)(c-v)}{1+\lambda(c-v)}\right)^{\frac{\lambda-\gamma}{\gamma}}\frac{r}{1 - I(\lambda+\gamma,c)}\left(\frac{1}{1+\lambda(c-v)}\right)\right\} \leq A^{\frac{\lambda}{\gamma}}. \end{gathered}$$

This concludes the proof since such a A obviously exists. □

2.3.3 Construction of the travelling waves in the regime $c \in (c^*, v_{\max})$.

Let $c \in (c^*, v_{\max})$, where c^* denotes the minimal speed of Proposition 2.16. In order to prove the existence of travelling waves, we will prove that the solution of the following evolution equation, corresponding to equation (2.1) in the moving frame at speed c, converges to a travelling wave as $t \to +\infty$:

$$\begin{cases} \partial_t g + (v-c)\partial_x g = M(v)\rho_g - g + r\rho_g\,(M(v)-g) \text{ in } \mathbb{R}\times V, \\ g(0,x,v) = \overline{f}(x,v) \text{ for all } (x,v)\in\mathbb{R}\times V. \end{cases} \tag{2.20}$$

The well-posedness of equation (2.20) immediately follows from Proposition 2.13. Let now derive some properties of the function g from Proposition 2.14.

Lemma 2.21. *For all $(t,x,v)\in\mathbb{R}_+\times\mathbb{R}\times V$, one has $\underline{f}(x,v)\le g(t,x,v)\le\overline{f}(x,v)$.*

Proof of Lemma 2.21. As $\underline{f}$ is a subsolution of (2.20) and $\overline{f}$ is a supersolution of (2.20), with $\underline{f}(x,v)\le\overline{f}(x,v)$ for all $(x,v)\in\mathbb{R}\times V$, this result is an immediate corollary of Proposition 2.14. □

Lemma 2.22. *For all $(t,v)\in\mathbb{R}_+\times V$, the function $x\in\mathbb{R}\mapsto g(t,x,v)$ is nonincreasing.*

Proof of Lemma 2.22. Take $h\ge 0$ and define $g_h(t,x,v)=g(t,x+h,v)$. Then as $\overline{f}$ is nonincreasing in x, one has $g_h(0,x,v)\le g(0,x,v)$ for all $(x,v)\in\mathbb{R}\times V$. Proposition 2.14 yields that $g_h(t,x,v)\le g(t,x,v)$ for all $(t,x,v)\in\mathbb{R}_+\times\mathbb{R}\times V$. □

Lemma 2.23. *For all $(x,v)\in\mathbb{R}\times V$, the function $t\in\mathbb{R}_+\mapsto g(t,x,v)$ is nonincreasing.*

Proof of Lemma 2.23. Take $\tau\ge 0$ and define $g_\tau(t,x,v)=g(t+\tau,x,v)$. Then Lemma 2.21 yields that $g_\tau(0,x,v)\le\overline{f}(x,v)=g(0,x,v)$ for all $(x,v)\in\mathbb{R}\times V$. Hence, Proposition 2.14 gives $g_\tau(t,x,v)\le g(t,x,v)$ for all $(t,x,v)\in\mathbb{R}_+\times\mathbb{R}\times V$. □

Lemma 2.24. *The family $(g(t,\cdot,\cdot))_{t\ge 0}$ is uniformly continuous with respect to $(x,v)\in\mathbb{R}\times V$. Moreover, for any $A\in(c^*,v_{\max})$, the continuity constants does not depend on $c\in(c^*,A)$.*

Proof of Lemma 2.24. We begin with the space regularity. Let $|h|<1$. The function $g(0,x,v)=\overline{f}(x,v)=\min\{M(v),e^{-\lambda x}F_\lambda(v)\}$ is such that $\log g(0,x,v)$ is Lipschitz-continuous with respect to x. Therefore there exists a constant $C_0>0$ such that for all $(x,v)\in\mathbb{R}\times V$, we have $g(0,x+h,v)\le(1+C_0|h|)g(0,x,v)$. As $1+C_0|h|>1$, it is easily checked that $(t,x,v)\mapsto(1+C_0|h|)g(t,x-h,v)$ is a supersolution of (2.20). Hence Proposition 2.14 yields that

$$g(t,x,v)\le(1+C_0|h|)g(t,x-h,v) \quad \text{for all } (t,x,v)\in\mathbb{R}_+\times\mathbb{R}\times V\,.$$

Hence the function $\log g$ is Lipschitz continuous with respect to x. Since the function $\log g$ is bounded from above, $g=\exp(\log g)$ is also Lipschitz continuous with respect to x. The Lipschitz constant is uniform with respect to $c\in(c^*,A)$ and $\lambda\in(0,1/(v_{\max}-c))$.

We now come to the velocity regularity. For the sake of clarity we first consider the case where M is $\mathcal{C}^1$ on V. The function $v\mapsto g(0,x,v)$ is $\mathcal{C}^1$ too. We introduce $g_v=\partial_v g$. It satisfies the following equation

$$\partial_t g_v + (v-c)\partial_x g_v + (1+r\rho_g)g_v = (1+r)M'(v)\rho_g - \partial_x g \text{ in } \mathbb{R}\times V\,.$$

Multiplying the equation by sign g_v we obtain

$$\partial_t|g_v| + (v-c)\partial_x|g_v| + (1+r\rho_g)|g_v| \leq (1+r)|M'(v)|\rho_g + |\partial_x g| \text{ in } \mathbb{R}\times V\,.$$

The l.h.s. is linear with respect to $|g_v|$ and satisfies the maximum principle. The r.h.s. is uniformly bounded since $0 \leq \rho_g \leq 1$ and g is uniformly Lipschitz with respect to x. Obviously the constant $(1+r)\sup_V |M'(v)| + \sup_{\mathbb{R}_+\times\mathbb{R}\times V} |\partial_x g|$ is a supersolution. We deduce that g_v is uniformly bounded over $\mathbb{R}_+ \times \mathbb{R} \times V$.

In the case where M is only continuous over the compact set V, thus uniformly continuous, we shall use the method of translations again. However we have to be careful since V is bounded. Let $0 < h < 1$. We introduce $H(v) = \max(v+h, v_{\max}) - v$. The function $g_H(t,x,v) = g(t,x,v+H(v)) - g(t,x,v)$ satisfies the following equation

$$\begin{aligned}\partial_t g_H + (v-c)\partial_x g_H + (1+r\rho_g)g_H \\ = (1+r)(M(v+H(v)) - M(v))\rho_g - H(v)\partial_x g(t,x,v+H(v))\,.\end{aligned}$$

Let $\varepsilon > 0$. There exists $\delta > 0$ such that for $0 < h < \delta$ we have $|g_H(0,x,v)| \leq \delta$ and $|M(v + H(v)) - M(v)| < \delta$. Moreover we have obviously $0 < H(v) < \delta$. We conclude using the same argument as in the $\mathcal{C}^1$ case. The modulus of uniform continuity is uniform with respect to $c \in (c^*, A)$ and $\lambda \in (0, 1/(v_{\max} - c))$. □

We are now in position to prove the existence of travelling waves of speed c, except for the minimal speed c^*.

Proof of the existence in Theorem 2.2 when $c > c^*$. Gathering Lemmas 2.21, 2.22 and 2.23, we know that

$$f(x,v) = \lim_{t\to+\infty} g(t,x,v),$$

is well-defined for all $(x,v) \in \mathbb{R}\times V$, that $f(\cdot,v)$ is nonincreasing in x for all v and that $\underline{f} \leq f \leq \overline{f}$.

Let now prove that f defines a travelling wave solution of (2.3). As g satisfies (2.20), converges pointwise and is bounded by the locally integrable function $\overline{f}$, it follows from Lebesgue's dominated convergence theorem that f satisfies (6.2) in the sense of distributions. Moreover, Lemma 2.24 ensures that f is continuous with respect to (x,v).

We next check the limits towards infinity. Let $f^{\pm}(v) = \lim_{x\to\pm\infty} f(x,v)$. Thanks to $f \leq \overline{f}$, the Lebesgue dominated convergence theorem gives $\rho_{f^\pm} = \int_V f^{\pm}(v)dv \leq 1$. On the other hand, we get

$$\left(M(v)\rho_{f^\pm} - f^{\pm}(v)\right) + r\rho_{f^\pm}\left(M(v) - f^{\pm}(v)\right) = 0 \tag{2.21}$$

in the sense of distributions. Integrating (2.21) over the compact set V, we deduce that holds $\rho_{f^\pm}\left(1-\rho_{f^\pm}\right) = 0$, *i.e.* that $\rho_{f^\pm} = 0$ or 1. As f is nonincreasing and $\underline{f} \leq f \leq \overline{f}$, one necessarily has $\rho_{f^+} = 0$ and $\rho_{f^-} = 1$. Finally, (2.21) gives $f^+(v) = 0$ and $f^-(v) = M(v)$ for all $v \in V$. This gives the appropriate limits. □

2.3.4 Construction of the travelling waves with minimal speed c^*.

Proof of the existence in Theorem 2.2 when $c = c^*$. Consider a decreasing sequence (c_n) converging towards c^*. We already know that for all n, equation (2.1) admits a travelling wave solution $u_n(t,x,v) = f_n(x - c_n t, v)$, with $f_n(-\infty, v) = M(v)$ and $f_n(+\infty, v) = 0$, and $z \mapsto f_n(z,v)$ is nonincreasing. Up to translation, we can assume that $\rho_{f_n}(0) = 1/2$. Moreover, Lemma 2.24 ensures that the functions $(f_n)_n$ are uniformly continuous with respect to $(x,v) \in \mathbb{R} \times V$ since the continuity stated in Lemma 2.24 is uniform with respect to $c \in (c^*, A)$ for any $A \in (c^*, v_{\max})$. Thanks to the Ascoli theorem and a diagonal extraction process, we can assume that the sequence $(f_n)_n$ converges locally uniformly in $(x,v) \in \mathbb{R} \times V$ to a function f. Clearly f satisfies (6.2) in the sense of distributions. Moreover, as f is nonincreasing with respect to x, one could recover the appropriate limits at infinity with the same arguments as in the proof of the existence of travelling waves with speeds $c > c^*$. □

2.3.5 Non-existence of travelling wave solutions in the subcritical regime $c \in [0, c^*)$.

Lemma 2.25. *Assume that* $\inf_V M(v) > 0$. *For all* $0 \le c < c^*$ *there exists* $c < c_0 < c^*$ *and a nonnegative, arbitrarily small, compactly supported function* $h(x,v)$ *which is a subsolution of*

$$(v - c^0)\partial_x f = M(v)\rho_f - f + r\rho_f\,(M(v) - f) \quad \text{in } \mathbb{R} \times V\,. \tag{2.22}$$

Proof of Lemma 2.25. For the sake of clarity we emphasize the dependence of the function I (2.15) upon the growth rate $r > 0$:

$$I_r(\lambda; c) = \int_V \frac{(1+r)M(v)}{1 + \lambda(c - v)}\, dv\,.$$

We denote by c_r^* the smallest speed such that there exists a solution $\lambda > 0$ of $I_r(\lambda, c) = 1$ (see Proposition 2.16).

Let $\delta > 0$. By continuity we can choose δ so small that $c < c^*_{r-\delta}$. We claim that there exists (c^0, λ^0) such that $I_{r-\delta}(\lambda^0; c^0) = 1$, with $c < c^0 < c^*_{r-\delta}$ and $\lambda^0 \in \mathbb{C} \setminus \mathbb{R}$. Indeed we know from the proof of Proposition 2.16 [Step 3] that $\lambda_r^* < 1/(v_{\max} - c_r^*)$ under the assumption $v \mapsto M(v)/(v_{\max} - v) \notin L^1(V)$. Using a continuity argument we also have the strict inequality $\lambda^*_{r-\delta} < 1/(v_{\max} - c^*_{r-\delta})$, uniformly with respect to δ. The complex function $\lambda \mapsto I_{r-\delta}(\lambda; c^*_{r-\delta})$ is analytic in a neighborhood of $\lambda^*_{r-\delta}$. Hence, the Rouché theorem yields that there exists $c^0 < c^*_{r-\delta}$ such that the equation $I_{r-\delta}(\lambda; c^0) = 1$ has a solution $\lambda^0 \in \mathbb{C}$ with λ^0 arbitrarily close to $\lambda^*_{r-\delta}$. We denote by $F^0(v)$ the corresponding velocity profile,

$$F^0(v) = \frac{(1 + r - \delta)M(v)}{1 + \lambda^0(c^0 - v)}\,, \qquad \int_V F^0(v)\, dv = 1\,.$$

By continuity we can choose c^0 and λ^0 such that $\mathrm{Re}\left(F^0(v)\right) > 0$ holds for all $v \in V$. By the very definition of $c^*_{r-\delta}$, we have $\lambda^0 \notin \mathbb{R}$. We denote $\lambda^0 = \lambda_R + i\lambda_I$. Recall that we have the strict inequality $\lambda^*_{r-\delta} < 1/(v_{\max} - c^*_{r-\delta})$, uniformly with respect to δ. Using a continuity argument we can impose that $\lambda_R < 1/(v_{\max} - c^0)$.

Now define the real function h^0 by

$$h^0(x,v) = \mathrm{Re}\left(e^{-\lambda^0 x} F^0(v)\right) = e^{-\lambda_R x}\left[\mathrm{Re}\left(F^0(v)\right)\cos(\lambda_I x) + \mathrm{Im}\left(F^0(v)\right)\sin(\lambda_I x)\right], \tag{2.23}$$

One has $h^0(0,v) > 0$ and $h^0(\pm\pi/\lambda_I, v) < 0$ for all $v \in V$. Thus, there exists an interval $[b_1, b_2] \subset \mathbb{R}$ and a bounded domain $D \subset [b_1, b_2] \times V$ such that :

$$\begin{cases} h^0(x,v) > 0 & \text{for all } (x,v) \in D, \\ h^0(x,v) = 0 & \text{for } (x,v) \in \partial D. \end{cases}$$

On the other hand, as $\lambda_R < 1/(v_{\max} - c^0)$, there exists a constant $C(\delta)$ such that

$$(\forall v \in V) \quad |h^0(x,v)| \leq e^{-\lambda_R b_1} |F^0(v)| = e^{-\lambda_R b_1} \frac{(1 + r - \delta) M(v)}{|1 + \lambda^0(c^0 - v)|} \leq C(\delta) M(v) \,.$$

Hence, one can choose $\overline{\kappa} > 0$ small enough such that

$$r\overline{\kappa} h^0(x,v) \leq \frac{\delta}{2} M(v) \quad \text{for all } (x,v) \in \mathbb{R} \times V \,.$$

For all $\kappa \in (0, \overline{\kappa})$ we deduce from $I_{r-\delta}(\lambda^0; c^0) = 1$ the following identities,

$$\begin{aligned} \kappa(v - c^0)\partial_x \left(e^{-\lambda^0 x} F^0(v) \right) + \kappa \left(e^{-\lambda^0 x} F^0(v) \right) &= \kappa e^{-\lambda^0 x} (1 + r - \delta) M(v) \\ &= \kappa (1 + r - \delta) M(v) \int_V e^{-\lambda^0 x} F^0(v') \, dv' \,. \end{aligned}$$

Taking real part on both sides, we get for $(x,v) \in D$,

$$\begin{aligned} (v - c)\partial_x \left(\kappa h^0 \right) + \kappa h^0 &= (1 + r - \delta) M(v) \int_V \kappa h^0(x, v') \, dv' \\ &= M(v) \rho_{\kappa h^0} + r M(v) \rho_{\kappa h^0} - \delta M(v) \rho_{\kappa h^0} \\ &\leq M(v) \rho_{\kappa h^0} + r \left(M(v) - \kappa h^0 \right) \rho_{\kappa h^0} \,. \end{aligned}$$

Hence κh^0 is a subsolution of (2.22) for all $\kappa \in (0, \overline{\kappa})$ on D. We deduce that the truncated function $h = \max(0, \kappa h^0)$ is a subsolution of (2.22) over $\mathbb{R} \times V$. □

Proof of the non-existence in Theorem 2.2. Assume that $f \in \mathcal{C}^0(\mathbb{R} \times V)$ is a travelling wave solution of (6.2) of speed $c \in (0, c^*)$. According to Lemma 2.25, there exists $c < c^0 < c^*$ and a nonnegative compactly supported subsolution h of (6.2) with speed c^0. As f is positive and continuous, we can decrease h so as to obtain $f \geq h$. Let $g_1(t,x,v) = f(x - ct, v)$ and $g_2(t,x,v) = h(x - c^0 t, v)$. These two functions are respectively a solution and a subsolution of (2.1). As $g_1(0,x,v) = f(x,v) \geq h(x,v) = g_2(0,x,v)$ for all $(x,v) \in \mathbb{R} \times V$, Proposition 2.14 implies

$$g_1(t,x,v) = f(x - ct, v) \geq h(x - c^0 t, v) = g_2(t,x,v) \quad \text{for all } (t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V.$$

Taking $x = c^0 t$ and letting $t \to +\infty$, we get

$$0 = \lim_{t \to +\infty} f\left((c^0 - c)t, v \right) \geq h(0,v) \,.$$

This is a contradiction. □

2.3.6 Proof of the spreading properties

Proof of Proposition 2.6. 1. Let $c > c^*$. Consider first the initial datum

$$\widetilde{g}^0(x,v) = \begin{cases} M(v) & \text{if} \quad x < x_R\,, \\ 0 & \text{if} \quad x \geq x_R\,, \end{cases}$$

and let $\widetilde{g}$ the solution of the Cauchy problem (2.13). Denote by f a travelling wave of minimal speed c^*. There exists $\kappa > 1$ such that $\widetilde{g}^0(x,v) \leq \kappa f(x,v)$ for all $(x,v) \in \mathbb{R} \times V$. It is straightforward to check that $g_1(t,x,v) = \kappa f(x - c^* t, v)$ is a supersolution of (2.1). Hence, the comparison principle of Proposition 2.14 implies that $\widetilde{g}(t,x,v) \leq g_1(t,x,v)$ for all $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V$. In particular we have,

$$\widetilde{g}(t,ct,v) \leq g_1(t,ct,v) = \kappa f\left((c - c^*)t, v\right) \qquad \text{for all } (t,v) \in \mathbb{R}_+ \times V.$$

As $f(+\infty, v) = 0$ for all $v \in V$ and $c > c^*$, we get $\lim_{t\to+\infty} \widetilde{g}(t,ct,v) = 0$ for all $v \in V$.

On the other hand, as $\widetilde{g}^0$ is nonincreasing with respect to $x \in \mathbb{R}$ it follows from the comparison principle that $x \mapsto \widetilde{g}(t,x,v)$ is nonincreasing (see Lemma 2.22). Thus $\widetilde{g}(t,x,v) \leq \widetilde{g}(t,ct,v)$ for all $x \geq ct$ and the conclusion follows.

For a general initial datum g^0 satisfying the hypotheses of Proposition 2.6, one has $g^0(x,v) \leq \widetilde{g}^0(x,v)$ for all $(x,v) \in \mathbb{R} \times V$ and thus $g(t,x,v) \leq \widetilde{g}(t,x,v)$ for all $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V$, from which the conclusion follows.

2. Let $c < c^*$. The same arguments as in the first step yield that we can assume that

$$g^0(x,v) = \begin{cases} \gamma M(v) & \text{if} \quad x < x_L\,, \\ 0 & \text{if} \quad x \geq x_L\,. \end{cases}$$

Let h a compactly supported subsolution of (2.22) associated with a speed $c^0 \in (c, c^*)$. Since h can be chosen arbitrarily small, up to translation of h, we can always assume that $h(x,v) \leq g^0(x,v)$. Let g_2 the solution of the Cauchy problem (2.13) associated with the initial datum $g_2(0,x,v) = h(x,v)$. The comparison principle yields $g(t,x,v) \geq g_2(t,x,v)$ for all $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V$.

Let $w(t,x,v) = g_2(t, x + c^0 t, v)$. This function satisfies

$$\begin{cases} \partial_t w + (v - c^0)\partial_x w = M(v)\rho_w - w + r\rho_w\left(M(v) - w\right) & \text{in } \mathbb{R}_+ \times \mathbb{R} \times V \\ w(0,x,v) = g(x,v) \quad \text{in } \mathbb{R} \times V. \end{cases} \tag{2.24}$$

Clearly h is a (stationary) subsolution of this equation. The comparison principle Proposition 2.14 yields that $t \mapsto w(t,x,v)$ is nondecreasing for all $(x,v) \in \mathbb{R} \times V$ (see also Lemma 2.23 for a similar proof).

Let $w_\infty(x,v) = \lim_{t\to+\infty} w(t,x,v)$. This function is clearly a weak solution of

$$(v - c^0)\partial_x w_\infty = M(v)\rho_{w_\infty} - w + r\rho_{w_\infty}\left(M(v) - w_\infty\right) \qquad \text{in } \mathbb{R} \times V.$$

Moreover, we have $w_\infty(x,v) \geq w(0,x,v) = h(x,v)$ and $w_\infty(x,v) \leq M(v)$.

Lemma 2.26 (Sliding lemma)**.** *We have $w_\infty \equiv M$.*

Proof of Lemma 2.26. # Step 1. First we prove that w_∞ is positive everywhere.

Take $(x_0, v_0) \in \mathbb{R} \times V$ such that $w_\infty(x_0, v_0) > 0$. As $\widetilde{w}(t,x,v) = w_\infty(x - c^0 t, v)$ satisfies (2.1), Proposition 2.14 yields $\widetilde{w}(t,x,v) > 0$ for all $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V$ such that $|x - x_0| < v_{\max} t$. As $c^0 < c^* \leq v_{\max}$, for all $(x,v) \in \mathbb{R} \times V$ one can take $t > 0$ large enough so that $|x + ct - x_0| < v_{\max} t$. Therefore $w(x,v) = \widetilde{w}(t, x + ct, v) > 0$. We thus conclude that w_∞ is positive over $\mathbb{R} \times V$.

 # Step 2. Next we prove that $\inf w_\infty > 0$.

Let $y \in \mathbb{R}$. Define $h_y(x,v) = h(x - y, v)$, and

$$\kappa_y = \sup\{\kappa \in (0,1),\ w_\infty \geq \kappa h_y \text{ in } \mathbb{R} \times V\}\,.$$

As h_y is compactly supported and w_∞ is positive over $\mathbb{R} \times V$ and continuous, we have $w_\infty \geq \kappa h_y$ when $\kappa > 0$ is small enough. Therefore $\kappa_y > 0$.

We argue by contradiction. Assume that $\kappa_y < 1$. The definition of κ_y yields that $u = w_\infty - \kappa_y h_y \geq 0$ and that $\inf_{\mathbb{R} \times V} u = 0$. As h_y is compactly supported, this infimum is indeed reached : there exists $(x_y, v_y) \in \mathbb{R} \times V$ such that $u(x_y, v_y) = 0$. Assume that $u \not\equiv 0$ and take $(x'_y, v'_y) \in \mathbb{R} \times V$ such that $w_\infty(x'_y, v'_y) > \kappa_y h_y(x'_y, v'_y)$.

We introduce $w_1(t,x,v) = w_\infty(x - c^0 t, v)$ and $w_2(t,x,v) = \kappa_y h_y(x - c^0 t, v)$. As $w_1(0, x'_y, v'_y) > w_2(0, x'_y, v'_y)$, Proposition 2.14 gives $w_1(t,x,v) > w_2(t,x,v)$ for all (t,x,v) in $\mathbb{R}_+ \times \mathbb{R} \times V$ such that $|x - x'_y| < v_{\max} t$, that is :

$$w_\infty(x - c^0 t, v) > \kappa_y h_y(x - c^0 t, v) \quad \text{ if } |x - x'_y| < v_{\max} t\,.$$

As $c^0 < c^* \leq v_{\max}$, for all $x \in \mathbb{R}$, one can take $t > 0$ large enough so that $|x + c^0 t - x'_y| < v_{\max} t$, leading to $w_\infty(x,v) > \kappa_y h_y(x,v)$ for all $(x,v) \in \mathbb{R} \times V$, a contradiction since equality holds at (x_y, v_y).

Hence, $w_\infty \equiv \kappa_y h_y$, which is also a contradiction since w_∞ is positive while h_y is compactly supported. We conclude that $\kappa_y = 1$, namely $w_\infty \geq h_y$. Evaluating this inequality at $x = y$, one gets $w_\infty(y,v) \geq h(0,v)$ for all $(y,v) \in \mathbb{R} \times V$. As $\inf_V g(0,v) > 0$ under the assumption $\inf_V M > 0$, we have proved in fact that

$$\inf_{\mathbb{R} \times V} w_\infty > 0\,.$$

 # Step 3. As $\inf_V M > 0$, we can define

$$\kappa^* = \sup\{\kappa \in (0,1),\ w_\infty(x,v) \geq \kappa M(v) \text{ for all } (x,v) \in \mathbb{R} \times V\}.$$

We know from the previous step that this quantity is positive. If $\kappa^* < 1$, then the same types of arguments as in Step 2 lead to a contradiction. Hence $\kappa^* = 1$, meaning that $w_\infty \geq M(v)$. As $w_\infty \leq M(v)$, we conclude that $w_\infty \equiv M(v)$. □

As a consequence of Lemma 2.26 we obtain

$$\lim_{t \to +\infty} g_2(t, x + c^0 t, v) = M(v) \quad \text{for all } (x,v) \in \mathbb{R} \times V.$$

This implies in particular that $\lim_{t \to +\infty} g(t, x + c^0 t, v) = M(v)$ for all (x,v) by a sandwiching argument. Moreover, as g^0 is nonincreasing with respect to x, $x \mapsto g(t,x,v)$ is nonincreasing and thus $g(t,x,v) \geq g(t, c^0 t, v)$ for all $x \leq c^0 t$, from which the conclusion follows since $c^0 > c$. □

2.4 Proof of the dependence results

Proof of Proposition 2.5(a). Recall that the dispersion relation giving the speed $c(\lambda)$ as a function of the exponential decay λ is $I(\lambda; c(\lambda)) = 1$, where I is defined in (2.15). Let introduce I_σ the function associated with the dilated velocity profile M_σ. The function I_σ clearly satisfies the scaling relation $I_\sigma(\lambda; c(\lambda)) = I(\sigma\lambda; \sigma^{-1}c(\lambda))$, therefore we get $c^*(M_\sigma) = \sigma c^*(M)$ from the very definition of c^*. □

Proof of Proposition 2.5(b). We use the symmetry of the kernel $M(v) = M(-v)$ to write

$$I(\lambda; c) = \int_0^{v_{\max}} \frac{(1+r)(1+\lambda c)}{(1+\lambda c)^2 - \lambda^2 v^2} 2M(v)\, dv\,.$$

Let define

$$f(v) = \frac{(1+r)(1+\lambda c)}{(1+\lambda c)^2 - \lambda^2 v^2}.$$

It is an increasing function over $(0, v_{\max})$, thus $f_\star = f$. The Hardy-Littlewood inequality [146, Chap. 3] yields

$$\int_0^{v_{\max}} M^\star(v) f_\star(v) dv \le \int_0^{v_{\max}} M(v) f(v) dv \le \int_0^{v_{\max}} M_\star(v) f_\star(v) dv\,.$$

The dispersion relation is nonincreasing with respect to c. It follows immediately that

$$c^*(M^\star) \le c^*(M) \le c^*(M_\star).$$

□

Proof of Proposition 2.5(c). We use the symmetry of the kernel $M(v) = M(-v)$. For $\lambda > 0$ the dispersion relation writes

$$(1+r)\int_0^{v_{\max}} \frac{(1+\lambda c(\lambda))}{(1+\lambda c(\lambda))^2 - \lambda^2 v^2} 2M(v)\, dv = 1\,. \tag{2.25}$$

Since the function $X \mapsto \left((1+\lambda c(\lambda))^2 - \lambda^2 X\right)^{-1}$ is convex on its domain of definition, Jensen's inequality yields

$$(1+r)\frac{(1+\lambda c(\lambda))}{(1+\lambda c(\lambda))^2 - \lambda^2 \left(2\int_0^{v_{\max}} v^2 M(v)\, dv\right)} \le 1\,.$$

We recognize the dispersion relation associated with the two-speed model [36]. We deduce

$$\lambda^2 c(\lambda)^2 + (1-r)\lambda c(\lambda) - D\lambda^2 - r \ge 0\,.$$

This second-order polynomial has a negative value at $c = 0$, therefore $c(\lambda)$ is necessarily greater than the vanishing value,

$$c(\lambda) \ge \frac{(r-1) + \sqrt{(r-1)^2 + 4(D\lambda^2 + r)}}{2\lambda}\,.$$

Minimizing with respect to $\lambda > 0$, we deduce that,

$$\begin{cases} c^* \ge \dfrac{2\sqrt{rD}}{1+r} & \text{if } r < 1\,, \\ c^* \ge \sqrt{D} & \text{if } r \ge 1\,. \end{cases}$$

On the other hand we clearly obtain from (2.25),

$$(1+r)\frac{(1+\lambda c(\lambda))}{(1+\lambda c(\lambda))^2-\lambda^2 v_{\max}^2} \geq 1\,.$$

By comparison of the relations, as in the proof of Proposition 2.5(b), we deduce that the speed corresponding to a given distribution $M(v)$ supported on $(-v_{\max}, v_{\max})$ is smaller than the speed corresponding to $\frac{1}{2}(\delta_{-v_{\max}} + \delta_{v_{\max}})$. This peculiar case is analysed in [36]. The minimal speed in this case is

$$c^*\left(\frac{1}{2}(\delta_{-v_{\max}} + \delta_{v_{\max}})\right) = \begin{cases} \dfrac{2\sqrt{r}}{1+r}v_{\max} & \text{if } r<1\,, \\ v_{\max} & \text{if } r \geq 1\,. \end{cases}$$

□

Proof of Proposition 2.5(d). The dispersion relation for the rescaled equation (2.3) reads

$$I_\varepsilon(\lambda;c) = (1+\varepsilon^2 r)\int_V \frac{1}{1+\varepsilon^2\lambda(c-v/\varepsilon)}M(v)\,dv\,. \tag{2.26}$$

The previous result guarantees that c_ε^* is bounded from above for $\varepsilon^2 r < 1$,

$$c_\varepsilon^* \leq \frac{2\sqrt{\varepsilon^2 r}}{1+\varepsilon^2 r}\left(\frac{v_{\max}}{\varepsilon}\right) \leq 2\sqrt{r}v_{\max}\,.$$

For a given $\lambda > 0$, we perform a Taylor expansion of (2.26) up to second order,

$$I_\varepsilon(\lambda;c) = 1 + \varepsilon^2(r - \lambda c + \lambda^2 D) + O(\varepsilon^3)\,,$$

uniformly for $c \in [0, 2\sqrt{r}v_{\max}]$, since V is bounded. Therefore, solving the relation dispersion for the minimal speed boils down to solving

$$r - \lambda c_\varepsilon(\lambda) + \lambda^2 D + O(\varepsilon) = 0\,.$$

We deduce

$$\lim_{\varepsilon\to 0} c_\varepsilon(\lambda) = \frac{r}{\lambda} + \lambda D\,.$$

Therefore the minimal speed verifies $\lim_{\varepsilon\to 0} c_\varepsilon^* = 2\sqrt{rD}$. □

2.5 Stability of the travelling waves

2.5.1 Linear stability

In this Subsection, we focus on the linearized problem around some travelling wave moving at speed $c \in [c^*, v_{\max})$. We recall that we consider a solution u of the equation associated with the linearization around a travelling wave :

$$\partial_t u + (v-c)\partial_z u + \left(1 + r\rho_f\right)u = ((1+r)M - rf)\int_V u'dv'. \tag{2.27}$$

where the notation $'$ always stands in the sequel for a function of the (t, z, v') variable. We shall prove stability of the wave in a suitable L^2 framework, inspired by [140, 104, 105, 36].

Proof of Theorem 2.7. We search for an ansatz $u = we^{\phi}$. The function w satisfies :

$$\partial_t w + (v-c)\partial_z w + \left((v-c)\partial_z\phi + 1 + r\rho_f\right) w = ((1+r)M - rf) \int_V e^{\phi'-\phi} w' dv', \qquad (2.28)$$

From (2.28), we shall derive the dissipation inequality (2.10). We test (2.28) against w to obtain the following energy estimate :

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}\times V} |w|^2 \, dzdv\right) + \int_{\mathbb{R}\times V} \left((v-c)\partial_z\phi + 1 + r\rho_f\right) |w|^2 \, dzdv$$
$$= \int_{\mathbb{R}\times V\times V'} ((1+r)M - rf)\, e^{\phi'-\phi} w w' dv dv' dz. \qquad (2.29)$$

We aim at choosing a weight ϕ such that the dissipation is coercive in L^2 norm. Let define the symmetric kernel K as follows

$$K(v,v') = \left((v-c)\partial_z\phi + 1 + r\rho_f\right)\delta_{v=v'}$$
$$-\frac{1}{2}\left(((1+r)M - rf)\, e^{\phi'-\phi} + \left((1+r)M' - rf'\right) e^{\phi-\phi'}\right), \qquad (2.30)$$

we seek a function ϕ such that

$$K(v,v') \geq A(z,v)\delta_{v=v'},$$

for a suitable positive function A, in the sense of kernel operators. For this purpose we focus on the eigenvalues of the kernel operator $A(z,v)\delta_{v=v'} - K(v,v')$.

Definition 2.27 (Weight ϕ)**.** *We introduce $\Lambda(z) \in \left[0, \frac{1}{v_{\max}-c}\right)$ the smallest solution of the following dispersion relation*

$$\int_V \frac{(1+r)M(v) - rf(z,v)}{1+\Lambda(z)(c-v)}\, dv = 1\,, \qquad (2.31)$$

and we define $\Gamma(z)$ through the differential equation

$$\frac{1}{2}\frac{\Gamma'(z)}{\Gamma(z)} = \Lambda(z)\,, \quad \Gamma(0) = 1\,. \qquad (2.32)$$

Finally we define

$$\phi(z,v) = \frac{1}{2}\ln\left(\frac{(1+r)M(v) - rf(z,v)}{\Gamma(z)}\right)\,, \qquad (2.33)$$

Recall that $0 \leq f \leq M$, so that the weight ϕ is well-defined as soon as Λ is well-defined. A small argumentation is required to prove that $\Lambda(z)$ is well-defined too. For a given c and z, define

$$G(\Lambda) = \int_V \frac{(1+r)M(v) - rf(z,v)}{1+\Lambda(c-v)}\, dv\,, \quad \Lambda \in \left[0, \frac{1}{v_{\max}-c}\right).$$

The function G is continuous, and satisfies the following properties,

$$G(0) = (1+r) - r\rho_f(z) = (1+r)\left(1-\rho_f(z)\right) + \rho_f(z) \in [1, 1+r]\,,$$
$$G(\lambda) = \int_V \frac{(1+r)M(v) - rf(z,v)}{1+\lambda(c-v)} dv = 1 - \int_V \frac{rf(z,v)}{1+\lambda(c-v)} dv \leq 1\,,$$

where λ is chosen such that $I(\lambda; c) = 1$. Thus we can define the smallest $\Lambda(z) \in [0,\lambda]$ such that $G(\Lambda(z)) = 1$.

Remark 2.28 (Asymptotic behavior of the weight)**.** *It is important to state clearly the asymptotic behavior of the weight as it determines the possible perturbations. The following estimates were established in Section 2.3,*

$$\lim_{z\to-\infty} f(z,v) = M(v), \quad \lim_{z\to+\infty} f(z,v) = 0.$$

We deduce from (2.31) *and the dispersion relation* (2.13) *that*

$$\lim_{z\to-\infty} \Lambda(z) = 0, \quad \lim_{z\to+\infty} \Lambda(z) = \lambda.$$

It yields from (2.33) *and* (2.32) *that*

$$\phi(z,v) \underset{z\to-\infty}{\sim} -\frac{1}{2}\log\left(\frac{\Gamma(z)}{M(v)}\right), \quad \phi(z,v) \underset{z\to+\infty}{\sim} -\lambda z\,.$$

The precise behavior of Γ *near* $-\infty$ *would require further analysis about the integrability of* Λ*. However, we believe it converges towards a positive constant. As compared to [69],* ϕ *combines the two weights in a single one, see [69, Eqs (3.6)-(3.7)]. As a consequence, the perturbation* $g^0 - f$ *must decay faster than the wave profile at* $+\infty$ *to have finite energy, as usual.*

Lemma 2.29. *Let* A *be defined as*

$$A(z,v) = \frac{r}{2}\left(\rho_f(z) + \frac{f(z,v)}{(1+r)M(v) - rf(z,v)}\right),$$

and $\mathbf{T}$ *be the operator associated with the symmetric kernel* $T(v,v') = A(z,v)\delta_{v=v'} - K(v,v')$*. The operator* $\mathbf{T}$ *is nonpositive.*

Proof of Lemma 2.29. We shall prove that 0 is the Perron eigenvalue of the operator $\mathbf{T}$. For that purpose we shall exhibit a positive eigenvector in the kernel of $\mathbf{T}$. The equation $\mathbf{T}(W) = 0$ reads

$$(\forall v \in V) \quad \int_V \left(A(z,v)\delta_{v=v'} - K(v,v')\right) W(v')dv' = 0\,.$$

Plugging the formula for $K(v,v')$ (2.30) into this expression we get,

$$\begin{aligned}&\left(A(z,v) - (v-c)\partial_z\phi(z,v) - 1 - r\rho_f(z)\right) W(v)\\ &\qquad + \frac{1}{2}\left((1+r)M(v) - rf(z,v)\right)\left(\int_V e^{\phi(z,v')-\phi(z,v)} W(v')\,dv'\right)\\ &\qquad\qquad + \frac{1}{2}\int_V \left((1+r)M(v') - rf(z,v')\right) e^{\phi(z,v)-\phi(z,v')} W(v')\,dv' = 0\,.\end{aligned}$$

From the Definitions (2.31)-(2.33) we have,

$$\partial_z\phi(z,v) = -\frac{r}{2}\frac{\partial_z f(z,v)}{(1+r)M(v) - rf(z,v)} - \Lambda(z)\,.$$

The weight ϕ and the function A are chosen such that

$$\begin{aligned}&A(z,v) - (v-c)\partial_z\phi(z,v) - 1 - r\rho_f(z)\\ &= \frac{r}{2}\left(\rho_f(z) + \frac{f(z,v)}{(1+r)M(v) - rf(z,v)} + (v-c)\frac{\partial_z f(z,v)}{(1+r)M(v) - rf(z,v)}\right)\\ &\qquad\qquad + (v-c)\Lambda(z) - 1 - r\rho_f(z)\\ &= \frac{r}{2}\left(2\rho_f(z)\right) + (v-c)\Lambda(z) - 1 - r\rho_f(z)\\ &= (v-c)\Lambda(z) - 1\,.\end{aligned}$$

Therefore the equation $\mathbf{T}(W) = 0$ is equivalent to

$$W(v) = \frac{1}{1+\Lambda(z)(c-v)} \left(\frac{1}{2} \left((1+r)M(v) - rf(z,v) \right) e^{-\phi(z,v)} X_1(z) + \frac{1}{2} e^{\phi(z,v)} X_2(z) \right) , \quad (2.34)$$

where the macroscopic quantities X_1 and X_2 are defined as follows,

$$X_1(z) = \int_V e^{\phi(z,v')} W(v')\, dv' , \quad X_2(z) = \int_V \left((1+r)M(v') - rf(z,v') \right) e^{-\phi(z,v')} W(v')\, dv' .$$

To resolve this eigenvalue problem, we seek proper values for X_1 and X_2. From (2.34) we deduce that they are solution of a 2×2 closed linear system, namely

$$\begin{cases} X_1(z) = \dfrac{1}{2} \left(\displaystyle\int_V \frac{(1+r)M(v) - rf(z,v)}{1+\Lambda(z)(c-v)}\, dv \right) X_1(v) \\ \qquad\qquad + \dfrac{1}{2} \left(\displaystyle\int_V \frac{e^{2\phi(z,v)}}{1+\Lambda(z)(c-v)}\, dv \right) X_2(z) \\ X_2(z) = \dfrac{1}{2} \left(\displaystyle\int_V \frac{\left((1+r)M(v) - rf(z,v) \right)^2 e^{-2\phi(z,v)}}{1+\Lambda(z)(c-v)}\, dv \right) X_1(v) \\ \qquad\qquad + \dfrac{1}{2} \left(\displaystyle\int_V \frac{(1+r)M(v) - rf(z,v)}{1+\Lambda(z)(c-v)}\, dv \right) X_2(z) \end{cases}$$

This system simplifies thanks to the choice of $\Lambda(z)$ (2.31). Indeed we have

$$\begin{aligned} &\int_V \frac{(1+r)M(v) - rf(z,v)}{1+\Lambda(z)(c-v)}\, dv = 1 \\ &\int_V \frac{e^{2\phi(z,v)}}{1+\Lambda(z)(c-v)}\, dv = \left(\int_V \frac{(1+r)M(v) - rf(z,v)}{1+\Lambda(z)(c-v)}\, dv \right) \frac{1}{\Gamma(z)} = \frac{1}{\Gamma(z)} \\ &\int_V \frac{\left((1+r)M(v) - rf(z,v) \right)^2 e^{-2\phi(z,v)}}{1+\Lambda(z)(c-v)}\, dv \\ &\qquad = \left(\int_V \frac{(1+r)M(v) - rf(z,v)}{1+\Lambda(z)(c-v)}\, dv \right) \Gamma(z) = \Gamma(z) . \end{aligned}$$

We are reduced to the following eigenvalue problem,

$$\begin{pmatrix} X_1(z) \\ X_2(z) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \Gamma(z)^{-1} \\ \Gamma(z) & 1 \end{pmatrix} \begin{pmatrix} X_1(z) \\ X_2(z) \end{pmatrix} .$$

Clearly, $(X_1(z), X_2(z)) = (1, \Gamma(z))$ is the unique solution up to multiplication. We obtain eventually that W is given (up to a multiplicative factor) by

$$\begin{aligned} W(v) &= \frac{1}{2} \frac{\left((1+r)M(v) - rf(z,v) \right) e^{-\phi(z,v)} + e^{\phi(z,v)} \Gamma(z)}{1+\Lambda(z)(c-v)} \\ &= \frac{\left[\left((1+r)M(v) - rf(z,v) \right) \Gamma(z) \right]^{1/2}}{1+\Lambda(z)(c-v)} > 0 . \end{aligned}$$

As a consequence, we have found that the symmetric operator $\mathbf{T}$, which is nonnegative out of the diagonal $v = v'$, possesses a positive eigenvector W associated with the eigenvalue 0. Therefore it is a nonpositive operator. This ends the proof of the Lemma. □

We can now conclude the proof of Theorem 2.7. Lemma 2.29 claims that for all $w \in L^2(\mathbb{R})$ such that $u = e^{\phi} w$ is solution to the linearized equation, we have

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}\times V} |w|^2 \, dzdv\right) + \int_{\mathbb{R}\times V} A(z,v)\, |w|^2 \, dzdv \leq 0 \, .$$

which proves the Proposition. □

Remark 2.30 (Non optimality of the weight)**.** *We believe that the weight* $\exp(\phi(z,v))$ *proposed in Definition 2.27 is not optimal with respect to the spectral property of the linearized operator* (2.27)*. Indeed the dissipation factor $A(z,v)$ is equivalent in the diffusion limit ($r \to r\varepsilon^2$) to $r\varepsilon^2 \rho_f(z)$, although we expect $2r\varepsilon^2\rho_f(z)$ [140, 69]. The missing factor* 2 *is responsible for the restriction $\gamma > 1/2$ in our nonlinear stability result, Corollary 2.8.*

Let us recall how to derive the spectral properties of the linearized equation in the diffusive limit, namely the linearized Fisher-KPP equation,

$$\partial_t u - c\partial_z u - D\partial_{zz} u = r(1 - 2\rho_f)u \, , \tag{2.35}$$

where $\rho_f(z)$ is the profile of the travelling wave in the frame $z = x - ct$. Applying the same procedure as in the proof of Theorem 2.7, we shall derive an equation for the weighted perturbation $w = e^{-\phi}u$, and optimize the dissipation with respect to the weight ϕ (see also [36]), as follows

$$\partial_t w - c\partial_z w - D\partial_{zz} w - 2D\partial_z\phi\partial_z w - D\partial_{zz}\phi w - (c\partial_z\phi + D|\partial_z\phi|^2)w = r(1-2\rho_f)w$$
$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}} |w|^2 \, dz\right) + D\int_{\mathbb{R}} |\partial_z w|^2 \, dz + \int_{\mathbb{R}} \left(2r\rho_f - r - c\partial_z\phi - D|\partial_z\phi|^2\right) |w|^2 \, dz = 0$$

The best choice is achieved when $\partial_z\phi$ is constant and minimizes $r + c\lambda + D\lambda^2$, i.e. *$\partial_z\phi = -c/(2D)$. In the case of the minimal speed $c = c^* = 2\sqrt{rD}$, we obtain the following dissipation formula for the linearized operator,*

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}} |w|^2 \, dz\right) + D\int_{\mathbb{R}} |\partial_z w|^2 \, dz + \int_{\mathbb{R}} 2r\rho_f |w|^2 \, dz = 0 \, . \tag{2.36}$$

Notice the factor 2 *which is apparently missing in the dissipation term* (2.10)*.*

A systematic way to find the correct weight is to derive the eigenvectors of the operator and its dual, then to use the framework of relative entropy (see [159] for a general presentation). This was done by Kirchgässner [140] who derived the so-called eichform *for* (2.35)*. The linearized operator $\mathcal{L}(u) = -c\partial_z u - \partial_{zz}u - (1-2\rho_f)u$ possesses obviously the nonpositive eigenvector $\eta = \partial_z\rho_f$, $\mathcal{L}(\eta) = 0$. The dual operator $\mathcal{L}^*(\varphi) = +c\partial_z\varphi - \partial_{zz}\varphi - (1-2\rho_f)\varphi$ possesses the nonpositive eigenvector $\psi = \partial_z\rho_f e^{cz}$, $\mathcal{L}^*(\psi) = 0$, as can be checked by direct calculation. Therefore the relative entropy identity for the convex function $H(p) = \frac{1}{2}|p|^2$ writes for the linearized system as follows,*

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}} \psi(z)\left(\frac{u(t,z)}{\eta(z)}\right)^2 \eta(z)\, dz\right) + \int_{\mathbb{R}} \psi(z)\left|\frac{\partial}{\partial z}\left(\frac{u(t,z)}{\eta(z)}\right)\right|^2 \eta(z)\, dz = 0 \, ,$$

which is equivalent to (2.36) *after straightforward computation (recall $w = e^{(c/2D)z}u$).*

A similar strategy could be performed here : the linearized operator

$$\mathcal{L}(u) = (v - c)\partial_z u + \left(1 + r\rho_f\right) u - ((1+r)M - rf)\int_V u' dv'$$

possesses the nonpositive eigenvector $\eta = \partial_z f$, $\mathcal{L}(\eta) = 0$ (recall $z \mapsto f(z,v)$ is nonincreasing). To derive the corresponding relative entropy identity, we should find an eigenvector ψ in the nullset of the dual operator

$$\mathcal{L}^*(\varphi) = -(v-c)\partial_z\varphi + \left(1 + r\rho_f\right)\varphi - \int_V \left((1+r)M' - rf'\right)\varphi' dv' \,.$$

Existence of such an eigenvector would follow from the Krein-Rutman Theorem. However we were not able to find an explicit formulation of ψ, and thus of the dissipation, which is necessary to derive a quantitative nonlinear stability estimate such as Corollary 2.8. This is the reason why we stick to the weight proposed in Definition 2.27 although we believe it is not the optimal one.

2.5.2 Nonlinear stability by a comparison argument.

Proof of Corollary 2.8. First, the comparison principle of Proposition 2.14 and (2.11) yield

$$\rho_u(t,z) \geq (\gamma - 1)\,\rho_f(t,z), \quad \forall (t,z) \in \mathbb{R}_+ \times \mathbb{R}. \tag{2.37}$$

Now, we write the nonlinear equation verified by the weighted perturbation $w = e^{-\phi}u$,

$$\partial_t w + (v-c)\partial_z w + \left((v-c)\partial_z\phi + 1 + r\rho_f\right) w = \left((1+r)M - rf\right)\int_V e^{\phi'-\phi} w' dv' - rw\rho_u \,, \tag{2.38}$$

and as for the linear stability problem we test (2.38) against w :

$$\frac{d}{dt}\left(\int_{\mathbb{R}\times V} \frac{|w|^2}{2} dzdv\right) + \int_{\mathbb{R}\times V} \left((v-c)\partial_z\phi + 1 + r\rho_f\right)|w|^2\, dzdv$$
$$= \int_{\mathbb{R}\times V\times V'} w\left((1+r)M - rf\right) e^{\phi'-\phi} w' dzdvdv' - \int_{\mathbb{R}\times V} r|w|^2 \rho_u dzdv \,.$$

Using (2.37) we deduce

$$\frac{d}{dt}\left(\int_{\mathbb{R}\times V} \frac{|w|^2}{2} dzdv\right) + \int_{\mathbb{R}\times V} \left((v-c)\partial_z\phi + 1 + \gamma r\rho_f\right)|w|^2\, dzdv$$
$$\leq \int_{\mathbb{R}\times V\times V'} w\left((1+r)M - rf\right) e^{\phi'-\phi} w' dvdv'dz.$$

This last equation is very similar to (2.29). Following the same steps as in the proof of Theorem 2.7, we find that using the same weight ϕ and setting

$$A(z,v) = \frac{r}{2}\left((2\gamma - 1)\rho_f + \frac{f}{(1+r)M(v) - rf}\right),$$

we obtain an estimate very similar to (2.10),

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}\times V} |u|^2 e^{-2\phi(z,v)} dzdv\right)$$
$$+ \int_{\mathbb{R}\times V} \frac{r}{2}\left[(2\gamma-1)\rho_f + \frac{f}{M(v) + r\left(M(v) - f\right)}\right]|u|^2 e^{-2\phi(z,v)} dzdv \leq 0 \,. \tag{2.39}$$

□

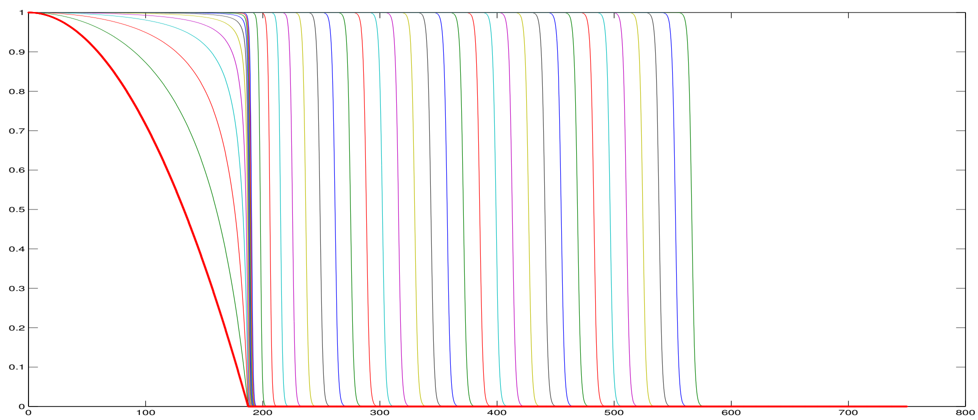


FIGURE 2.1 – Numerical simulation of equation (2.1) with initial datum being chosen as $g^0(x < 0, v) = M(v)$ and $g^0(x > 0, v) = M(v)\left(1 - \alpha x^2\right)_+$. The density distribution $M(v)$ is a truncated Gaussian function on a compact velocity set. We plot the evolution of the macroscopic density ρ_g (initial condition in red bold). After short time the density has accumulated towards a steep profile. Then the front starts to propagate with constant speed.

2.6 Numerics

In this Section, we show the outcome of numerical simulations to illustrate our results, and to motivate the last Section about accelerating fronts. We use a simple explicit numerical scheme for approximating (2.1). The free transport operator is discretized using an upwind scheme.

We show in Figure 2.1 the expected asymptotic behavior when the velocity space is bounded. The solution of the Cauchy problem converges towards a travelling front with minimal speed.

Next we investigate the case $V = \mathbb{R}$. Of course, numerical simulations require that the support of M is truncated. We opt for the following strategy : the velocity set is truncated $V_A = [-A, A]$, and the distribution M is renormalized accordingly. For any $A > 0$ we observe the asymptotic regime of a travelling front with finite speed, as expected. However, the asymptotic spreading speed diverges as $A \to +\infty$. In fact, we observe that the envelope of the spreading speed scales approximately as $\langle c \rangle = \mathcal{O}(t^{1/2})$. Hence the front is accelerating like the power law $\langle x \rangle = \mathcal{O}(t^{3/2})$.

2.7 Superlinear spreading and accelerating fronts ($V = \mathbb{R}$)

We assume in this Section that $V = \mathbb{R}$ and that $M(v) > 0$ for all $v \in \mathbb{R}$. We prove superlinear spreading. We deduce as a Corollary that there cannot exist a travelling wave solution of (2.1). We also give some quantitative features about the spreading of the density when M is a

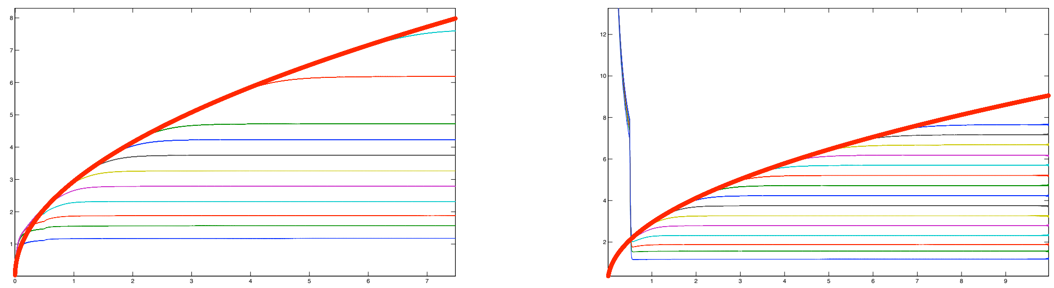


FIGURE 2.2 – Numerical simulations of equation (2.1) with initial datum being chosen as (left) $g^0(x < 0, \cdot) = M(\cdot)$ and $g^0(x > 0, \cdot) = 0$, and (right) the same initial condition as in Figure 2.1. The distribution M is a Gaussian function. Each plot corresponds to the evolution of speed of the front for some truncation $V = [-A, A]$, for (left) $A = [(1 : 9), 15, 20]$, and (right) $A = (1 : 15)$. The curves are ordered from bottom to top : the speed of the front increases with A. We plot in red bold the function $t \mapsto t^{1/2}$. We observe that it fits very well with the envelop of the family of curves. As a consequence, the front propagation scales approximately as $\langle x \rangle = \mathcal{O}(t^{3/2})$.

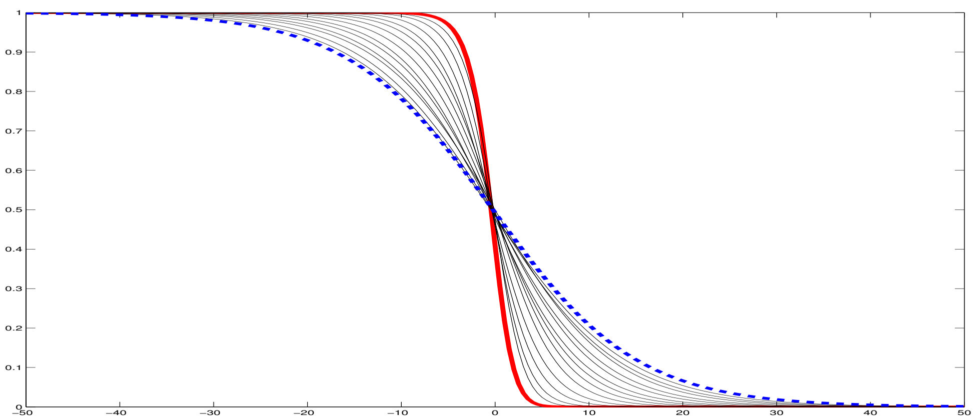


FIGURE 2.3 – Same numerical simulation as Figure 2.2 with the same initial datum as in Figure 2.1. We superpose various macroscopic profiles ρ_g obtained after long time simulations of the scheme, for different truncation levels $A = (1 : 15)$. Time is the same for all profiles. It is sufficiently large to guarantee that we have reached the asymptotic regime (Figure 2.2, right). All profiles are translated such that $\rho_g(T, 0) = \frac{1}{2}$. We observe that the exponential decay is monotonically decreasing with A. This indicates that the solution corresponding to $V = \mathbb{R}$ should flatten when $t \to \infty$.

Gaussian distribution. In accordance with numerical simulations, we prove a sharp spreading rate, namely $\mathcal{O}\left(t^{3/2}\right)$. To this end, we construct explicit sub- and supersolutions from which we estimate the spreading (respectively from below and above).

Before we go to the proof, let us give some heuristics concerning the superlinear spreading rate. Reaction-diffusion fronts with KPP nonlinearity are *pulled fronts* : the spreading rate is determined by the dynamics of small populations at the far edge of the front. In the kinetic model with unbounded velocities, individuals with arbitrary large speeds go at the far edge of the front. There, their density grows exponentially, and pull the accelerating front.

2.7.1 Nonexistence of travelling waves and superlinear spreading

Proof of Proposition 2.9. Let $\underline{A} > 0$ so that $(1+r)\int_{-\underline{A}}^{\underline{A}} M(v)dv = 1$. For all $A > \underline{A}$ we define the renormalized truncated kernel and the associated growth rate,

$$M_A(v) = \frac{1_{[-A,A]}(v)}{\int_{-A}^{A} M} M(v) \quad \text{and} \quad r_A = (1+r)\int_{-A}^{A} M(v)dv - 1 \in (0,r)\,.$$

As M_A is compactly supported, we can apply the results proved when V is bounded in order to construct appropriate subsolutions.

Before we proceed with subsolutions we investigate the dispersion relation in the limit $A \to +\infty$. Define for all $c \in (0, A)$ and $\lambda \in (0, 1/(A-c))$:

$$I_A(\lambda; c) = (1+r_A)\int_{\mathbb{R}} \frac{M_A(v)}{1+\lambda(c-v)}dv = (1+r)\int_{-A}^{A} \frac{M(v)}{1+\lambda(c-v)}dv$$

and c_A^* the corresponding minimal speed defined in Lemma 2.18.

Lemma 2.31. *One has* $\lim_{A\to+\infty} c_A^* = +\infty$.

Proof of Lemma 2.31. For all $A > \underline{A}$, let $\lambda_A \in (0, 1/(A-c_A^*))$ such that

$$I_A(\lambda_A; c_A^*) = (1+r)\int_{-A}^{A} \frac{M(v)}{1+\lambda_A(c_A^* - v)}dv = 1. \tag{2.40}$$

If c_A^* does not diverge to $+\infty$ as $A \to +\infty$, then it is bounded along a sequence $(A_n)_n$ and one has $\lim \lambda_{A_n} = 0$ simply by comparison $\lambda_{A_n} \le 1/(A_n - c_{A_n}^*)$. Applying Fatou's lemma to (2.40), one gets

$$\begin{aligned}(1+r)\int_{\mathbb{R}} \liminf_{n\to+\infty} \frac{M(v)1_{(-A_n,A_n)}(v)}{1+\lambda_{A_n}(c_{A_n}^* - v)}dv \\ = (1+r)\int_{\mathbb{R}} M(v)dv = 1 + r \le \liminf_{n\to+\infty} I_{A_n}(\lambda_{A_n}, c_{A_n}^*) = 1\,,\end{aligned}$$

a contradiction. □

Let g_A the solution of

$$\begin{cases} \partial_t g_A + v\partial_x g_A = M_A(v)\rho_{g_A} - g_A + r_A \rho_{g_A}\left(M_A(v) - g_A\right) & \text{in } \mathbb{R}_+ \times \mathbb{R} \times [-A, A], \\ g_A(0,x,v) = g^0(x,v) & \text{in } \mathbb{R} \times [-A, A], \end{cases} \tag{2.41}$$

and $\widetilde{g}_A = \frac{r_A}{r} g_A$. Clearly, $M_A(v) \leq \frac{M(v)}{\int_{-A}^{A} M}$ for all $v \in V$. Hence, multiplying (2.41) by $\frac{r_A}{r}$, we get

$$\begin{aligned}
\partial_t \widetilde{g}_A + v\partial_x \widetilde{g}_A &\leq \frac{M(v)}{\int_{-A}^{A} M} \rho_{\widetilde{g}_A} - \widetilde{g}_A + r_A \rho_{\widetilde{g}_A} \left(\frac{M(v)}{\int_{-A}^{A} M} - g_A \right) \\
&\leq (1 + r_A) \frac{M(v)}{\int_{-A}^{A} M} \rho_{\widetilde{g}_A} - \widetilde{g}_A - r_A \rho_{\widetilde{g}_A} g_A \\
&= (1 + r) M(v) \rho_{\widetilde{g}_A} - \widetilde{g}_A - r_A \rho_{\widetilde{g}_A} g_A \\
&= (1 + r) M(v) \rho_{\widetilde{g}_A} - \widetilde{g}_A - r \rho_{\widetilde{g}_A} \widetilde{g}_A \\
&= M(v) \rho_{\widetilde{g}_A} - \widetilde{g}_A + r \rho_{\widetilde{g}_A} \left(M(v) - \widetilde{g}_A \right).
\end{aligned}$$

Extending $\widetilde{g}_A$ by 0 outside of $\mathbb{R}_+ \times \mathbb{R} \times [-A, A]$, as $\widetilde{g}_A(0, x, v) = \frac{r_A}{r} g^0(x, v) \leq g^0(x, v)$, we get that $\widetilde{g}_A$ is a subsolution of (2.1) and it follows from the maximum principle stated in Proposition 2.14 that $g(t, x, v) \geq \widetilde{g}_A(t, x, v)$ for all $(t, x, v) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$.

On the other hand, we know from Proposition 2.6 that for all $c < c_A^*$:

$$\lim_{t \to +\infty} \left(\sup_{x \leq ct} |M_A(v) - \widetilde{g}_A(t, x, v)| \right) = \lim_{t \to +\infty} \left(\sup_{x \leq ct} \left(M_A(v) - \widetilde{g}_A(t, x, v) \right) \right) = 0.$$

Hence, as $M(v) \geq g(t, x, v) \geq \widetilde{g}_A(t, x, v)$ and $M_A(v) \geq M(v)$ for all $v \in [-A, A]$, one gets for all $v \in [-A, A] : 0 \leq \lim_{t \to +\infty} \left(\sup_{x \leq ct} \left(M(v) - g(t, x, v) \right) \right) \leq 0$. Therefore we conclude

$$\lim_{t \to +\infty} \left(\sup_{x \leq ct} |M(v) - g(t, x, v)| \right) = 0 \quad \text{for all } c < c_A^* \text{ and } A > \underline{A}.$$

the conclusion follows from the fact that $\lim_{A \to +\infty} c_A^* = +\infty$. □

2.7.2 Upper bound for the spreading rate in the gaussian case

We construct below supersolutions for (2.1) when $V = \mathbb{R}$ and M is a Gaussian distribution.

Proposition 2.32. *Let $V = \mathbb{R}$ and $M(v) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{v^2}{2\sigma^2}\right)$. For $1 \leq b \leq a$ define*

$$\rho(t, x) = M\left(\frac{x}{t + a}\right) e^{r(t+a)} \quad \text{and} \quad g^0(x, v) = \frac{1}{b} M\left(\frac{x}{b}\right) M(v) e^{ra}. \tag{2.42}$$

Let g be defined by

$$g(t, x, v) = g^0(x - vt, v) e^{-t} + \int_0^t (1 + r) M(v) \rho(s, x - v(t - s)) e^{-(t-s)} ds.$$

Then $\overline{g}(t, x, v) = \min\{M(v), g(t, x, v)\}$ is a supersolution of (2.1), that is :

$$\partial_t \overline{g} + v\partial_x \overline{g} \geq \left(M(v)\rho_{\overline{g}} - \overline{g} \right) + r\rho_{\overline{g}} \left(M(v) - \overline{g} \right), \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R} \times V.$$

Proof of Proposition 2.32. We shall prove that g is a supersolution of (2.1). Indeed, it will follow that $\overline{g}$ is a supersolution since it is the minimum of two supersolutions. From the Duhamel formula, we deduce that

$$\partial_t g + v\partial_x g + g = (1+r)M(v)\rho,$$

To prove that g is a subsolution we must prove in fact that

$$(1+r)M(v)\rho \geq (1+r)M(v)\rho_g - r\rho_g g.$$

This is sufficient to prove that the inequality $\rho \geq \rho_g$ holds true. Computing the expression of ρ_g we obtain

$$\rho_g(t,x) = \underbrace{\int_V g^0(x-vt,v)e^{-t}dv}_{=A} + \underbrace{\int_0^t (1+r)e^{-(t-s)+r(s+a)} \int_V M(v)M\left(\frac{x-v(t-s)}{s+a}\right) dvds}_{=B}$$

We first deal with the estimate of B. We claim the following inequality holds true : for all $x \in \mathbb{R}$ and $s \in [0,t]$,

$$\int_V M(v)M\left(\frac{x-v(t-s)}{s+a}\right) dv \leq M\left(\frac{x}{t+a}\right) \tag{2.43}$$

In fact one has

$$\begin{aligned}
\int_V M(v)M\left(\frac{x-v(t-s)}{s+a}\right) dv &= \int_V \frac{1}{2\pi\sigma^2} \exp\left(-\frac{v^2}{2\sigma^2} - \frac{\left(\frac{x-v(t-s)}{s+a}\right)^2}{2\sigma^2}\right) dv \\
&= \frac{1}{\sqrt{2\pi}\sigma} \frac{s+a}{[(s+a)^2+(t-s)^2]^{\frac{1}{2}}} \exp\left(-\frac{1}{2\sigma^2}\frac{x^2}{(s+a)^2+(t-s)^2}\right) \\
&\leq \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}\frac{x^2}{(t+a)^2}\right),
\end{aligned}$$

since

$$\forall s \in [0,t], \quad (t+a)^2 \geq (s+a)^2 + (t-s)^2 \geq (s+a)^2.$$

This yields

$$\begin{aligned}
B(t,x) &\leq (1+r)\left(\int_0^t e^{-(t-s)+r(s+a)} ds\right) M\left(\frac{x}{t+a}\right) \\
&= e^{ra-t}\left(e^{(1+r)t} - 1\right) M\left(\frac{x}{t+a}\right) \\
&= \left(e^{(1+r)t} - 1\right) e^{-t+ra}\rho(t,x)e^{-r(t+a)} \\
&= \left(1 - e^{-(1+r)t}\right)\rho(t,x)
\end{aligned}$$

To estimate A, we plug in the formula for ρ (2.42),

$$\left(\frac{A}{\rho}\right)(t,x) = \sqrt{2\pi}\sigma \exp\left(\frac{x^2}{2\sigma^2(t+a)^2} - (1+r)t - ra\right)\int_V g^0(x-vt,v)dv\,.$$

We compute the last integral using the formula for the initial condition g^0 (2.42),

$$\int_V g^0(x-vt,v)dv = \frac{1}{\sqrt{2\pi}\sigma}\frac{1}{(t^2+b^2)^{\frac{1}{2}}}\exp\left(-\frac{1}{2\sigma^2}\frac{x^2}{t^2+b^2}\right)e^{ra},$$

Thus, for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$:

$$\begin{aligned}\left(\frac{A}{\rho}\right)(t,x) &= \frac{1}{(t^2+b^2)^{\frac{1}{2}}}\exp\left(-\frac{x^2}{2\sigma^2(t+a)^2}\left[\frac{(t+a)^2}{t^2+b^2}-1\right]\right)\exp\left(-(1+r)t\right)\\ &\le \exp\left(-(1+r)t\right) \qquad (2.44)\end{aligned}$$

as long as $b \ge 1$ and $(t+a)^2 \ge t^2+b^2$, that is $a \ge b \ge 1$. This concludes the proof. □

Proof of Theorem 2.11. Let $\varepsilon > 0$. For all $t \ge 0$, we define the zone

$$\Gamma_t = \left\{x \in \mathbb{R} \mid |x| \ge \sigma\left(1+\varepsilon\right)\sqrt{2r}(t+a)^{3/2}\right\}.$$

From the definition of $\overline{g}$, we deduce that $\overline{g}$ is a supersolution such that

$$\rho_{\overline{g}} \le \min\left(1,\rho_g\right) \le \min\left(1,\rho\right).$$

However, for all $t > 0$ and $x \in \Gamma_t$, we have

$$\begin{aligned}\rho(t,x) &\le \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{2r\sigma^2\left(1+\varepsilon\right)^2(t+a)^3}{2\sigma^2(t+a)^2} + r(t+a)\right)\\ &= \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-r(t+a)\left(\left(1+\varepsilon\right)^2-1\right)\right).\end{aligned}$$

It yields that

$$\lim_{t\to+\infty}\left(\sup_{x\in\Gamma_t}\rho(t,x)\right) = 0.$$

□

Computations are made easier above since the class of Gaussian distributions is stable by convolution. This is also the case for the class of Cauchy distributions. Therefore we are able to derive an inequality similar to (2.43) in the latter case. Let us comment this case. Because the distribution M has an infinite variance, we learn from [154] that the correct macroscopic limit leads to a nonlocal fractional Laplacian operator. On the other hand, we expect from [46, 45, 64] an exponentially fast propagation in the fractional diffusion regime. Similarly as for our previous results, we can reasonably expect that the spreading rate is faster in the kinetic model than in the macroscopic limit. Therefore we expect a spreading rate faster than exponential. In fact the supersolution that we are able to derive confirms this expectation.

In the following Proposition, we construct a supersolution that spreads with rate $\mathcal{O}(te^{rt/2})$. However, we leave the complete analysis of spreading in the case of the Cauchy distribution for future work.

Proposition 2.33. *Let $V = \mathbb{R}$ and $M(v) = \frac{1}{\pi}\frac{\sigma}{\sigma^2+v^2}$. For $1 \le b \le a - \frac{1}{4}$, define*

$$\rho(t,x) = M\left(\frac{x}{t+a}\right)e^{r(t+a)} \quad \text{and} \quad g^0(x,v) = \frac{1}{b}M\left(\frac{x}{b}\right)M(v)e^{ra}\,. \tag{2.45}$$

Let g be defined by

$$g(t,x,v) = g^0(x-vt,v)e^{-t} + \int_0^t (1+r)M(v)\rho(s,x-v(t-s))e^{-(t-s)}ds\,.$$

Then $\overline{g}(t,x,v) = \min\{M(v)\,,g(t,x,v)\}$ is a supersolution of (2.1)*, that is :*

$$\partial_t\overline{g} + v\partial_x\overline{g} \ge \left(M(v)\rho_{\overline{g}} - \overline{g}\right) + r\rho_{\overline{g}}\left(M(v) - \overline{g}\right), \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V\,.$$

Proof of Proposition 2.33. The proof is the same as for Proposition 2.32. We just show the main computations in the case of the Cauchy distribution. To prove (2.43) we use the residue method,

$$\begin{aligned}\int_V M(v)M\left(\frac{x-v(t-s)}{s+a}\right)dv &= \int_V \frac{\sigma^2}{\pi^2}\frac{1}{\sigma^2+v^2}\cdot\frac{1}{\sigma^2+\left(\frac{x-v(t-s)}{s+a}\right)^2}dv\\ &= \frac{\sigma}{\pi}\frac{(s+a)(t+a)}{x^2+\sigma^2(t+a)^2}\\ &\le M\left(\frac{x}{t+a}\right)\end{aligned}$$

The analog computation for proving (2.44) goes as follows. First we have

$$\left(\frac{A}{\rho}\right)(t,x) = \pi\left(1+\left(\frac{x}{t+a}\right)^2\right)\exp\left(-(1+r)t - ra\right)\int_V f^0(x-vt,v)dv$$

Thanks to the expression of the initial condition, we compute the latest integral :

$$\int_V f^0(x-vt,v)dv = \frac{1}{\pi}\frac{t+b}{x^2+(t+b)^2}e^{ra},$$

Thus,

$$\left(\frac{A}{\rho}\right)(t,x) = \frac{t+b}{(t+a)^2}\frac{x^2+(t+a)^2}{x^2+(t+b)^2}\exp\left(-(1+r)t\right) \le \exp\left(-(1+r)t\right)$$

which holds true if $b \ge 1$ and $a \ge b + \frac{1}{4}$. □

2.7.3 Lower bound for the spreading rate in the gaussian case

We construct below subsolutions for (2.1) when $V = \mathbb{R}$ and the distribution M is a Gaussian distribution. Contrary to the previous Section 2.7.2, the strategy does not rely on a specific computational trick (*i.e.* Gaussian distributions are stable by convolution). We rather build a typical subsolution based on the dispersion property of the kinetic transport-scattering operator. This construction heavily relies on preliminary results obtained in [31]. We shall motivate the construction of the subsolution based on these ideas.

The first fact is that the subsolution we build here is essentially not regular. It is discontinuous along the line $x = vt$, so that it is not affected by the free transport operator. It is also discontinuous along the line $v = -K$ for some (large) K. However, this does not cause any further trouble due to the absence of derivatives with respect to velocity. Moreover the necessary truncation at a certain level $\gamma \in (0,1)$ yields C^1 discontinuities. This is not a problem since the PDE is of order one, as opposed to classical reaction-diffusion equations for which such a rough troncature is not possible when seeking subsolutions due to the presence of second-order derivatives. Of course we pay much attention to the nonlocal contributions (integral with respect to the velocity) where this truncature causes additional difficulties.

The second point to highlight is that long-range dispersion happens via the free transport operator and the redistribution with respect to the velocity (scattering). It is obviously a matter of (small) densities having large velocities. This is exemplified when noticing that the function defined by

$$g_2(t,x,v) = \exp\left(-\frac{x}{v}\right) M(v)\,, \quad \text{if } v > \frac{x}{t}\,,$$

and zero elsewhere, is a solution of

$$\partial_t g_2 + v\partial_x g_2 + g_2 = 0\,,$$

and thus a subsolution of (2.1) under the condition that $g \leq M$ everywhere. This branch of the solution (restricted to $v > x/t$) will contribute to dropping the mass after redistribution through the nonlocal "source" term $(1+r)M(v)\rho_g$ in the area $-K < v < x/t$. There is some technical issue due to the fact that g_2 is unbounded for $x < 0$. We will circumvent by truncating the density.

Another technical issue stems from the fact that we require negative velocities (up to $v > -K$ for K large enough) in order to maintain enough local redistribution through the nonlocal source term $(1+r)M(v)\rho_g$. This yields an artificial linear transport to the left side (with velocity $-K$). Nonetheless, as superlinear spreading is expected, this backward linear transport term will not affect the conclusion. Our strategy consists in working in the moving frame $y = x + Kt$.

We are now in position to define a proper subsolution. Let K, L be two positive (large) bounds on the velocity. Let $\gamma \in (0,1)$ be a truncation level. We stress that the subsolution will automatically satisfy

$$\underline{g} \leq \gamma M(v)\,.$$

Therefore we are led to find such a function $\underline{g}$ verifying the following inequality,

$$\begin{cases} \partial_t \underline{g} + v\partial_x \underline{g} + \underline{g} \leq (1 + (1-\gamma)r)M(v)\rho_{\underline{g}}\,, \\ \underline{g}(0,x,v) = \gamma M(v)\mathbf{1}_{x<A} \end{cases} \tag{2.46}$$

As mentioned above, we shall set $\underline{g} = 0$ for $v < -K$. To define the subsolution for $v > -K$, we set the problem in the moving frame $y = x + Kt$. Equation (2.46) writes

$$\begin{cases} \partial_t \underline{g} + (v+K)\partial_y \underline{g} + \underline{g} \leq (1 + (1-\gamma)r)M(v)\rho_{\underline{g}}\,, \\ \underline{g}(0,y,v) = \gamma M(v)\mathbf{1}_{y<A} \end{cases} \tag{2.47}$$

We define $\underline{g}$ piecewise :

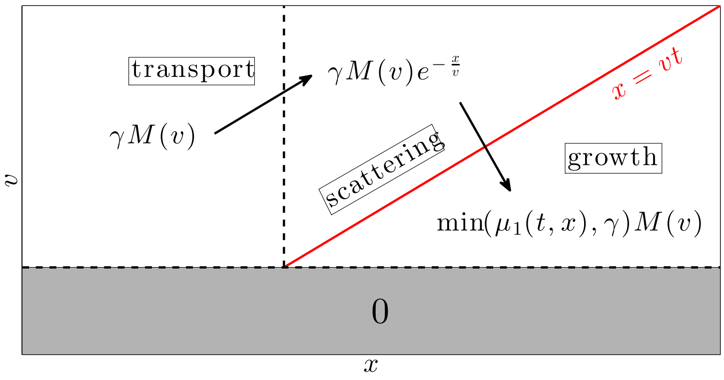


FIGURE 2.4 – Schematic view of the subsolution $\underline{g}$. It is defined piecewise. We have set $K = 0$ for the sake of clarity. The mechanism which drives the subsolution can be described as follows. The free transport operator sends very few particles with very high velocity at the edge of the front. They are redistributed, and their density grows exponentially fast. The mass in the lower branch $\{v < x/t\}$, $\mu_1(t,x)$, is computed analytically.

– For $y \geq 0$,

$$\underline{g}(t,y,v) = \begin{cases} \gamma \exp\left(-\dfrac{y}{v+K}\right) M(v)\,, & \text{if} \quad v+K > \dfrac{y}{t} \\ \min\left(\mu_1(t,y),\gamma\right) M(v)\,, & \text{if} \quad 0 < v+K < \dfrac{y}{t} \\ 0\,, & \text{if} \quad v+K < 0 \end{cases} \tag{2.48}$$

– For $y \leq 0$,

$$\underline{g}(t,y,v) = \begin{cases} \gamma M(v)\,, & \text{if} \quad v+K > 0 \\ 0\,, & \text{if} \quad v+K < 0 \end{cases} \tag{2.49}$$

For a schematic view of the subsolution and the growth-dispersion process, see Figure 2.4. The partial mass contained in the mid-zone $(-K, y/t - K)$ is denoted by $\mu_1(t,y)$. It is defined as the solution to the following ODE,

$$\partial_t \mu_1 + \mu_1 = (1 + (1-\gamma)r)\left(\min\left(\mu_1,\gamma\right) \int_{-K}^{L} M(v)\,dv + \mu_2\right)\,, \tag{2.50}$$

with the initial datum

$$\mu_1(0,y) = \gamma \mathbf{1}_{y<A}\,. \tag{2.51}$$

Finally, the source term μ_2 is defined as the partial mass contained in the branch $v > y/t - K$:

$$\mu_2(t,y) = \gamma \int_{y/t-K}^{\infty} \exp\left(-\frac{y}{v+K}\right) M(v)\,dv\,. \tag{2.52}$$

We stress out that there is a minor discrepancy between the requirement for being a subsolution (2.46) and the definition of (2.50). Namely, the integral contribution runs over $v \in (-K, L)$, although it should naively be $v \in (-K, y/t - K)$. However it is mandatory for the sequel that μ_1 is nonincreasing with respect to y, which is not obvious if L is replaced with $y/t - K$ in (2.50). Note that μ_2 is indeed nondecreasing with respect to y, so that μ_1 defined by (2.50)-(2.51) is clearly nondecreasing with respect to y as well. A simple way to eliminate this discrepancy is to guarantee that $\mu_1(t,y) \geq \gamma$ when $L > y/t - K$, which is the wrong sign of the discrepancy. This is somehow expected since we know *a posteriori* that the front is located in the region $y = \mathcal{O}(t^{3/2})$, such that the unsaturated area is such that $L < y/t - K$ for large time. For small time, it will be guaranteed by tuning the range of the initial datum, namely the parameter A.

The remainder of this Subsection is organized as follows : we first establish some technical estimates of μ_2. Then we deduce that $L > y/t - K$ implies $\mu_1 \geq \gamma$. As a consequence, we establish that $\underline{g}$ is a subsolution of (2.1). Finally, the technical estimates are used again to prove that $\underline{g}$ (in fact, μ_1) exhibits superlinear propagation with the expected scaling $y = \mathcal{O}(t^{3/2})$.

We introduce the modified growth rate

$$\tilde{r} = (1 + (1-\gamma)r) \int_{-K}^{L} M(v)\, dv - 1\,.$$

Note that $\tilde{r} < r$, and it can be chosen arbitrary close to r, by varying γ, K, L.

Lemma 2.34. *The function μ_1 is given by the Duhamel formula in the area $\{\mu_1 < \gamma\}$:*

$$\mu_1(t,y) = e^{\tilde{r}t}\mu_1(0,y) + (1 + (1-\gamma)r) \int_0^t e^{\tilde{r}(t-s)} \mu_2(s,y)\, ds\,. \tag{2.53}$$

Proof of Lemma 2.34. We notice that μ_1 is nondecreasing with respect to t, since the source term μ_2 is nonnegative. Therefore the formula (2.53) is valid up to $\mu_1 = \gamma$. □

Lemma 2.35. *The following estimate holds true,*

$$\mu_2(t,y) \geq \begin{cases} \dfrac{1}{r_1(t,y)} \exp\left(-\dfrac{1}{2\sigma^2}\left(\dfrac{y}{t} - K\right)^2 - t \right), & \text{if } \dfrac{y}{t} > v^*(y) \\ \dfrac{1}{r_2(y)} \exp\left(-\dfrac{y}{v^*(y)} - \dfrac{(v^*(y)-K)^2}{2\sigma^2} \right), & \text{if } \dfrac{y}{t} < v^*(y). \end{cases} \tag{2.54}$$

where $v^*(y)$ is a velocity satisfying $v^*(y) \sim_{y\to+\infty} \sigma^{2/3}y^{1/3}$ and r_1 and r_2 are lower order corrections (as compared to the exponential decay), and are described below in the proof.

Proof of Lemma 2.35. Before we start with the technical estimates, let us explain briefly why (2.54) is very much expected. We consider $K = 0$ for the sake of simplicity. The function $-\log\left(e^{-\frac{x}{v}}M(v)\right) = \frac{x}{v} + \frac{v^2}{2\sigma^2} + \frac{1}{2}\log(2\pi\sigma^2)$ admits a global minimum with respect to $v > 0$, attained at $v^* = \sigma^{2/3}x^{1/3}$. It should be discussed whether this minimum lies in the area $v > \frac{x}{t}$ or not. In any case, the integral μ_2 is close to the value of the corresponding exponential maximum, up to lower order corrections included in $r_{1,2}$.

We consider now a general K. To estimate μ_2, let us rewrite

$$\mu_2(t,y) = \frac{\gamma}{\sigma\sqrt{2\pi}} \int_{\frac{y}{t}}^{\infty} e^{-\frac{y}{v}} e^{-\frac{(v-K)^2}{2\sigma^2}}\, dv\,.$$

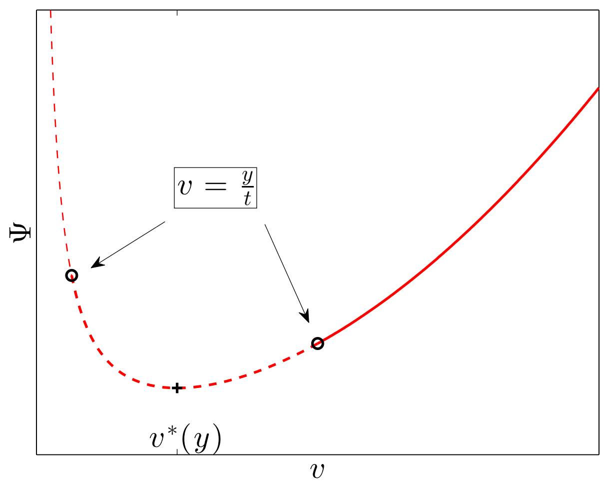


FIGURE 2.5 – Scheme of the proof of the estimate (2.54) : the relative positions of $\frac{y}{t}$ and $v^*(y)$ have to be discussed. In any case, we restrict to the nondecreasing branch $v > v^*(y)$ to estimate $\mu_2(y)$, and we apply a suitable change of variables in this branch.

We observe that $\Psi := v \mapsto \frac{y}{v} + \frac{(v-K)^2}{2\sigma^2}$ has a minimum on $\mathbb{R}^+$, attained at the velocity $v^*(y)$ given by the nonnegative root of

$$v^2(v-K) = \sigma^2 y, \tag{2.55}$$

see Figure 2.5. As a byproduct of this first order condition, we get directly that $v^*(y) \geq K$. We get also that

$$v^*(y) \sim \sigma^{2/3} y^{1/3}, \quad \text{and accordingly,} \quad \Psi(v^*(y)) \sim \frac{3}{2}\left(\frac{y}{\sigma}\right)^{2/3}, \quad \text{as } y \to +\infty. \tag{2.56}$$

This equivalents will be of crucial importance in the following Lemmas to estimate properly the propagation. We can now uniquely define the change of variables $\varphi : (\Psi(v^*(y)), +\infty) \to (v^*(y), +\infty)$ defined by

$$\frac{(\varphi(u)-K)^2}{2\sigma^2} + \frac{y}{\varphi(u)} = u. \tag{2.57}$$

Moreover, φ is increasing as the inverse function of an increasing function. Multipying by $\varphi(u)$ and differentiating (2.57) yields

$$\forall u \in (\Psi(v^*(y)), +\infty), \quad \varphi'(u) = \frac{\varphi(u)}{\frac{1}{2\sigma^2}(3\varphi^2(u) - 4K\varphi(u) + K^2) - u}.$$

We distinguish naturally between two cases, depending on the relative positions of $\frac{y}{t}$ and $v^*(y)$.

 # Step 1 : The case $\frac{y}{t} \geq v^*(y)$. We directly apply the change of variables :

$$\frac{\gamma}{\sigma\sqrt{2\pi}} \int_{\frac{y}{t}}^{\infty} e^{-\frac{y}{v}} e^{-\frac{(v-K)^2}{2\sigma^2}} dv = \frac{\gamma}{\sigma\sqrt{2\pi}} \int_{\Psi\left(\frac{y}{t}\right)}^{\infty} e^{-u} \varphi'(u)\, du,$$

We need some estimate from below of $\varphi'(u)$. We first deduce from $\varphi'(u) \geq 0$ that

$$3\varphi^2(u) - 4K\varphi(u) + K^2 \geq 2\sigma^2 u\,.$$

It yields that necessarily

$$\varphi(u) \leq \frac{2}{3}K - \sqrt{\frac{K^2}{9} + \frac{2\sigma^2}{3}u}\,, \quad \text{or} \quad \varphi(u) \geq \frac{2}{3}K + \sqrt{\frac{K^2}{9} + \frac{2\sigma^2}{3}u}\,.$$

The first alternative is impossible since $\varphi(u) \geq v^*(y) \geq K$. On the other hand, one deduce from the very definition (2.57) and $\varphi(u) \geq K$ that

$$\begin{aligned}
\frac{1}{2\sigma^2}\left(3\varphi^2(u) - 4K\varphi(u) + K^2\right) - u &= \frac{3}{2\sigma^2}\left(\left(\varphi(u) - \frac{2K}{3}\right)^2 - \frac{K^2}{9}\right) - u\,,\\
&\leq \frac{3}{2\sigma^2}\left(\varphi(u) - K\right)^2 - u\,,\\
&= 3u - 3\tfrac{y}{\varphi(u)} - u\,,\\
&\leq 2u.
\end{aligned}$$

We deduce that

$$\varphi'(u) \geq \frac{\frac{2}{3}K + \sqrt{\frac{K^2}{9} + \frac{2\sigma^2}{3}u}}{2u} \geq \left(\frac{2\sigma^2}{3}\right)^{\frac{1}{2}} \frac{1}{2\sqrt{u}}\,.$$

We obtain as a consequence,

$$\mu_2(t,y) \geq \frac{\gamma}{\sigma\sqrt{2\pi}} \int_{\Psi\left(\frac{y}{t}\right)}^{\infty} e^{-u} \left(\frac{2\sigma^2}{3}\right)^{\frac{1}{2}} \frac{du}{2\sqrt{u}} = \frac{\gamma}{\sqrt{3\pi}} \int_{\left(\Psi\left(\frac{y}{t}\right)\right)^{\frac{1}{2}}}^{\infty} e^{-v^2} dv.$$

Next, we apply a quantitative estimate for the remainder of the gaussian integral [1, p. 298] :

$$\forall x \geq 0, \quad \int_x^{\infty} e^{-x^2} dx > \frac{e^{-x^2}}{x + \sqrt{x^2+2}}\,. \tag{2.58}$$

Consequently we obtain

$$\mu_2(t,y) \geq \frac{\gamma}{\sqrt{3\pi}} \frac{e^{-\Psi\left(\frac{y}{t}\right)}}{\left(\Psi\left(\frac{y}{t}\right)\right)^{\frac{1}{2}} + \left(\Psi\left(\frac{y}{t}\right) + 2\right)^{\frac{1}{2}}}.$$

 # Step 2 : The case $\frac{y}{t} \leq v^*(y)$. There, we simply neglect the decreasing part of Ψ (see Figure 2.5). The result is a direct consequence of the previous calculation. Indeed, we have :

$$\mu_2(t,y) = \frac{\gamma}{\sigma\sqrt{2\pi}} \left(\int_{\frac{y}{t}}^{v^*(y)} e^{-\frac{y}{v}} e^{-\frac{(v-K)^2}{2\sigma^2}}\, dv + \int_{v^*(y)}^{\infty} e^{-\frac{y}{v}} e^{-\frac{(v-K)^2}{2\sigma^2}}\, dv\right). \tag{2.59}$$

After neglecting the first contribution, and following the same lines as in Step 1 for the second integral, we get eventually :

$$\mu_2(t,y) \geq \frac{\gamma}{\sqrt{3\pi}} \frac{e^{-\Psi(v^*(y))}}{\left(\Psi\left(v^*(y)\right)\right)^{\frac{1}{2}} + \left(\Psi\left(v^*(y)\right) + 2\right)^{\frac{1}{2}}}.$$

□

Lemma 2.36. *There exists A_0 such that for all $A \geq A_0$, the following estimate holds true,*

$$\left(\forall (t,y) \in \mathbb{R}^+ \times \mathbb{R}^+\right) \quad \frac{y}{t} - K < L \implies \mu_1(t,y) \geq \gamma\,.$$

Proof of Lemma 2.36. Because μ_1 is nonincreasing with respect to y, it is sufficient to prove that

$$\forall t \in \mathbb{R}^+, \quad \mu_1(t,(K+L)t) \geq \gamma.$$

We recall the definition of μ_1 in the area $\{\mu_1 < \gamma\}$,

$$\mu_1(t,(K+L)t) = \gamma e^{\tilde{r}t}\mathbf{1}_{(K+L)t<A} + (1+(1-\gamma)r)\int_0^t e^{\tilde{r}(t-s)}\mu_2(s,(K+L)t)\,ds\,. \tag{2.60}$$

We observe that for $t < \frac{A}{K+L}$, the condition $\mu_1(t,(K+L)t) \geq \gamma$ is fulfilled due to the initial datum. Then we estimate the integral term in the r.h.s. of (2.60). The following estimate is crucial since it moreless contains the superlinear propagation behavior. Let $\alpha \in (0,1)$ to be chosen later. We have

$$\int_0^t e^{\tilde{r}(t-s)}\mu_2(s,(K+L)t)\,ds \geq \mu_2(\alpha t,(K+L)t)\int_{\alpha t}^t e^{\tilde{r}(t-s)}ds.$$

As a consequence, when t is large enough such that

$$\frac{K+L}{\alpha} = \frac{(K+L)t}{\alpha t} \leq v^*\left((K+L)t\right)\,,$$

we have

$$\mu_1(t,(K+L)t) \geq \mu_2(\alpha t,(K+L)t)\int_{\alpha t}^t e^{\tilde{r}(t-s)}ds \geq \frac{1}{\tilde{r}}\left(e^{\tilde{r}(1-\alpha)t}-1\right)\frac{\gamma}{\sqrt{3\pi}}\frac{e^{-\Psi(v^*((K+L)t))}}{r_2((K+L)t)} \tag{2.61}$$

Thus, it is enough to guarantee that the following estimate holds true,

$$\frac{1}{\tilde{r}}\left(e^{\tilde{r}(1-\alpha)t}-1\right)\frac{1}{\sqrt{3\pi}}\frac{e^{-\Psi(v^*((K+L)t))}}{r_2((K+L)t)} \geq 1,$$

for t large enough, say $t > T_0$. It is indeed the case since (2.56) implies that

$$\log\left(\mu_1(t,(K+L)t)\right) \geq \tilde{r}(1-\alpha)t + O(t^{3/2}) + O(\log t)\,.$$

The conclusion is straightforward, provided we choose A large enough such that $T_0 = \frac{A}{K+L}$ satisfies

$$v^*(A) > \frac{K+L}{\alpha}\,.$$

Finally we notice that α is still a free parameter in the range $(0,1)$. It will be fixed in the next Lemma when optimizing the propagation behavior. □

Theorem 2.37. *Let the constants K, L, γ chosen as above. Let $\alpha < \frac{\tilde{r}}{\tilde{r}+2}$, and choose A accordingly. The function $\underline{g}$ defined by* (2.48)-(2.49) *is a subsolution of* (2.1). *Moreover it exhibits a superlinear spreading with rate $\mathcal{O}(t^{3/2})$. More precisely, the point $y(t)$ such that $\mu_1(t,y(t)) = \frac{\gamma}{2}$ is such that $y(t) \geq \sigma(\alpha t)^{3/2}$ for t sufficiently large.*

Proof of Theorem 2.37. We first observe that for all $y > 0$ we have $v^*(y) \geq \sigma^{2/3} y^{1/3}$ (2.55). On the other hand, there exists Y_0 such that for $y \geq Y_0$ we have

$$\Psi\left(v^*(y)\right) \leq 2\sigma^{-2/3} y^{2/3}\,.$$

We define the zone

$$\mathcal{Y}_t = \left\{ y \,:\, Y_0 \leq y \leq \sigma\left(\alpha t\right)^{3/2} \right\}\,.$$

For $y \in \mathcal{Y}_t$ we have immediately

$$\frac{y}{\alpha t} \leq \sigma^{2/3} y^{1/3} \leq v^*\left(y\right)\,,$$

where we have used that $y \geq Y_0$ to justify the last inequality. We recall the estimation of μ_1 (2.61) which holds true for $y \in \mathcal{Y}_t$:

$$\begin{aligned} \mu_1(t,y) &\geq \mu_2(\alpha t, y) \int_{\alpha t}^{t} e^{\tilde{r}(t-s)} ds \\ &\geq \frac{1}{\tilde{r}}\left(e^{\tilde{r}(1-\alpha)t} - 1\right) \frac{\gamma}{\sqrt{3\pi}} \frac{e^{-\Psi(v^*(y))}}{r_2(y)} \\ &\geq \frac{1}{\tilde{r}}\left(e^{\tilde{r}(1-\alpha)t} - 1\right) \frac{\gamma}{\sqrt{3\pi}} \frac{e^{-2\sigma^{-2/3} y^{2/3}}}{r_2(y)}\,. \end{aligned}$$

This yields

$$(\forall y \in \mathcal{Y}_t) \quad \log(\mu_1(t,y)) \geq \left(\tilde{r}(1-\alpha) - 2\alpha\right) t + O(\log t)\,.$$

As a consequence, choosing $\alpha \in (0,1)$ such that

$$\alpha < \frac{\tilde{r}}{\tilde{r}+2}\,,$$

ensures that for sufficiently large times, $\mu_1(t,y) \geq \gamma$, and thus the front has already passed through $\mathcal{Y}_t$. □

Acknowledgements. The authors wish to thank Jimmy Garnier and Emmanuel Grenier for enlightening discussions concerning the correct spreading rate in the gaussian case.

Appendix

We give in this Appendix the proof of Propositions 2.13 and 2.14. Well-posedness relies on a fixed point argument which is also used for the comparison principle. We first state two Lemmas.

Lemma 2.38. *Let $a, b \in \mathcal{C}_b^0\left(\mathbb{R}_+ \times \mathbb{R} \times V\right)$ and $g^0 \in \mathcal{C}_b^0\left(\mathbb{R}, L^1(V)\right)$. Then there exists a unique function $g \in \mathcal{C}_b^0\left(\mathbb{R}_+ \times \mathbb{R}, L^1(V)\right)$ such that*

$$\begin{cases} \partial_t g + v\partial_x g + a(t,x,v) g = b(t,x,v)\rho_g & \text{in} \quad \mathbb{R}_+ \times \mathbb{R} \times V, \\ g(0,x,v) = g^0(x,v) & \text{in} \quad \mathbb{R} \times V, \end{cases} \tag{2.62}$$

in the sense of distributions. This solution also satisfy the Duhamel formula :

$$g(t,x,v) = g^0(x - vt, v)e^{-\int_0^t a(s,x-(t-s)v,v)ds} + \int_0^t e^{-\int_s^t a(\tau,x-(t-\tau)v,v)d\tau} b\left(s, x - v(t-s), v\right) \rho_g\left(s, x - v(t-s)\right) ds\,. \quad (2.63)$$

Moreover, if $b \geq 0$ and $g^0 \geq 0$, then $g \geq 0$ in $\mathbb{R}_+ \times \mathbb{R} \times V$.

Proof of Lemma 2.38. For $T > 0$ we define the operator

$$\begin{array}{rccc} A_T : & \mathcal{C}_b^0\left((0,T) \times \mathbb{R}, L^1(V)\right) & \to & \mathcal{C}_b^0\left((0,T) \times \mathbb{R}, L^1(V)\right) \\ & g & \mapsto & \widetilde{g} \end{array} \quad (2.64)$$

where

$$\widetilde{g}(t,x,v) = g^0(x - vt, v)e^{-\int_0^t a(s,x-(t-s)v,v)ds} + \int_0^t e^{-\int_s^t a(\tau,x-(t-\tau)v,v)d\tau} b\left(s, x - v(t-s), v\right) \rho_g\left(s, x - v(t-s)\right) ds\,. \quad (2.65)$$

Take $g_1, g_2 \in \mathcal{C}_b^0\left((0,T) \times \mathbb{R}, L^1(V)\right)$ and define $\widetilde{g}_1 = A_T g_1$ and $\widetilde{g}_2 = A_T g_2$. Assume that $a \not\equiv 0$ over $(0,T) \times \mathbb{R} \times V$. For all $(t,x) \in (0,T) \times \mathbb{R}$, one has :

$$\begin{aligned} &\int_V \left|\widetilde{g}_1(t,x,v) - \widetilde{g}_2(t,x,v)\right| dv \\ &\leq \int_V \int_0^t e^{-\int_s^t a(\tau,x-(t-\tau)v,v)d\tau} b\left(s, x - v(t-s), v\right) \left|\rho_{g_1} - \rho_{g_2}\right| \left(s, x - v(t-s)\right) dvds \\ &\leq \int_0^t e^{(t-s)\|a\|_{L^\infty}} \|b\|_{L^\infty} ds \times \sup_{(t,x)\in(0,T)\times\mathbb{R}} \int_V \left|g_1(t,x,v) - g_2(t,x,v)\right| dv \\ &\leq \frac{1}{\|a\|_{L^\infty}} \left(e^{T\|a\|_{L^\infty}} - 1\right) \|b\|_{L^\infty} \times \sup_{(t,x)\in(0,T)\times\mathbb{R}} \int_V \left|g_1(t,x,v) - g_2(t,x,v)\right| dv\,. \end{aligned}$$

Hence, there exists $T_0 > 0$ such that for all $T \in (0, T_0)$, A_T is a contraction over the space $\mathcal{C}_b^0\left((0,T) \times \mathbb{R}, L^1(V)\right)$. If $a \equiv 0$ on $(0,T) \times \mathbb{R} \times V$, then such an estimate can be derived similarly. Hence, A_T admits a unique fixed point, which satisfies (2.63) over $(0,T) \times \mathbb{R} \times V$. This gives the local existence and uniqueness of the solution of (2.63). Moreover, as T_0 does not depend on the initial datum g^0, the global existence follows.

If $b \geq 0$ and $g^0 \geq 0$, then A_T preserves the cone of nonnegative functions and thus applying the fixed point theorem in this cone, we get the nonnegativity of g. □

Lemma 2.39. *Assume that b is everywhere positive and that V is an interval. Then if $g^0 \in \mathcal{C}_b^0(\mathbb{R}_+ \times \mathbb{R} \times V)$ is nonnegative and if there exists $(x_0, v_0) \in \mathbb{R} \times V$ such that $g^0(x_0, v_0) > 0$, letting g the unique solution of* (2.62)*, one has $g(t,x,v) > 0$ for all $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V$ such that $|x - x_0| < v_{\max} t$.*

Proof of Lemma 2.39. First, assume by contradiction that there exists $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ such that $\rho_g(t,x) = 0$, with $|x - x_0| < v_{\max} t$. Then integrating (2.63) over V, one gets

$$0 = \rho_g(t,x) = \int_{v\in V} g^0(x - vt, v)e^{-\int_0^t a(s,x-(t-s)v,v)ds} dv + \int_{v\in V} \int_0^t e^{-\int_s^t a(\tau,x-(t-\tau)v,v)d\tau} b\left(s, x - v(t-s), v\right) \rho_g\left(s, x - v(t-s)\right) dsdv.$$

Hence, $\rho_g(s, x - v(t-s)) = 0$ for all $v \in V$ and $s \in (0,t)$. Letting $s \to 0$, one gets $\rho_g(0, x - vt) = 0$ for all $v \in V$. As $|x - x_0| < v_{\max} t$ and V is an interval, one can take $v \in V$ such that $x - vt = x_0$, leading to $\rho_g(0, x_0) = 0$. This is a contradiction since, as g is continuous, nonegative and $g(0, x_0, v_0) > 0$, one has $\rho_g(0, x_0) > 0$. Hence $\rho_g(t, x, v) > 0$ for all $(t, x, v) \in (0, T) \times R \times V$ such that $|x - x_0| < v_{\max} t$.

Next, as

$$g(t,x,v) = g^0(x - vt, v) e^{-\int_0^t a(s, x-(t-s)v, v) ds} + \int_0^t e^{-\int_s^t a(\tau, x-(t-\tau)v, v) d\tau} b(s, x - v(t-s), v) \rho_g(s, x - v(t-s)) \, ds \,,$$

it follows from the first step that $g(t,x,v) > 0$ as soon as there exists $s \in (0,t)$ such that $|x - x_0 - v(t-s)| < v_{\max} s$, which also reads : $|x - x_0| < v_{\max} t$. □

Proof of Proposition 2.14. Define $w = g_1 - g_2$. As in the proof of Lemma 6 in [69], we first remark that this function satisfies

$$\partial_t w + v \partial_x w + \left(1 + r\rho_{g_1}\right) w \geq \left(M(v) + r\left(M(v) - g_2\right)\right) \rho_w \text{ in } \mathbb{R}_+ \times \mathbb{R} \times V, \tag{2.66}$$

with $w(0, x, v) \geq 0$ for all $(x, v) \in \mathbb{R} \times V$. We define $a = 1 + r\rho_{g_1}$ and $b = M(v) + r\left(M(v) - g_2\right)$. Writing the integral formulation as in the proof of Lemma 2.38 gives

$$w(t,x,v) \geq \int_0^t e^{-\int_s^t a(\tau, x-(t-\tau)v, v) d\tau} b(s, x - v(t-s), v) \rho_w(s, x - v(t-s)) \, ds \,,$$

and thus $w \geq A_T w$ in $(0, T) \times \mathbb{R} \times V$ for some operator A_T which is monotone and contractive when T is small enough. It follows that $w \geq A_T^n w$ for all $n \geq 1$. Since A_T is contractive the sequence $\left(A_T^n w\right)_n$ converges to 0. We conclude that $w \geq 0$, meaning that $g_1 \geq g_2$.

Next, assume that $\inf_V M > 0$, V is an interval, and that there exists (x_0, v_0) such that $g_2(0, x_0, v_0) > g_1(0, x_0, v_0)$. We can follow the proof of Lemma 2.39, where b defined above is positive everywhere. We deduce that $w(t,x,v) > 0$ as soon as $|x - x_0| < v_{\max} t$. □

Chapitre 3

Une équation eikonale cinétique

Dans cette note en collaboration avec Vincent Calvez, nous analysons une équation cinétique linéaire de transport avec un opérateur de relaxation BGK. Nous étudions la limite hyperbolique de grande échelle $(t, x) \to (t/\varepsilon, x/\varepsilon)$. Nous obtenons à la limite une nouvelle équation de Hamilton-Jacobi, qui est l'analogue de l'équation eikonale classique obtenue à partir de l'équation de la chaleur avec petite diffusion. Il est alors intéressant de constater que la limite hydrodynamique ne commute pas avec l'asymptotique des grandes déviations. Nous démontrons le caractère bien posé de l'équation vérifiée par la phase, ainsi que la convergence vers une solution de viscosité de l'équation de Hamilton-Jacobi. Ceci est un travail préliminaire en vue d'analyser la propagation de fronts de réaction pour des équations cinétiques.

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Version française abrégée

Nous considérons un modèle cinétique linéaire avec un opérateur de relaxation BGK, posé sur un ensemble de vitesses V symétrique et borné. On analyse le comportement de l'équation dans la limite hyperbolique de grande échelle $(t,x) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$,

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon}\left(M(v)\rho^\varepsilon - f^\varepsilon\right), \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V. \tag{3.1}$$

Nous démontrons que la phase φ^ε définie par la relation $f^\varepsilon(t,x,v) = M(v)e^{-\frac{\varphi^\varepsilon(t,x,v)}{\varepsilon}}$ converge (localement) uniformément, lorsque $\varepsilon \to 0$, vers une fonction $\varphi^0(t,x)$ indépendante de v. De surcroît, la fonction φ^0 est solution de viscosité de l'équation de Hamilton-Jacobi suivante,

$$\int_V \frac{M(v)}{1 - \partial_t \varphi^0(t,x) - v \cdot \nabla_x \varphi^0(t,x)} dv = 1, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{3.2}$$

Cette équation peut se réécrire sous la forme canonique $\partial_t \varphi^0 + H(\nabla_x \varphi^0) = 0$ pour un hamiltonien effectif $H(p)$ qui est lipschitzien et convexe. Comme dans [15], cet Hamiltonien est relié à la résolution d'un problème aux valeurs propres dans l'espace des vitesses V, ce dernier s'écrivant comme suit : trouver un vecteur propre $Q(v)$ et une valeur propre $H(p)$ tels que

$$(1 + H(p) - v \cdot p)\, Q(v) = \int_V M(v')Q(v')dv'.$$

La démonstration du passage à la limite de (3.1) vers (3.2) s'appuie sur une série d'estimations *a priori* qui démontre que φ^ε appartient à l'espace de Sobolev $W^{1,\infty}$, avec un contrôle uniforme en $\varepsilon > 0$ (Proposition 3.2 ci-dessous). Dans un deuxième temps, nous démontrons que toute fonction test $\psi^0(t,x)$ de classe $\mathcal{C}^2$ telle que $\varphi^0 - \psi^0$ admet un maximum local en (t^0, x^0), vérifie

$$\int_V \frac{M(v)}{1 - \partial_t \psi^0(t^0,x^0) - v \cdot \nabla_x \psi^0(t^0,x^0)} dv \leq 1.$$

Ceci démontre que φ^0 est une sous-solution de viscosité l'équation de Hamilton-Jacobi (3.2). Un raisonnement identique montre qu'il s'agit aussi d'une sur-solution de viscosité. La démonstration se base sur la construction d'un correcteur microscopique $\eta(t,x,v)$ défini de façon ad-hoc par la relation

$$\forall (v,v') \in V \times V, \quad e^{\eta(t,x,v)} - e^{\eta(t,x,v')} = \left(v' - v\right) \cdot \nabla_x \psi^0(t,x).$$

3.1 Large-scale limit and derivation of the Hamilton-Jacobi equation

We consider the following kinetic equation with BGK relaxation operator :

$$\partial_t f + v \cdot \nabla_x f = M(v)\rho - f, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V, \tag{3.3}$$

where $f(t,x,v)$ denotes the density of particles moving with speed $v \in V$ at time t and position x. The function $\rho(t,x)$ denotes the macroscopic density of particules :

$$\rho(t,x) = \int_V f(t,x,v)\, dv\,, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n\,.$$

Here V denotes a bounded symmetric subset of $\mathbb{R}^n$. We assume that the Maxwellian M is symmetric and satisfies the following moment identities :

$$\int_V M(v)dv = 1\,, \qquad \int_V vM(v)dv = 0\,, \qquad \int_V v^2 M(v)dv = \theta^2\,.$$

In this paper we focus on the large scale hyperbolic limit $(t,x) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$, $\varepsilon \to 0$. The kinetic equation (3.3) reads as follows in the new scaling :

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon}\left(M(v)\rho^\varepsilon - f^\varepsilon\right), \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V\,. \tag{3.4}$$

Clearly, the velocity distribution relaxes rapidly towards the Maxwellian distribution. This motivates the introduction of the following Hopf-Cole transformation :

$$f^\varepsilon(t,x,v) = M(v) e^{-\frac{\varphi^\varepsilon(t,x,v)}{\varepsilon}}\,.$$

where we expect the phase φ^ε to become independent of v as $\varepsilon \to 0$. To avoid technical complications due to ill-prepared data, we set $\varphi^\varepsilon(0,x,v) = \varphi_0(x) \geq 0$ as an initial data for (3.4). The equation satisfied by φ^ε reads :

$$\partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon = \int_V M(v')\left(1 - e^{\frac{\varphi^\varepsilon - \varphi^{\varepsilon\prime}}{\varepsilon}}\right) dv'\,, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V\,, \tag{3.5}$$

Theorem 3.1. *Let $V \subset \mathbb{R}^n$ be bounded and symmetric, and $M \in L^1(V)$ be nonnegative and symmetric. Then φ^ε converges (locally) uniformly towards φ^0, where φ^0 does not depend on v. Moreover φ^0 is the viscosity solution of the following Hamilton-Jacobi equation :*

$$\int_V \frac{M(v)}{1 - \partial_t \varphi^0(t,x) - v \cdot \nabla_x \varphi^0(t,x)} dv = 1\,, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n\,. \tag{3.6}$$

The denominator of the integrand is positive for all $v \in V$.

The last observation in Theorem 3.1 is not compatible with an unbounded velocity set.

As in [15, 98], the Hamilton-Jacobi equation is connected with an eigenvalue problem in the velocity space V : Find an eigenvector $Q(v)$ such that :

$$\left(1 - \partial_t \varphi^0 - v \cdot \nabla_x \varphi^0\right) Q(v) = \int_V M(v') Q(v') dv'\,.$$

This eigenproblem can be solved explicitly, and yields formula (A.4).

Thanks to monotonicity properties, we can boil down to the classical framework of first order Hamilton-Jacobi equations. Indeed, writing equation (A.4) as

$$G(\partial_t \varphi^0, \nabla_x \varphi^0) = 0,$$

we observe that G is increasing with respect to the first variable. Hence the equation is equivalent to $\partial_t \varphi^0 + H(\nabla_x \varphi^0) = 0$, where the effective Hamiltonian H is defined through the implicit formula,

$$\int_V \frac{M(v)}{(1 + H(p) - v \cdot p)}\, dv = 1\,. \tag{3.7}$$

Differentiating (3.7) we obtain,

$$\int_V \frac{M(v)}{(1 + H(p) - v \cdot p)^2} \left(\nabla H(p) - v\right)\, dv = 0\,.$$

We deduce $\|\nabla H\|_\infty \leq V_{\max}$. This is in accordance with the underlying kinetic equation, since ∇H can be interpreted as the group speed, which is bounded by the maximal speed of the particles. Differentiating (3.7) twice we obtain

$$\left(\int_V \frac{M(v)}{(1 + H(p) - v \cdot p)^2}\, dv \right) \mathrm{D}^2 H(p) \\ = 2 \int_V \frac{M(v)}{(1 + H(p) - v \cdot p)^3} \left(\nabla H(p) - v\right) \otimes \left(\nabla H(p) - v\right)\, dv\,.$$

We deduce that the effective Hamiltonian is convex.

As an example, we can compute the effective Hamiltonian H in one dimension for a constant Maxwellian $M \equiv \frac{1}{2}$ on $V = (-1, 1)$. We obtain $H(p) = \frac{p - \tanh(p)}{\tanh(p)}$. It is equivalent to $\theta^2 |p|^2$ for small p ($\theta^2 = \frac{1}{3}$). Another example where the effective Hamitonian is explicit is given by the Maxwellian $M(v) = \frac{1}{2} \left(\delta_1 + \delta_{-1}\right)$, though it is not a L^1 function. This corresponds to a two velocities model (also known as the *telegraph equation*, see [93, 36]). In this case we obtain the relativistic hamiltonian $H(p) = \frac{\sqrt{1 + 4p^2} - 1}{2}$.

Interestingly enough, we obtain a Hamilton-Jacobi equation which differs from the classical eikonal equation. The latter could have been expected from the following argumentation. The formal limit of equation (3.4) at order $O(\varepsilon)$ is the heat equation with small diffusivity :

$$\partial_t \rho^\varepsilon = \varepsilon \theta^2 \Delta_x \rho^\varepsilon\,, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n\,.$$

It is well-known that the phase $\phi^\varepsilon = -\varepsilon \log \rho^\varepsilon$ satisfies in the limit $\varepsilon \to 0$ the classical eikonal equation in the sense of viscosity solutions [87, 101, 90, 88, 102, 98] :

$$\partial_t \phi^0 + \theta^2 |\nabla_x \phi^0|^2 = 0\,. \tag{3.8}$$

Interestingly, the hydrodynamic limit and the large deviation approach do not commute. We only have asymptotic equivalence between the two approaches for small $|p|$ as can be seen directly on (3.7) by Taylor expansion : $H(p) \sim \theta^2 |p|^2$.

In Figure 3.1 we show numerical simulations of the kinetic eikonal equation (A.4), with a constant Maxwellian on $V = (-1, 1)$, and we compare it with the classical eikonal equation (3.8).

We end this introduction by listing some possible extensions of Theorem 3.1 for other choices of transport and scattering operators. We will develop a more general framework in a future work.

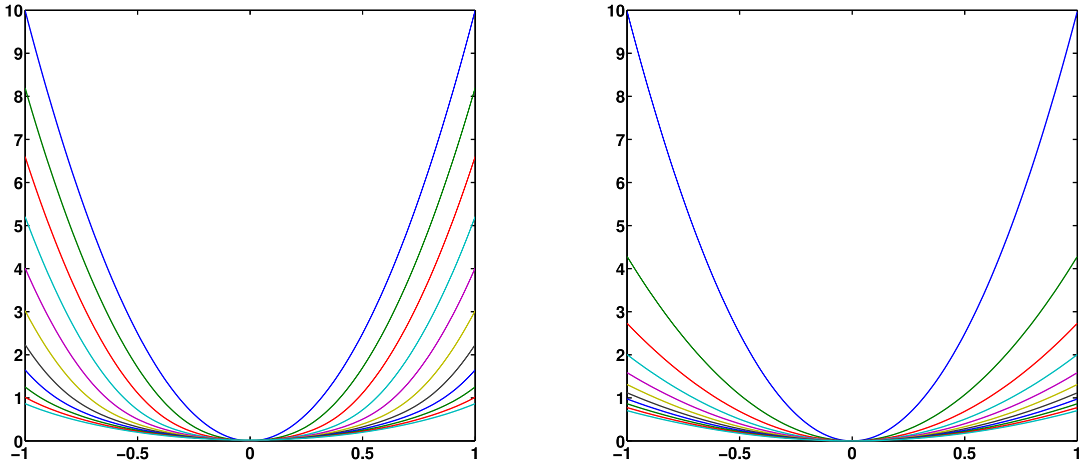

FIGURE 3.1 – Numerical simulations of the Hamilton-Jacobi equation $\partial_t\varphi + H(\partial_x\varphi) = 0$. (*left*) The kinetic eikonal equation (A.4) where $M(v) = \frac{1}{2}\mathbf{1}_{(-1,1)}$. (*right*) The classical eikonal equation $H(p) = \theta^2|p|^2$ (3.8). In both cases the initial condition is a parabola.

1. In the case of the Vlasov-Fokker-Planck equation,

$$\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon - \nabla_x V(x)\cdot\nabla_v f^\varepsilon = \frac{1}{\varepsilon}\nabla_v\cdot\left(\nabla_v f^\varepsilon + v f^\varepsilon\right)\,, \qquad (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n,$$

   we obtain simply the eikonal equation $\partial_t\phi^0 + |\nabla_x\phi^0|^2 = 0$ in the WKB expansion $f^\varepsilon = M(v)e^{-\frac{\varphi^\varepsilon}{\varepsilon}}$, where $M(v)$ is a Gaussian.

2. It is challenging to replace the BGK operator in (3.4) by a convolution operator $L(f) = K * f - f$, where K is a probability kernel [20]. However in this case we are not able to solve explicitly the eigenproblem in the cell V.

3. In a forthcoming work we will investigate the propagation of reaction fronts in kinetic equations, following [88, 93, 69, 36].

3.2 Proof of Theorem 3.1

First let us mention that the solution φ^ε remains nonnegative for all times. We proceed in two steps. First we prove uniform estimates with respect to $\varepsilon > 0$. It allows to extract a uniformly converging subsequence. Second we identify the limit as the viscosity solution of equation (A.4) using the maximum principle. The second step relies on the construction of a suitable corrector $\eta(t,x,v)$ [90, 89].

Step 1. Existence and uniform bounds.

Proposition 3.2. *Let $V \subset \mathbb{R}^n$ be a bounded subset. Assume $M \in L^1(V)$ and $\varphi_0 \in W^{1,\infty}(\mathbb{R}^n)$. The kinetic equation* (A.3) *has a unique solution $\varphi^\varepsilon \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^n \times V)$. Furthermore, the solution satisfies the following uniform estimates :*

$$0 \leq \varphi^\varepsilon(t,\cdot) \leq \|\varphi_0\|_\infty \,, \tag{3.9}$$

$$\|\nabla_x \varphi^\varepsilon(t,\cdot)\|_\infty \leq \|\nabla_x \varphi_0\|_\infty \,, \tag{3.10}$$

$$\|\nabla_v \varphi^\varepsilon(t,\cdot)\|_\infty \leq t\|\nabla_x \varphi_0\|_\infty \,, \tag{3.11}$$

$$\|\partial_t \varphi^\varepsilon(t,\cdot)\|_\infty \leq V_{\max}\|\nabla_x \varphi_0\|_\infty \,. \tag{3.12}$$

Proof. We obtain a unique solution φ^ε from a fixed point method on the Duhamel formulation of (A.3) :

$$\varphi^\varepsilon(t,x,v) = \varphi_0(x-tv) + \int_0^t \int_V M(v') \left(1 - e^{\frac{\varphi^\varepsilon(t-s,x-sv,v) - \varphi^\varepsilon(t-s,x-sv,v')}{\varepsilon}}\right) dv' ds, \tag{3.13}$$

We obtain directly,

$$\forall \varepsilon > 0, \qquad 0 \leq \varphi^\varepsilon(t,x,v) \leq \varphi_0(x-tv) + t\,.$$

This ensures that φ^ε is uniformly bounded on $[0,T] \times \mathbb{R}^n \times V$. To prove the bound (3.9), we define $\psi^\varepsilon_\delta(t,x,v) = \varphi^\varepsilon(t,x,v) - \delta t - \delta^4|x|^2$. For any $\delta > 0$, ψ^ε_δ attains a maximum at point $(t_\delta, x_\delta, v_\delta)$. Suppose that $t_\delta > 0$. Then, we have

$$\partial_t \varphi^\varepsilon(t_\delta,x_\delta,v_\delta) \geq \delta, \qquad \nabla_x \varphi^\varepsilon(t_\delta,x_\delta,v_\delta) = 2\delta^4 x_\delta.$$

As a consequence, we have at the maximum point $(t_\delta, x_\delta, v_\delta)$:

$$0 \geq \int_V M(v') \left(1 - e^{\frac{\psi^\varepsilon_\delta(t_\delta,x_\delta,v_\delta) - \psi^\varepsilon_\delta(t_\delta,x_\delta,v')}{\varepsilon}}\right) dv' \geq \delta + 2v_\delta \delta^4 x_\delta \geq \delta - 2V_{\max}\delta^4|x_\delta|\,. \tag{3.14}$$

Moreover, the maximal property of $(t_\delta, x_\delta, v_\delta)$ also implies

$$\|\varphi^\varepsilon\|_\infty - \delta^4|x_\delta|^2 \geq \varphi^\varepsilon(t_\delta,x_\delta,v_\delta) - \delta t_\delta - \delta^4|x_\delta|^2 \geq \varphi^\varepsilon(0,0,v_\delta) \geq 0\,.$$

We obtain a contradiction with (A.8) since $\delta^{-6}/(2V_{\max}) \leq |x_\delta|^2 \leq \delta^{-4}\|\varphi^\varepsilon\|_\infty$ cannot hold for sufficiently small $\delta > 0$. As a consequence $t_\delta = 0$, and we have,

$$\forall (t,x,v) \in [0,T] \times \mathbb{R}^n \times V, \quad \varphi^\varepsilon(t,x,v) \leq \varphi^0(x_\delta,v_\delta) + \delta t + \delta^4|x|^2 \leq \|\varphi_0\|_\infty + \delta t + \delta^4|x|^2.$$

Passing to the limit $\delta \to 0$, we obtain (3.9). To find the bound (3.10), we use the same ideas on the difference $\varphi^\varepsilon_h(t,x,v) = \varphi^\varepsilon(t,x+h,v) - \varphi^\varepsilon(t,x,v)$. The equation for φ^ε_h reads as follows,

$$\partial_t \varphi^\varepsilon_h + v \cdot \nabla_x \varphi^\varepsilon_h = \int_V M(v') e^{\frac{\varphi^\varepsilon - \varphi^{\varepsilon\prime}}{\varepsilon}} \left(1 - e^{\frac{\varphi^\varepsilon_h - \varphi^{\varepsilon\prime}_h}{\varepsilon}}\right) dv'.$$

Using the same argument as above with a $\delta-$correction, we conclude that

$$\forall (t,x,v) \in [0,T] \times \mathbb{R}^n \times V, \qquad \varphi^\varepsilon_h(t,x,v) \leq \sup_{(x,v)\in\mathbb{R}\times V} \left|\varphi^0(x+h,v) - \varphi^0(x,v)\right|$$

The same argument applies to $-\varphi^\varepsilon_h$,

$$\partial_t \left(-\varphi^\varepsilon_h\right) + v \cdot \nabla_x \left(-\varphi^\varepsilon_h\right) = -\int_V M(v') e^{\frac{\varphi^\varepsilon - \varphi^{\varepsilon\prime}}{\varepsilon}} \left(1 - e^{-\frac{\left(-\varphi^\varepsilon_h\right) - \left(-\varphi^{\varepsilon\prime}_h\right)}{\varepsilon}}\right) dv'.$$

so that the r.h.s has the right sign when $-\varphi_h^\varepsilon$ attains a maximum. Finally,

$$\forall (t,x,v) \in [0,T] \times \mathbb{R}^n \times V, \qquad |\varphi_h^\varepsilon(t,x,v)| \leq \sup_{(x,v)\in\mathbb{R}\times V} \left|\varphi^0(x+h,v) - \varphi^0(x,v)\right| \leq \left\|\nabla_x \varphi^0\right\|_\infty |h|.$$

from which the estimate (3.10) follows.

To obtain regularity in the velocity variable (3.11), we differentiate (A.3) with respect to v,

$$(\partial_t + v\cdot\nabla_x)\,(\nabla_v\varphi^\varepsilon) = -g_\varepsilon(\varphi^\varepsilon)\nabla_v\varphi^\varepsilon - \nabla_x\varphi^\varepsilon,$$

where $g_\varepsilon(\varphi^\varepsilon) = \frac{1}{\varepsilon}\int_V M(v')e^{\frac{\varphi^\varepsilon - \varphi^{\varepsilon\prime}}{\varepsilon}} dv' \geq 0$. Multiplying by $\frac{\nabla_v\varphi^\varepsilon}{|\nabla_v\varphi^\varepsilon|}$, we obtain

$$\begin{aligned}(\partial_t + v\cdot\nabla_x)\,(|\nabla_v\varphi^\varepsilon|) &= -g_\varepsilon(\varphi^\varepsilon)|\nabla_v\varphi^\varepsilon| - \left(\nabla_x\varphi^\varepsilon\cdot\frac{\nabla_v\varphi^\varepsilon}{|\nabla_v\varphi^\varepsilon|}\right) \\ &\leq \|\nabla_x\varphi_0\|_\infty\,.\end{aligned} \tag{3.15}$$

from which we deduce (3.11) since $\nabla_v\varphi_0 = 0$ by hypothesis.

Finally, the bound (3.12) is obtained similarly as the bound on $\nabla_x\varphi^\varepsilon$ (3.10), using the difference $\varphi_s^\varepsilon(t,x,v) = \varphi^\varepsilon(t+s,x,v) - \varphi^\varepsilon(t,x,v)$. We obtain

$$\forall (t,x,v) \in [0,T] \times \mathbb{R}^n \times V, \qquad |\varphi_s^\varepsilon(t,x,v)| \leq \sup_{(x,v)\in\mathbb{R}\times V} \left|\varphi^\varepsilon(s,x,v) - \varphi^0(x,v)\right|\,.$$

We use the Duhamel formulation (A.7) to estimate the last contribution :

$$\left|\varphi^\varepsilon(s,x,v) - \varphi^0(x,v)\right| \leq |\varphi_0(x-sv) - \varphi_0(x)| + o(s)\,.$$

The estimate (3.12) follows. □

Step 2. Viscosity solution procedure.

From Proposition 3.2 we deduce that the familly $(\varphi^\varepsilon)_\varepsilon$ is locally uniformly bounded in the space $W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^n \times V)$. Then, from the Ascoli-Arzelà theorem, we can extract a locally uniformly converging subsequence. We denote by φ^0 the limit. Furthermore, from the fact that $\int_V M(v')e^{\frac{\varphi^\varepsilon-\varphi^{\varepsilon\prime}}{\varepsilon}}dv'$ is uniformly bounded on $[0,T]\times\mathbb{R}^n\times V$, we deduce that φ^0 does not depend on v.

Let $\psi^0 \in \mathcal{C}^2(\mathbb{R}_+\times\mathbb{R}^n)$ be a test function such that $\varphi^0 - \psi^0$ has a local maximum at (t^0, x^0). We want to show that ψ^0 is a subsolution of (A.4), yielding that φ^0 is a viscosity subsolution [66]. The supersolution case can be performed similarly. Thereby, we define a corrective term η not depending on ε [90] : $\psi^\varepsilon = \psi^0 + \varepsilon\eta$. The corrector η is defined up to an additive constant. We choose the renormalization $\int_V M(v')e^{-\eta'}dv' = 1$. We define η as follows,

$$\forall (v,v') \in V\times V\,,\quad e^{\eta(t,x,v)} - e^{\eta(t,x,v')} = (v'-v)\cdot\nabla_x\psi^0(t,x)\,. \tag{3.16}$$

The corrector η is well defined. In fact, we can choose any $v_0 \in V$ and define $e^{\eta(t,x,v)} = \mu_0 + (v_0 - v)\cdot\nabla_x\psi^0(t,x)$. There is a unique positive $\mu_0 = e^{\eta(t,x,v_0)}$ under the condition

$$\int_V M(v')e^{-\eta'}dv' = 1.$$

The uniform convergence ensures that $\varphi^\varepsilon - \psi^\varepsilon$ has a maximum at $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$, where $(t^\varepsilon, x^\varepsilon)$ is close to (t^0, x^0). As V is a bounded set, the sequence (v^ε) has an accumulation point, say v^*. We can extract a subsequence (without relabelling) such that $(t^\varepsilon, x^\varepsilon, v^\varepsilon) \to (t^0, x^0, v^*)$. We have at $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$:

$$1 - \partial_t \psi^\varepsilon - v^\varepsilon \cdot \nabla_x \psi^\varepsilon = 1 - \partial_t \varphi^\varepsilon - v^\varepsilon \cdot \nabla_x \varphi^\varepsilon = \int_V M(v') e^{\frac{\varphi^\varepsilon - \varphi^{\varepsilon\prime}}{\varepsilon}} dv' \,.$$

From the maximum property of $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$, the last inequality implies at this point :

$$1 - \partial_t \psi^\varepsilon (t^\varepsilon, x^\varepsilon, v^\varepsilon) - v^\varepsilon \cdot \nabla_x \psi^\varepsilon (t^\varepsilon, x^\varepsilon, v^\varepsilon) \geq \int_V M(v') e^{\eta(t^\varepsilon, x^\varepsilon, v^\varepsilon) - \eta(t^\varepsilon, x^\varepsilon, v')} dv'.$$

Passing to the limit, we obtain at (t^0, x^0) :

$$1 - \partial_t \psi^0 (t^0, x^0) - v^* \cdot \nabla_x \psi^0 (t^0, x^0) \geq \int_V M(v') e^{\eta(t^0, x^0, v^*) - \eta(t^0, x^0, v')} dv' = e^{\eta(t^0, x^0, v^*)} \,.$$

From the very definition of the corrector η (3.16), this writes also :

$$\forall v \in V \,, \quad 1 - \partial_t \psi^0 (t^0, x^0) - v \cdot \nabla_x \psi^0 (t^0, x^0) \geq e^{\eta(t^0, x^0, v)} \,.$$

Therefore we obtain at point (t^0, x^0),

$$\int_V \frac{M(v)}{1 - \partial_t \psi^0 (t^0, x^0) - v \cdot \nabla_x \psi^0 (t^0, x^0)} dv \leq \int_V M(v) e^{-\eta(t^0, x^0, v)} dv = 1 \,.$$

We conclude that ψ^0 is a subsolution of (A.4).

□

Chapitre 4

L' approche Hamilton-Jacobi pour la propagation dans des équations cinétiques

Dans cet article, nous utilisons la théorie des solutions de viscosité pour les équations de Hamilton-Jacobi pour étudier des phénomènes de propagation dans des équations cinétiques. On étudie la limite hyperbolique de certains modèles cinétiques grâce à une transformation de Hopf-Cole cinétique. Les modèles étudiés décrivent des particules qui effectuent des sauts en vitesses, et qui se reproduisent, la reproduction étant modélisée par un terme de réaction monostable. L'opérateur de *scattering* est supposé vérifier un principe du maximum. Quand l'espace des vitesses est borné, nous montrons que sous des hypothèses raisonnables la phase converge vers la solution de viscosité d'un problème d'obstacle de Hamilton-Jacobi dont l'Hamiltonien effectif est obtenu via la résolution d'un certain problème spectral dans la variable cinétique. Dans le cas de vitesses non-bornées, le fait que le problème spectral n'ait pas de solution peut conduire à des comportements qualitatifs très différents. Par exemple, un phénomène d'accélération du front peut apparaitre. Nous pensons néanmoins que lorsque le problème spectral a une solution le théorème de convergence peut être étendu.

Contents

4.1 Introduction

In this paper, we aim to study propagation phenomena in some kinetic models. The main motivation for this work comes from the study of pulse waves in bacterial colonies of *Escherichia coli*. Kinetic models have been proposed to describe the run-and-tumble motion of individual bacteria at the mesoscopic scale. It has been shown recently that these kinetic models are much more accurate than their diffusion approximations, see [185] and the references therein for details. In this work, and contrary to works on chemotaxis models, we focus on propagation driven by growth effects (*à la* Fisher-KPP). This is one major difference between the initial motivation and this paper.

We consider a population of cells which is described by a probability density f on $\mathbb{R}^+ \times \mathbb{R}^n \times V$, where V denotes the velocity space, which is a symmetric subset of $\mathbb{R}^n$. We assume that the velocity of cells changes randomly following a velocity-jump process given by some linear operator L analogous to the scattering operator in radiative transfer theory. We model the cell division with a kinetic nonlinearity of monostable type. Our kinetic model reads

$$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V, \qquad \partial_t f + v \cdot \nabla_x f = L(f) + r\rho \left(M(v) - f \right), \tag{4.1}$$

where $r \geq 0$ stands for a *growth parameter* and

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \qquad \rho(t,x) := \int_V f(t,x,v)dv,$$

is the macroscopic density in position x at time t. The *linear* operator $L : L^1(V) \mapsto L^1(V)$ acting only on the velocity variable describes the tumbling in the velocity space and is *mass preserving*, that is

$$\forall \varphi \in L^1_+(V), \qquad \int_V L(\varphi)(v)dv = 0.$$

We assume that $\mathrm{Ker}(L) = \mathrm{Span}(M)$, where the distribution $M \in \mathrm{Ker}(L)$ is assumed to be nonnegative and satisfies

$$\int_V M(v)dv = 1, \qquad \int_V vM(v)dv = 0, \qquad \int_V v^2 M(v)dv < +\infty.$$

We note that 0 and M are thus stationary solutions of (4.1).

A first attempt to understand the long time behavior of kinetic equations such as (4.1) is to perform scaling limits. Due to the unbiased velocity jump process contained in our model, the diffusive limit seems particularly relevant at first glance. This issue has been particularly studied in the particular case of a BGK equation without any growth term (see [12] and the references therein). As a corollary, the Fisher-KPP equation can be obtained as a parabolic limit of (4.1) when $r > 0$. The long time behavior of this latter parabolic equation is now well understood since the pioneering works of Kolmogorov-Petrovskii-Piskunov [143] and Aronson-Weinberger [10]. For nonincreasing inital data with sufficiently fast decay at infinity, the solution behaves asymptotically as a travelling front. It is thus natural to study propagation phenomena for kinetic equations such as (4.1).

Let us emphasize that travelling wave solutions for kinetic equations raised a lot of interest recently. Caflisch and Nicolaenko construct weak shock profiles solutions of the Boltzmann equation using a micro-macro decomposition [47]. Liu and Yu's main result in [148] is the establishment of the positivity of shock profiles for the Boltzmann equation. In [44], a compactness argument as in [114] also proves existence and positivity of big waves for a nonlinear BGK equation. The Caflisch and Nicolaenko micro-macro decomposition has been used to construct waves in a parabolic regime for a particular version of (4.1) for the Fisher-KPP equation [69]. In [35], travelling waves have been constructed in the full kinetic regime. Golse [114] uses compactness properties to prove existence of big waves for the kinetic Perthame-Tadmor model.

An important technique to derive the propagating behavior in reaction-diffusion equations is to revisit the Schrödinger WKB expansion to study hyperbolic limits [103, 88]. Let us quickly present this approach on the standard Fisher-KPP equation, as it contains all the heuristic ideas needed to understand the present work. This equation reads

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \qquad \partial_t \rho - D\Delta_{xx}\rho = r\rho(1-\rho), \tag{4.2}$$

where here x is the *space* variable, and r, D are positive parameters. In the hyperbolic limit $(t,x) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$, we make the so-called *WKB ansatz* :

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \qquad \rho^\varepsilon(t,x) = e^{-\frac{\varphi^\varepsilon(t,x)}{\varepsilon}}, \tag{4.3}$$

so that the *phase* φ^ε is nonnegative and satisfies the following viscous Hamilton-Jacobi equation

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \qquad \partial_t \varphi^\varepsilon + D|\nabla_x \varphi^\varepsilon|^2 + r = \varepsilon D \Delta_x \varphi^\varepsilon + r\rho^\varepsilon \tag{4.4}$$

The theory of viscosity solutions concerns the locally uniform convergence of φ^ε towards φ^0, the viscosity solution of the following so-called *variational Hamilton-Jacobi equation*

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \qquad \min\left\{\partial_t \varphi^0 + D|\nabla_x \varphi^0|^2 + r, \varphi^0\right\} = 0. \tag{4.5}$$

One can find rigorous justifications in [88] and complements in [14, 15, 196, 68]. This limit phase contains all the information we need to understand the propagating behavior. More precisely, it is possible to prove [90, 19, 98] that in the hyperbolic limit $\varepsilon \to 0$, the population is contained in the nullset of the phase φ^0. The main interests of this technique is that φ^ε can be expected to be more uniformly regular than ρ^ε, and that the full theory of Hamilton-Jacobi equations and Lagrangian dynamics can be used to understand the limit equation (4.5). As an example, studying the nullset of φ^0, we recover the propagation at the minimal speed $c^* = 2\sqrt{rD}$ for the previous Fisher-KPP equation. This fruitful WKB technique has also much been used to describe the evolution of dominant phenotypical traits in a given population (see [149, 37] and the references therein) and also to describe propagation in reaction-diffusion models of kinetic types [34].

In [33], the authors have proposed a preliminary work on a BGK equation which combines Hamilton-Jacobi equations and kinetic equations to perform the WKB approach. This latter work shows that it is necessary to stay at the kinetic level to understand the large deviation regime ; One misses something while performing the WKB approach on a macroscopic approximation of the BGK equation.

In this work, we develop the results announced in [33] for a wider class of linear kinetic equations. We derive rigorously the hydrodynamic limit of (4.1) in some special situations given by the hypothesis below. Unless otherwise stated in the sequel, we suppose that L takes the form :

$$\forall v \in V, \qquad L(f)(v) = P(f)(v) - \Sigma(v) f(v),$$

where $\Sigma \in W^{1,\infty}(V)$ and P is a linear operator that satisfies some structural assumptions that we specify below. The examples of such operators to keep in mind are the following

Example 4.1. *Our analysis is able to cover local and non-local situations :*

1. *Elliptic operators with Neumann boundary conditions on ∂V, e.g. the Laplacian : $L(f) = P(f) = \Delta f$, $\Sigma \equiv 0$.*
2. *Kernel operators : $P(f) = \int_V K(v,v') f(v') dv'$ and $\Sigma(v) = \int_V K(v',v) dv'$, where K is a nonnegative kernel ($K \in L^\infty_+(V \times V)$).*

As for the Fisher-KPP equation (7.2), we perform the hyperbolic scaling $(t,x,v) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$ in (4.1). Note that at this moment we do not rescale the velocity variable. By analogy with (4.3), our *kinetic WKB ansatz* writes

$$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V, \qquad f^\varepsilon(t,x,v) = M(v) e^{-\frac{\varphi^\varepsilon(t,x,v)}{\varepsilon}}. \tag{4.6}$$

We assume that initially

$$\forall (x,v) \in \mathbb{R} \times V, \qquad 0 \le f^\varepsilon(0,x,v) \le M(v).$$

As a consequence, thanks to the maximum principle of Hypothesis (H1) below, the phase φ^ε is well defined and remains nonnegative for all times. Plugging (4.6) in (4.1), one obtains the following equation for φ^ε :

$$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V, \qquad \partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon = -\frac{L\left(M(v) e^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)}{M(v) e^{-\frac{\varphi^\varepsilon}{\varepsilon}}} - r\rho^\varepsilon \left(e^{\frac{\varphi^\varepsilon}{\varepsilon}} - 1\right). \tag{4.7}$$

To perform the limiting equation, we would rather define the operator

$$\mathcal{L}(f) = L(f) + r\left(M(v)\rho - f\right),$$

and the associated decomposition

$$\mathcal{P}(f) := P(f) + rM(v)\rho, \qquad \overline{\Sigma} := \Sigma + r.$$

We can now transform (4.7) on the following form

$$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V, \qquad \partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon + r = -\frac{\mathcal{L}\left(M(v)e^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)}{M(v)e^{-\frac{\varphi^\varepsilon}{\varepsilon}}} + r\rho^\varepsilon. \tag{4.8}$$

This formulation is the kinetic equivalent of what was (4.4) for the Fisher-KPP case. We shall assume that for all $\varepsilon > 0$, there exists a unique solution $\varphi^\varepsilon \in \mathcal{C}_b^1(\mathbb{R}^+ \times \mathbb{R}^n \times V)$ of the Cauchy problem associated to (4.8) given some initial condition $\varphi^\varepsilon(0,x,v) = \varphi_0(x) \in \mathcal{C}_b^1(\mathbb{R}^n)$. We stress out that if boundary conditions are needed in the velocity variable, they are implicitly contained in the definition of the operator Ł.

We now formulate our convergence results. For this purpose, let us specify the assumptions on the different operators involved and on the velocity set V.

(H0) The velocity set $V \subset \mathbb{R}^n$ is ***bounded***.
This hypothesis is very helpful to prove Theorem 4.4 and will be discussed and extended in Section 4.6.

(H1) The operator P satisfies a ***maximum principle***, which will be used in the following way in the sequel :
Suppose that $Q : V \mapsto \mathbb{R}$ is nonnegative and that $u : V \mapsto \mathbb{R}$ attains a maximum in $v^0 \in V$. Then

$$P(Qu)(v^0) \leq P(Q)(v^0)u(v^0).$$

This first hypothesis is rather standard and strong but nevertheless crucial in viscosity solution procedures. It is structural and not technical. It is also helpful for space and time Lipschitz estimates, see Section 4.2. To facilitate Lipschitz estimates in velocity, we will assume a maximum principle for the differentiated operator in velocity. Indeed, in light of the WKB ansatz (4.6), let us assume the following

(H2) There exists an operator $\mathfrak{U}^\varepsilon$, acting only on the velocity variable, satisfying (H1) with $\mathfrak{U}^\varepsilon(1) \leq 0$ and $\mathfrak{B}^\varepsilon$ a bounded (uniformly in ε) function such that,

$$\nabla_v \left(\frac{\mathcal{P}\left(Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)}{Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}} \right) = \mathfrak{B}^\varepsilon - \mathfrak{U}^\varepsilon\left(\nabla_v \varphi^\varepsilon\right),$$

Example 4.2. *Let us specify Hypothesis (H2) on our typical examples. For a kernel operator of the form*

$$\forall v \in V, \qquad L(f)(v) = \int_V K(v,v')f(v')dv' - \left(\int_V K(v,v')dv'\right) f(v),$$

the operator $\mathcal{P}$ is defined by

$$\forall v \in V, \qquad \mathcal{P}(f)(v) = \int_V \left(K(v,v') + rM(v)\right) f(v')dv'.$$

Consequently,

$$\frac{\mathcal{P}\left(Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)}{Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}} = \int_V \psi(\cdot,v') \left[e^{\frac{\varphi^\varepsilon(\cdot)-\varphi^\varepsilon(v')}{\varepsilon}}\right] dv', \quad \text{with} \quad \psi(v,v') = \left(\frac{K(v,v')}{M(v)} + r\right) M(v').$$

As a consequence, we have

$$\mathfrak{B}^\varepsilon(v) = \int_V \nabla_v \psi(v,v') \left[e^{\frac{\varphi^\varepsilon(v)-\varphi^\varepsilon(v')}{\varepsilon}} \right] dv', \quad \mathfrak{U}^\varepsilon\left(\nabla_v \varphi^\varepsilon\right) = -\frac{1}{\varepsilon}\left(\int_V \psi(v,v') e^{\frac{\varphi^\varepsilon(v)-\varphi^\varepsilon(v')}{\varepsilon}} dv' \right) \nabla_v \varphi^\varepsilon.$$

Hypothesis (H2) will be satisfied after Proposition 4.5 (i), (ii), (iii) and suitable regularity on ψ that we shall assume. As an example of an elliptic operator, let us consider

$$L(f) = P(f) = \Delta f,$$

with Neumann boundary conditions on ∂V. The stationary density M satisfies $\nabla_v M = 0$ on V. We thus have

$$\frac{\mathcal{P}\left(M e^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)}{M e^{-\frac{\varphi^\varepsilon}{\varepsilon}}} = -\frac{1}{\varepsilon}\Delta\varphi^\varepsilon + \frac{1}{\varepsilon^2}|\nabla_v \varphi^\varepsilon|^2 + r\int_V M(v') e^{\frac{\varphi^\varepsilon(\cdot)-\varphi^\varepsilon(v')}{\varepsilon}} dv',$$

so that

$$\nabla_v \left(\frac{\mathcal{P}\left(M e^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)}{M e^{-\frac{\varphi^\varepsilon}{\varepsilon}}} \right) = -\frac{1}{\varepsilon}\Delta\left(\nabla_v \varphi^\varepsilon\right) + \frac{2}{\varepsilon^2}\nabla_v \varphi^\varepsilon \cdot \nabla_v\left(\nabla_v \varphi^\varepsilon\right) + \frac{r}{\varepsilon}\left(\int_V M(v') e^{\frac{\varphi^\varepsilon(v)-\varphi^\varepsilon(v')}{\varepsilon}} dv' \right) \nabla_v \varphi^\varepsilon.$$

Thus Hypothesis (H2) is well satisfied, with $\mathfrak{B} = 0$ and

$$\mathfrak{U}\left(\nabla_v \varphi^\varepsilon\right) = \frac{1}{\varepsilon}\Delta_v\left(\nabla_v \varphi^\varepsilon\right) - \frac{2}{\varepsilon^2}\nabla_v \varphi^\varepsilon \cdot \nabla_v\left(\nabla_v \varphi^\varepsilon\right) - \frac{r}{\varepsilon}\left(\int_V M(v') e^{\frac{\varphi^\varepsilon(v)-\varphi^\varepsilon(v')}{\varepsilon}} dv' \right) \nabla_v \varphi^\varepsilon.$$

We finally need to state a structural hypothesis on $\mathcal{P}$ in order to characterize the behavior with respect to v in the limit. Roughly speaking, we need coercivity.

(H3) There exists a linear operator $\overline{\mathfrak{U}}$ which satisfies the maximum principle of Hypothesis (H1), a continuous and nonnegative Hamiltonian $\mathfrak{N} : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}^+$ such that every viscosity solution of $\mathfrak{N}\left(u, \nabla_v u\right) = 0$ is constant, and $\alpha, \beta > 0$, such that the following inequality holds true

$$\forall v \in V, \qquad \mathfrak{N}\left(\varphi^\varepsilon, \nabla_v \varphi^\varepsilon\right) - \varepsilon^\alpha \overline{\mathfrak{U}}\left(\varphi^\varepsilon\right) \leq \varepsilon^\beta \left| \frac{\mathcal{P}\left(M e^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)}{M e^{-\frac{\varphi^\varepsilon}{\varepsilon}}} \right|.$$

Example 4.3. *For a kernel operator, one has*

$$\mathfrak{N}(u, \nabla_v u) = \int_V \psi(v,v') \left| u(v) - u(v') \right|_+ dv', \qquad \overline{\mathfrak{U}} \equiv 0.$$

For the Laplacian equation, one has $\mathfrak{N}(u, \nabla_v u) = |\nabla_v u|^2$ and $\overline{\mathfrak{U}} \equiv \Delta$.

Let us now state our kinetic convergence result in the Theorem 4.4 below. The main difficulty in the kinetic framework is to understand what to do with the velocity variable in the limit $\varepsilon \to 0$. Roughly speaking, we will show that up to extraction, φ^ε converges towards a viscosity solution of an Hamilton-Jacobi equation, whose effective Hamiltonian is obtained through an eigenvalue problem in the velocity variable that we write in (H4) below. In fact, the limiting phase φ^0 will be independent from the velocity variable, but the kinetic nature of the ε-problem is contained in this following spectral problem. We notice finally that, the roles of the velocity variable v and the spectral problem in (H4) below are respectively similar to the ones of the fast variable and the cell problem in homogenization theory.

(H4) ***Spectral problem.*** For all $p \in \mathbb{R}^n$, there exists a unique $\mathcal{H}(p)$ such that there exists a positive normalized eigenvector $Q_p \in L^1(V)$ such that

$$\forall v \in V, \qquad \mathcal{L}(Q_p)(v) + (v \cdot p)\, Q_p(v) = \mathcal{H}(p) Q_p(v). \tag{4.9}$$

Moreover, $\mathcal{H}$ and Q_p are smooth functions of p.

Section 4.4 is devoted to giving relevant conditions on the operator $\mathcal{L}$ which ensure that (4.9) has a solution. We also provide there some classical examples. We are now ready to state the main result :

Theorem 4.4. ***Hamilton-Jacobi limit.***

Let V be a symmetric subset of $\mathbb{R}^n$ satisfying (H0), $M \in L^1(V)$ be nonnegative and symmetric and $r \geq 0$. Suppose that the initial data is well-prepared,

$$\forall (x,v) \in \mathbb{R}^n \times V, \qquad \varphi^\varepsilon(0,x,v) = \varphi_0(x),$$

and that the Hypotheses (H1), (H2), (H3) and (H4) are satisfied. Then, $(\varphi^\varepsilon)_\varepsilon$ converges locally uniformly towards φ^0, where φ^0 does not depend on v. Moreover φ^0 is the unique viscosity solution of one of the following Hamilton-Jacobi equations :

(i) If $r = 0$, then φ^0 solves the standard Hamilton-Jacobi problem

$$\begin{cases} \partial_t \varphi^0 + \mathcal{H}\left(\nabla_x \varphi^0\right) = 0, & \forall (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n, \\ \varphi^0(0,x) = \varphi_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{4.10}$$

(ii) If $r > 0$, then the limiting equation is the following constrained Hamilton-Jacobi equation

$$\begin{cases} \min\left\{\partial_t \varphi^0 + \mathcal{H}\left(\nabla_x \varphi^0\right) + r, \varphi^0\right\} = 0, & \forall (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n, \\ \varphi^0(0,x) = \varphi_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{4.11}$$

where in both cases $\mathcal{H}(p)$ is an Hamiltonian given by (H4).

We point out that the assumption concerning the non-dependency on v of the initial data $\varphi^\varepsilon(t=0,\cdot)$ in Theorem 4.4 is to avoid a boundary layer in $t = 0$ when $\varepsilon \to 0$. The result can be easily extended to the case of an initial condition with small velocity perturbations, that is $\lim_{\varepsilon \to 0} \varphi^\varepsilon(0,x,v) = \varphi_0(x)$ uniformly in $(x,v) \in \mathbb{R} \times V$.

Our paper is organized as follows. The following Section 4.2 proves $W^{1,\infty}$ type estimates on φ^ε after assuming Hypothesis (H1) and (H2). In Section 4.3, we provide the proof of Theorem 4.4. We dedicate Section 4.4 to solving the eigenvalue problem of Hypothesis (H4) which gives the Hamiltonian $\mathcal{H}$ in some particular situations. We conclude this first part of results with a Section 4.5, giving refined asymptotics on φ^ε, and recalling some elements to study the speed of propagation of the fronts when the constrained Hamilton-Jacobi equation (7.23) is derived, following [88, 103, 92]. The last Section 4.6 is devoted to discussing the results when the velocity set is unbounded. We put forward the fact that when the spectral problem of Hypothesis (H4) is not solvable, a front acceleration can occur. Finally, we show two cases for which Hypothesis (H4) holds and where we expect the convergence result to be also true in the whole space despite additional difficulties.

4.2 The phase φ^ε is uniformly Lipschitz.

In this Section, we derive some *a priori* estimates on φ^ε mainly thanks to the maximum principle contained in Hypothesis (H1) and (H2).

Proposition 4.5. *Let $r \geq 0$ and $\varphi^\varepsilon \in \mathcal{C}_b^1\left(\mathbb{R}^+ \times \mathbb{R} \times V\right)$ a solution of equation* (4.8)*. Suppose that (H0) and the structural assumptions on Ł, (H1) and (H2), hold. Then the phase φ^ε is uniformly locally Lipschitz. Precisely the following a priori bounds hold :*
$\exists\, C > 0, \forall t \in \mathbb{R}^+,$

$$(i) \qquad 0 \leq \varphi^\varepsilon(t,\cdot) \leq \|\varphi^0\|_\infty, \qquad (ii) \quad \|\nabla_x \varphi^\varepsilon(t,\cdot)\|_\infty \leq \|\nabla_x \varphi^0\|_\infty,$$
$$(iii) \quad \|\partial_t \varphi^\varepsilon(t,\cdot)\|_\infty \leq V_{max}\|\nabla_x \varphi_0\|_\infty, \quad (iv) \qquad \|\nabla_v \varphi^\varepsilon(t,\cdot)\|_\infty \leq Ct.$$

Proof of Proposition 4.5. Let us first prove (i). We define $\psi_\delta^\varepsilon(t,x,v) = \varphi^\varepsilon(t,x,v) - \delta t - \delta^4|x|^2$. As V is bounded and ψ_δ^ε is coercive in the space-time variable, for any $\delta > 0$, ψ_δ^ε attains a maximum at point $(t_\delta, x_\delta, v_\delta)$. Suppose that $t_\delta > 0$. Then, we have

$$\partial_t \varphi^\varepsilon(t_\delta, x_\delta, v_\delta) \geq \delta, \qquad \nabla_x \varphi^\varepsilon(t_\delta, x_\delta, v_\delta) = 2\delta^4 x_\delta.$$

Moreover, thanks to the maximum principle of hypothesis (H1) for the operator P, we get :

$$P\left(Me^{-\frac{\varphi^\varepsilon(t_\delta,x_\delta,\cdot)}{\varepsilon}}\right)(v_\delta) \geq P(M)(v_\delta)e^{-\frac{\varphi^\varepsilon(t_\delta,x_\delta,v_\delta)}{\varepsilon}}.$$

We also have

$$M(v_\delta)\int_V M(v')e^{-\frac{\varphi^\varepsilon(t_\delta,x_\delta,v')}{\varepsilon}}dv' \geq M(v_\delta)e^{-\frac{\varphi^\varepsilon(t_\delta,x_\delta,v_\delta)}{\varepsilon}}.$$

As a consequence we deduce after summing the two previous inequalities

$$\mathcal{P}\left(Me^{-\frac{\varphi^\varepsilon(t_\delta,x_\delta,\cdot)}{\varepsilon}}\right)(v_\delta) \geq \mathcal{P}(M)(v_\delta)e^{-\frac{\varphi^\varepsilon(t_\delta,x_\delta,v_\delta)}{\varepsilon}}.$$

Recalling the fact that $\mathcal{L}\left(M\right) = 0$, it yields

$$\begin{aligned}
\mathcal{L}\left(Me^{-\frac{\varphi^\varepsilon(t_\delta,x_\delta,\cdot)}{\varepsilon}}\right)(v_\delta) &= \mathcal{P}\left(Me^{-\frac{\varphi^\varepsilon(t_\delta,x_\delta,\cdot)}{\varepsilon}}\right)(v_\delta) - \overline{\Sigma}(v_\delta)M(v_\delta)e^{-\frac{\varphi^\varepsilon(t_\delta,x_\delta,v_\delta)}{\varepsilon}},\\
&\geq \left[\mathcal{P}(M)(v_\delta) - \overline{\Sigma}(v_\delta)M(v_\delta)\right]e^{-\frac{\varphi^\varepsilon(t_\delta,x_\delta,v_\delta)}{\varepsilon}},\\
&\geq \mathcal{L}\left(M\right)(v_\delta)e^{-\frac{\varphi^\varepsilon(t_\delta,x_\delta,v_\delta)}{\varepsilon}},\\
&= 0.
\end{aligned}$$

As a consequence, we have at the maximum point $(t_\delta, x_\delta, v_\delta)$:

$$0 \geq -\frac{\mathcal{L}\left(Me^{-\frac{\varphi^\varepsilon(t_\delta,x_\delta,\cdot)}{\varepsilon}}\right)(v_\delta)}{M(v_\delta)e^{-\frac{\varphi^\varepsilon}{\varepsilon}}} + r\left(\rho^\varepsilon - 1\right) \geq \delta + 2\delta^4 v_\delta \cdot x_\delta \geq \delta - 2\delta^4 V_{max}|x_\delta|. \tag{4.12}$$

From what we deduce

$$|x_\delta| \geq \frac{1}{2\delta^3 V_{max}}. \tag{4.13}$$

Moreover, the maximal property of $(t_\delta, x_\delta, v_\delta)$ also implies

$$\|\varphi^\varepsilon\|_\infty - \delta^4|x_\delta|^2 \geq \varphi^\varepsilon(t_\delta, x_\delta, v_\delta) - \delta t_\delta - \delta^4|x_\delta|^2 \geq \varphi^\varepsilon(0,0,0) \geq 0\,,$$

and this gives

$$|x_\delta| \leq \frac{\|\varphi^\varepsilon\|_\infty^{\frac{1}{2}}}{\delta^2}. \tag{4.14}$$

Gathering (4.14) and (4.13), we obtain a contradiction since both cannot hold for sufficiently small $\delta > 0$. As a consequence $t_\delta = 0$, and we have,

$$\forall (t,x,v) \in [0,T] \times \mathbb{R}^n \times V, \quad \varphi^\varepsilon(t,x,v) \leq \varphi^0(x_\delta, v_\delta) + \delta t + \delta^4|x|^2 \leq \|\varphi_0\|_\infty + \delta t + \delta^4|x|^2.$$

Passing to the limit $\delta \to 0$, we obtain the claim (i).

We now come to the proof of (ii). We also use maximum principle arguments, which are possible without any supplementary hypothesis on the structure of the operator $\mathcal{L}$ since this latter operator just acts on the velocity variable. Differentiating equation (4.8) with respect to the space variable, we obtain

$$(\partial_t + v\cdot\nabla_x)\,(\nabla_x\varphi^\varepsilon) = \frac{1}{\varepsilon}\left(\frac{\mathcal{L}\left(Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}\nabla_x\varphi^\varepsilon\right) - \mathcal{L}\left(Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)\nabla_x\varphi^\varepsilon}{Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}}\right) + r\nabla_x\rho^\varepsilon. \tag{4.15}$$

Let us expand the contributions of the r.h.s. :

$$\begin{aligned}
\nabla_x\rho^\varepsilon &= -\frac{1}{\varepsilon}\int_V M(v')e^{-\frac{\varphi^\varepsilon(v')}{\varepsilon}}\nabla_x\varphi^\varepsilon(v')dv' \\
&= -\frac{1}{\varepsilon}\int_V M(v')e^{-\frac{\varphi^\varepsilon(v')}{\varepsilon}}\left(\nabla_x\varphi^\varepsilon(v') - \nabla_x\varphi^\varepsilon(v)\right)dv' - \frac{\rho^\varepsilon}{\varepsilon}\nabla_x\varphi^\varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}\left(Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}\nabla_x\varphi^\varepsilon\right) - \mathcal{L}\left(Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)\nabla_x\varphi^\varepsilon = \\
L\left(Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}\nabla_x\varphi^\varepsilon\right) - L\left(Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)\nabla_x\varphi^\varepsilon \\
+ rM(v)\left(\int_V Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}\left(\nabla_x\varphi^\varepsilon(v') - \nabla_x\varphi^\varepsilon(v)\right)dv'\right).
\end{aligned}$$

We can rewrite (4.15) as follows

$$\begin{aligned}
(\partial_t + v\cdot\nabla_x)\,(\nabla_x\varphi^\varepsilon) &= \frac{1}{\varepsilon}\left(\frac{L\left(Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}\nabla_x\varphi^\varepsilon\right) - L\left(Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)\nabla_x\varphi^\varepsilon}{Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}}\right) \\
&+ \frac{r}{\varepsilon}\left(e^{\frac{\varphi^\varepsilon}{\varepsilon}} - 1\right)\left(\int_V M(v')e^{-\frac{\varphi^\varepsilon(v')}{\varepsilon}}\left(\nabla_x\varphi^\varepsilon(v') - \nabla_x\varphi^\varepsilon(v)\right)dv'\right) - \frac{r\rho^\varepsilon}{\varepsilon}\nabla_x\varphi^\varepsilon. \quad (4.16)
\end{aligned}$$

We now test (4.16) on $\mathrm{sgn}\left(\partial_{x_i}\varphi^\varepsilon\right) e_i$:

$$\begin{aligned}
(\partial_t + v\cdot\nabla_x)\left(|\partial_{x_i}\varphi^\varepsilon|\right) + \frac{r}{\varepsilon}\rho^\varepsilon|\partial_{x_i}\varphi^\varepsilon| \le \\
\frac{1}{\varepsilon}\left(\frac{L\left(Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}\partial_{x_i}\varphi^\varepsilon\right)\mathrm{sgn}\left(\partial_{x_i}\varphi^\varepsilon\right) - L\left(Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)|\partial_{x_i}\varphi^\varepsilon|}{Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}}\right) \\
-\frac{r}{\varepsilon}\left(e^{\frac{\varphi^\varepsilon}{\varepsilon}} - 1\right)\int_V M(v')e^{-\frac{\varphi^\varepsilon(v')}{\varepsilon}}\left(|\partial_{x_i}\varphi^\varepsilon(v)| - \mathrm{sgn}\left(\partial_{x_i}\varphi^\varepsilon(v)\right)\partial_{x_i}\varphi^\varepsilon(v')\right)dv', \quad (4.17)
\end{aligned}$$

As for the uniform bound on φ^ε, we conclude by performing a $\delta-$correction argument. Define, for a positive δ, the auxiliary function $\psi^\varepsilon_{\delta,i} = |\partial_{x_i}\varphi^\varepsilon| - \delta t - \delta^4|x|^2$. It attains a maximum in $(t,x,v)_\delta$. Let us now consider the two r.h.s of (4.17) separately. Thanks to the maximum principle of hypothesis (H1) and the fact that L only operates on the $v-$variable, one has, in $(t,x,v)_\delta$,

$$L\left(Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}\partial_{x_i}\varphi^\varepsilon\right)\mathrm{sgn}\left(\partial_{x_i}\varphi^\varepsilon\right) - L\left(Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)|\partial_{x_i}\varphi^\varepsilon| \le 0.$$

To prove that the second part of the r.h.s on (4.17) is nonpositive, we write

$$\begin{aligned}
\forall v' \in V, \qquad & \mathrm{sgn}\left(\partial_{x_i}\varphi^\varepsilon(v_\delta)\right)\partial_{x_i}\varphi^\varepsilon(v') - \delta t_\delta - \delta^4|x_\delta|^2 \\
& \le |\partial_{x_i}\varphi^\varepsilon(v')| - \delta t_\delta - \delta^4|x_\delta|^2 \le |\partial_{x_i}\varphi^\varepsilon(v_\delta)| - \delta t_\delta - \delta^4|x_\delta|^2,
\end{aligned}$$

which gives the property

$$\forall v' \in V, \qquad |\partial_{x_i}\varphi^\varepsilon(v_\delta)| - \mathrm{sgn}\left(\partial_{x_i}\varphi^\varepsilon(v_\delta)\right)\partial_{x_i}\varphi^\varepsilon(v') \ge 0.$$

Combining these two inequalities give, at the point of maximum :

$$\delta + 2\delta^4 v_\delta\cdot x_\delta + \frac{r\rho^\varepsilon}{\varepsilon}|\partial_{x_i}\varphi^\varepsilon| \le 0.$$

The conclusion is similar to the uniform bound of φ^ε : The maximum cannot be attained elsewhere than in $t_\delta = 0$, and the estimate (ii) is proved.

With exactly the same method, we get that necessarily $\|\partial_t\varphi^\varepsilon\|_\infty \le |\partial_t\varphi^\varepsilon(0)|$. But, passing to the limit $t\to 0$ in (4.7), and since φ^0 does not depend on v, one gets $|\partial_t\varphi^\varepsilon(0)| \le V_{max}\|\nabla_x\varphi_0\|_\infty$. This gives (iii).

We finally come to the proof of the bound on the velocity gradient. This proof clearly requires a supplementary assumption on the operator $\mathcal{L}$ to be able to write an useful equation on $|\nabla_v\varphi^\varepsilon|$. We have made the choice of a maximum principle for the derivative operator. Again, differentiating (4.8) with respect to v and using Hypothesis (H2) yield

$$\begin{aligned}
(\partial_t + v\cdot\nabla_x)\left(\nabla_v\varphi^\varepsilon\right) + \nabla_x\varphi^\varepsilon &= -\nabla_v\left(\frac{\mathcal{P}\left(Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)}{Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}}\right) + \nabla_v\Sigma \\
&= \nabla_v\Sigma - \mathfrak{B} + \mathfrak{U}\left(\nabla_v\varphi^\varepsilon\right)
\end{aligned}$$

We now test against $\frac{\nabla_v\varphi^\varepsilon}{|\nabla_v\varphi^\varepsilon|}$ and recall (ii) :

$$(\partial_t + v\cdot\nabla_x)\left(|\nabla_v\varphi^\varepsilon|\right) \le \|\nabla_x\varphi^0\|_\infty + \|\nabla_v\Sigma\|_\infty + \|\mathfrak{B}\|_\infty + \mathfrak{U}\left(\nabla_v\varphi^\varepsilon\right)\cdot\frac{\nabla_v\varphi^\varepsilon}{|\nabla_v\varphi^\varepsilon|}.$$

Let us define C a constant such that $\|\nabla_x \varphi^0\|_\infty + \|\nabla_v \Sigma\|_\infty + \|\mathfrak{B}\|_\infty \leq C$, which is possible after (H3). Thanks to the maximum principle (H1) satisfied by $\mathfrak{U}$, we deduce that at a maximum point in velocity of $|\nabla_v \varphi^\varepsilon|$:

$$\mathfrak{U}\left(\nabla_v \varphi^\varepsilon\right) \cdot \frac{\nabla_v \varphi^\varepsilon}{|\nabla_v \varphi^\varepsilon|} \leq \mathfrak{U}\left(1\right) |\nabla_v \varphi^\varepsilon| \leq 0.$$

by (H2). As a consequence,

$$\|\nabla_v \varphi^\varepsilon(t,\cdot)\|_\infty \leq \|\nabla_v \varphi^\varepsilon(t=0,\cdot)\|_\infty + Ct = Ct,$$

as we supposed that the initial data does not depend on v, and this proves (iv). □

4.3 Hamilton - Jacobi dynamics - Proof of Theorem 4.4.

In this Section, we present the proof of our main result, Theorem 4.4. We divide the proof into two parts. We first show that the structural assumptions on the operator $\mathcal{L}$ make φ^ε converge locally uniformly up to a subsequence towards a function independent of the velocity variable, which is the first point of Theorem 4.4. Then, we perform our kinetic Hamilton-Jacobi procedure to identify the limit as a solution of one of the Hamilton-Jacobi equations (4.10) or (7.23).

4.3.1 Convergence of φ^ε.

For the convenience of the reader, we enlighten the convergence property in the following

Proposition 4.6. *Suppose that (H0), (H1), (H2) and (H3) hold. Then, up to a subsequence, the phase φ^ε converges locally uniformly in $\mathbb{R}^+ \times \mathbb{R}^n \times V$ towards φ^0, which does not depend on v.*

Proof of Proposition 7.3. Given the assumptions (H0), (H1) and (H2), we deduce from Proposition 4.5, Ascoli's theorem that in all compact subsets of $\mathbb{R}^+ \times \mathbb{R}^n \times V$, we can extract from φ^ε a converging subsequence. The limit φ^0 is uniquely defined on the whole space after increasing extraction on compacts.

The uniform bounds of Proposition 4.5 also give that $\left|\frac{\mathcal{L}\left(M(v)e^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)}{M(v)e^{-\frac{\varphi^\varepsilon}{\varepsilon}}}\right|$ is uniformly bounded. Since Σ is also bounded by assumption, $\frac{\mathcal{P}\left(M(v)e^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)}{M(v)e^{-\frac{\varphi^\varepsilon}{\varepsilon}}}$ is also uniformly bounded by a constant C. We thus deduce from (H3) that for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$,

$$\forall v \in V, \qquad \mathfrak{N}\left(\varphi^\varepsilon, \nabla_v \varphi^\varepsilon\right) \leq \varepsilon^\alpha \overline{\mathfrak{U}}\left(\varphi^\varepsilon\right) + \varepsilon^\beta C.$$

Hence, since $\overline{\mathfrak{U}}$ satisfies the maximum principle (H1), one obtains when $\varepsilon \to 0$ that $u := \varphi^0(t,x,\cdot)$ is a viscosity sub-solution of

$$\mathfrak{N}\left(u, \nabla_v u\right) = 0.$$

Since $\mathfrak{N}$ is positive, u is also a super-solution, and is thus constant thanks to Hypothesis (H3). □

4.3.2 Identification of the limit.

In this Subsection, we present the viscosity procedure which identifies the viscosity limit of φ^ε. We will follow the same steps as in the seminal paper of Evans and Souganidis [88]. In addition with a relevant use of corrected tests functions, see [90]. Indeed, the resolution of the spectral problem of Hypothesis (H4) is of main importance to define a corrector in the viscosity procedure, see (4.19) and (7.32).

Since we already know that $\varphi^\varepsilon \geq 0$, the remaining properties to be proven to get the result of Theorem 4.4 are gathered in the two following steps :

 # Step 1 : Viscosity supersolution.

The statement of the supersolution property does not depend explicitely on the growth part.

Lemma 4.7. *Assume $r \geq 0$. Then φ^0 satisfies*

$$\forall (t,x) \in \mathbb{R}^{+*} \times \mathbb{R}^n, \qquad \partial_t \varphi^0 + \mathcal{H}\left(\nabla_x \varphi^0\right) + r \geq 0. \tag{4.18}$$

in the viscosity sense.

Proof of Lemma 4.7. Let $\psi^0 \in \mathcal{C}^2\left(\mathbb{R}^+ \times \mathbb{R}^n\right)$ be a test function such that $\varphi^0 - \psi^0$ has a strict local minimum a (t^0, x^0) with $t^0 > 0$. We want to show that

$$\partial_t \psi^0(t^0,x^0) + \mathcal{H}\left(\nabla_x \psi^0(t^0,x^0)\right) + r \geq 0.$$

We define the corrected test functions [90, 66] by

$$\forall (t,x,v) \in \mathbb{R}^{+*} \times \mathbb{R}^n \times V, \qquad \psi^\varepsilon(t,x,v) := \psi^0(t,x) + \varepsilon \eta(t,x,v), \tag{4.19}$$

with a correcting term η that comes after Hypothesis (H4). Indeed, we set :

$$\forall (t,x,v) \in \mathbb{R}^{+} \times \mathbb{R}^n \times V, \qquad \eta(t,x,v) = -\ln\left(\frac{Q_{[\nabla_x \psi^0(t,x)]}(v)}{M(v)}\right). \tag{4.20}$$

The definition of the correcting function gives that $\varphi^\varepsilon - \psi^\varepsilon$ converges locally uniformly towards $\varphi^0 - \psi^0$. As a consequence, there exists a sequence $(t^\varepsilon, x^\varepsilon) \in \mathbb{R}^{+*} \times \mathbb{R}^n$ of strict local minima in (t,x) which converges towards (t^0, x^0) and a sequence $v^\varepsilon \in V$ such that $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$ minimizes $\varphi^\varepsilon - \psi^\varepsilon$. At the point $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$, using the spectral problem of (H4) with $p^\varepsilon = \nabla_x \psi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon)$, one obtains :

$$\partial_t \psi^\varepsilon + \mathcal{H}\left(p^\varepsilon\right) + r \;=\; \partial_t \psi^\varepsilon + v^\varepsilon \cdot p^\varepsilon + \frac{\mathcal{L}\left(Q_{p^\varepsilon}\right)}{Q_{p^\varepsilon}} + r.$$

We notice that at the point $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$, the following holds

$$\partial_t \varphi^\varepsilon = \partial_t \psi^\varepsilon, \qquad \nabla_x \varphi^\varepsilon = \nabla_x \psi^\varepsilon = p^\varepsilon.$$

Thus,

$$\begin{aligned} \partial_t \psi^\varepsilon + \mathcal{H}\left(p^\varepsilon\right) + r &= \partial_t \varphi^\varepsilon + v^\varepsilon \cdot \nabla_x \varphi^\varepsilon + r + \frac{\mathcal{L}\left(Q_{p^\varepsilon}\right)}{Q_{p^\varepsilon}}, \\ &= \frac{\mathcal{L}\left(Q_{p^\varepsilon}\right)}{Q_{p^\varepsilon}} - \frac{\mathcal{L}\left(M e^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)}{M e^{-\frac{\varphi^\varepsilon}{\varepsilon}}} + r\rho^\varepsilon, \end{aligned}$$

recalling (4.8). Recall that potential boundary conditions are included in the formulation of the operators. Simplifying the latter and using $\rho^\varepsilon \geq 0$, we obtain at the point $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$:

$$\partial_t \psi^\varepsilon + \mathcal{H}\left(\nabla_x \psi^\varepsilon\right) + r \quad \geq \quad \frac{\mathcal{P}\left(Q_{p^\varepsilon}\right)}{Q_{p^\varepsilon}} - \frac{\mathcal{P}\left(Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)}{Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}}.$$

But, from the minimal character of $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$ and the maximum principle satisfied by $\mathcal{P}$ we deduce that the following holds at the point $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$:

$$\begin{aligned}
-\frac{\mathcal{P}\left(Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}\right)}{Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}} &= -\frac{\mathcal{P}\left(Me^{-\frac{\varphi^\varepsilon-\psi^\varepsilon}{\varepsilon}(t^\varepsilon,x^\varepsilon,\cdot)}e^{-\frac{\psi^0(t^\varepsilon,x^\varepsilon)}{\varepsilon}}e^{-\eta(t^\varepsilon,x^\varepsilon,\cdot)}\right)}{Me^{-\frac{\varphi^\varepsilon-\psi^\varepsilon}{\varepsilon}}e^{-\frac{\psi^0(t^\varepsilon,x^\varepsilon)}{\varepsilon}}e^{-\eta}}, \\
&= -\frac{\mathcal{P}\left(Me^{-\eta(t^\varepsilon,x^\varepsilon,\cdot)}e^{-\frac{\varphi^\varepsilon-\psi^\varepsilon}{\varepsilon}(t^\varepsilon,x^\varepsilon,\cdot)}\right)}{Me^{-\eta}e^{-\frac{\varphi^\varepsilon-\psi^\varepsilon}{\varepsilon}}}, \\
&= -\frac{e^{-\frac{\varphi^\varepsilon-\psi^\varepsilon}{\varepsilon}}\mathcal{P}\left(Me^{-\eta(t^\varepsilon,x^\varepsilon,\cdot)}\right)}{Me^{-\eta}e^{-\frac{\varphi^\varepsilon-\psi^\varepsilon}{\varepsilon}}}, \\
&\geq -\frac{\mathcal{P}\left(Me^{-\eta(t^\varepsilon,x^\varepsilon,\cdot)}\right)}{Me^{-\eta}}.
\end{aligned}$$

One deduces, at the point $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$:

$$\partial_t \psi^\varepsilon + \mathcal{H}\left(\nabla_x \psi^\varepsilon\right) + r \;\geq\; \frac{\mathcal{P}\left(Q_{p^\varepsilon}\right)}{Q_{p^\varepsilon}} - \frac{\mathcal{P}\left(Me^{-\eta}\right)}{Me^{-\eta}}$$

Here comes the specification of the corrector η. We obtain, at the point $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$:

$$\partial_t \psi^\varepsilon + \mathcal{H}\left(\nabla_x \psi^\varepsilon\right) + r \geq \frac{\mathcal{P}\left(Q_{p^\varepsilon}\right)}{Q_{p^\varepsilon}} - \frac{\mathcal{P}\left(Q_{\nabla_x \psi^0(t^\varepsilon,x^\varepsilon)}\right)}{Q_{[\nabla_x \psi^0(t^\varepsilon,x^\varepsilon)]}}.$$

As the sequence v^ε is *bounded* by (H0), passing to the limit $\varepsilon \to 0$ thanks to the local uniform convergence yields

$$\partial_t \psi^0(t^0, x^0) + \mathcal{H}\left(\nabla_x \psi^0(t^0, x^0)\right) + r \geq 0.$$

□

 # Step 2 : Viscosity Subsolution.

Here comes a slight distinction between the cases $r > 0$ and $r = 0$. Indeed, one gets less information (but enough) when the nonlinearity is present since the limit equation is an obstacle problem (7.23), similarly to [88].

Lemma 4.8. *Suppose that $r > 0$. On $\left\{\varphi^0 > 0\right\} \cap (\mathbb{R}^{+*} \times \mathbb{R}^n)$, the function φ^0 solves the following equation in the viscosity sense :*

$$\forall (t,x) \in \left\{\varphi^0 > 0\right\} \cap \left(\mathbb{R}^{+*} \times \mathbb{R}^n\right), \qquad \partial_t \varphi^0 + \mathcal{H}\left(\nabla_x \varphi^0\right) + r \leq 0.$$

In the case $r = 0$, this same subsolution property holds in the full space $\mathbb{R}^{+*} \times \mathbb{R}^n$.

Proof of Lemma 4.8. Let $\psi^0 \in \mathcal{C}^2(\mathbb{R}^{+*} \times \mathbb{R}^n)$ be a test function such that $\varphi^0 - \psi^0$ has a local maximum a (t^0, x^0). We want to show that

$$\partial_t \psi^0(t^0, x^0) + \mathcal{H}\left(\nabla_x \psi^0(t^0, x^0)\right) + r \leq 0.$$

Following the same steps as for Lemma 4.7, there exists $(t^\varepsilon, x^\varepsilon, v^\varepsilon) \in \mathbb{R}^{+*} \times \mathbb{R}^n \times V$ with $(t^\varepsilon, x^\varepsilon) \to (t^0, x^0)$ and a bounded sequence v^ε such that at the point $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$:

$$\partial_t \psi^\varepsilon + \mathcal{H}\left(\nabla_x \psi^\varepsilon\right) + r \leq \frac{\mathcal{P}\left(Q_{[\nabla_x \psi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon)]}\right)(v^\varepsilon)}{Q_{[\nabla_x \psi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon)]}(v^\varepsilon)} - \frac{\mathcal{P}\left(Q_{\nabla_x \psi^0(t^\varepsilon, x^\varepsilon)}\right)(v^\varepsilon)}{Q_{[\nabla_x \psi^0(t^\varepsilon, x^\varepsilon)]}(v^\varepsilon)} + r\rho^\varepsilon(t^\varepsilon, x^\varepsilon) \tag{4.21}$$

If $r = 0$, one obtains directly with the uniform convergences that φ^0 is a subsolution of $\partial_t u + \mathcal{H}(\nabla_x u) = 0$ in (t_0, x_0), as for Lemma 4.7.

We now come to the case $r > 0$. Suppose now that $\varphi^0(t^0, x^0) > 0$, we have by the uniform convergence of φ^ε (up to extraction) that for sufficiently small ε, $\forall v \in V, \varphi^\varepsilon(t^\varepsilon, x^\varepsilon, v) > 0$. The Lebesgue dominated convergence theorem gives

$$\lim_{\varepsilon \to 0} \rho^\varepsilon(t^\varepsilon, x^\varepsilon) = \lim_{\varepsilon \to 0} \int_V M(v) e^{-\frac{\varphi^\varepsilon(t^\varepsilon, x^\varepsilon, v)}{\varepsilon}} dv = 0.$$

As a consequence, passing to the limit $\varepsilon \to 0$ in (4.21) yields

$$\partial_t \psi^0(t^0, x^0) + \mathcal{H}\left(\nabla_x \psi^0(t^0, x^0)\right) + r \leq 0,$$

and the Lemma 4.8 is proved. □

4.3.3 Uniqueness of the viscosity solution.

Before referring to an uniqueness property for (4.10) and (7.23), we have to check the initial conditions in the viscosity sense. We perform the proof in the variational case $(r > 0)$, the other one is similar. One has to check, in the viscosity sense

$$\min\left(\min\left\{\partial_t \varphi^0 + \mathcal{H}\left(\nabla_x \varphi^0\right) + r, \varphi^0\right\}, \varphi^0 - \varphi_0\right) \leq 0, \qquad \text{in } \{t = 0\} \times \mathbb{R}^n, \tag{4.22}$$

and

$$\max\left(\min\left\{\partial_t \varphi^0 + \mathcal{H}\left(\nabla_x \varphi^0\right) + r, \varphi^0\right\}, \varphi^0 - \varphi_0\right) \geq 0, \qquad \text{in } \{t = 0\} \times \mathbb{R}^n. \tag{4.23}$$

Since (7.25) can be derived on the same model, we compute (7.24) only. Let $\psi^0 \in \mathcal{C}^2(\mathbb{R}^+ \times \mathbb{R})$ be a test function such that $\varphi^0 - \psi^0$ has a strict local maximum in $(0, x_0)$. We have to prove that either

$$\varphi^0(0, x_0) - \varphi_0(x_0) \leq 0,$$

or if $\varphi^0(0, x_0) > 0$, then

$$\partial_t \psi^0(0, x_0) + \mathcal{H}\left(\nabla_x \psi^0(0, x_0)\right) + r \leq 0.$$

Suppose then that

$$\varphi^0(0, x_0) > \max\left(\varphi_0(x_0), 0\right). \tag{4.24}$$

Using the same arguments as in **# Step 2** above, we have a sequence $(t_\varepsilon, x_\varepsilon)$ which tends to $(0, x_0)$ as $\varepsilon \to 0$ and a converging sequence v_ε such that $(t_\varepsilon, x_\varepsilon, v_\varepsilon)$ maximizes $\varphi^\varepsilon - \psi^\varepsilon$. The key point to be noticed is that there exists a sequence $\varepsilon_n \to 0$ and a subsequence $(t_{\varepsilon_n}, x_{\varepsilon_n}, v_{\varepsilon_n})$ of $(t_\varepsilon, x_\varepsilon, v_\varepsilon)$ such that $t_{\varepsilon_n} > 0$.

Indeed, suppose $t_\varepsilon = 0$ when ε is sufficiently small. Then for all (t, x, v) in some neighborhood of $(0, x_\varepsilon, v_\varepsilon)$, one has

$$\varphi^\varepsilon (0, x_\varepsilon, v_\varepsilon) - \psi^\varepsilon (0, x_\varepsilon, v_\varepsilon) \geq \varphi^\varepsilon (t, x, v) - \psi^\varepsilon (t, x, v).$$

Passing to the limit $\varepsilon \to 0$ thanks to the local uniform convergence and setting $(t, x) = (0, x_0)$, we get

$$\varphi_0(x_0) - \psi^0 (0, x_0) \geq \varphi^0(0, x_0) - \psi^0(0, x_0),$$

and this contradicts (4.24). The conclusion is then similar as in **# Step 2** above since along $(t_{\varepsilon_n}, x_{\varepsilon_n}, v_{\varepsilon_n})$, Equation (4.21) holds.

From Section 4.4, the Hamiltonian $\mathcal{H}$ is a Lipschitz function of p. As a consequence, we know from [91, 88] that there exists a unique viscosity solution of (4.10) and (7.23). It yields that all the sequence φ^ε converges locally uniformly to φ^0.

4.4 The eigenvalue problem (H4).

In this Section, we discuss the spectral problem of Hypothesis (H4). Existence basically relies on compactness, positivity, and the Krein-Rutman theory. As a complement, we also provide some qualitative properties of the resulting Hamiltonian. In the next Proposition, we treat the case when P is compact and strongly positive. This is natural for kernel operators.

Proposition 4.9. *Let V be a bounded velocity domain. Suppose that $P : \mathcal{C}^0(V) \mapsto \mathcal{C}^0(V)$ is a linear, compact, and strongly positive operator. Moreover, if $r = 0$ we require that there exists a constant c such that $P(f) \geq cM(v) \int_V f dv$. Then the spectral problem of Hypothesis (H4) has a solution.*

Proof of Proposition 4.9. Let us first recall and define

$$\mathcal{P}(f) = P(f) + rM(v) \int_V f(v) dv, \qquad \overline{\Sigma} = \Sigma + r.$$

Note that since V is bounded, $\mathcal{P}$ is also a compact operator. For all $p \in \mathbb{R}^n$, we are seeking $\mathcal{H}(p)$ such that there exists a positive function $Q \in \mathcal{C}^0(V)$ such that :

$$\forall v \in V, \qquad \mathcal{P}(Q)(v) = (\bar{\Sigma}(v) + \mathcal{H}(p) - v \cdot p)\, Q(v). \tag{4.25}$$

As in similar problems [179, 125], we will use the Krein-Rutman Theorem [145]. To make it appear, we denote $\mathcal{A}_\lambda(v) := \bar{\Sigma}(v) + \lambda - v \cdot p$. Note that since V is bounded, one can guaranty the positivity of $\mathcal{A}_\lambda$ for all $\lambda > \lambda^* := \sup_{v \in V} (v \cdot p - \Sigma(v))$. We now consider the following operator T :

$$\forall \Phi \in \mathcal{C}^0(V), \qquad \forall v \in V, \qquad T(\Phi)(v) = \frac{\mathcal{P}(\Phi)(v)}{\mathcal{A}_\lambda(v)}.$$

Then, the relation (6.6) writes :

$$\forall v \in V, \qquad T(\Phi)(v) = \Phi(v). \tag{4.26}$$

To solve this eigenvalue problem, we are now ready to apply the Krein-Rutman Theorem [145]. Indeed, T is also a strongly positive compact operator. We work on the total cone of positive continuous functions $K = \mathcal{C}^0_+(V)$ to find $\Phi_\lambda \in K$ which solves :

$$\forall v \in V, \qquad T(\Phi_\lambda)(v) = \mu_\lambda \Phi_\lambda(v),$$

where μ_λ is thus the principal eigenvalue of the operator T. We assume w.l.o.g. that $\int_V \Phi_\lambda(v')dv' = 1$.

We can do the same for the adjoint operator of T, which is given by

$$\forall \Psi \in \mathcal{C}^0(V), \qquad T^*(\Psi) = \mathcal{P}^*\left(\frac{\Psi}{\mathcal{A}_\lambda}\right).$$

From the same reasons as before for the direct problem, we can solve this latter eigenvalue problem to have both

$$T(\Phi_\lambda) = \mu_\lambda \Phi_\lambda, \qquad T^*(\Psi_\lambda) = \mu_\lambda \Psi_\lambda,$$

and the normalization $\langle \Psi_\lambda | \Phi_\lambda \rangle = \int_V \Psi_\lambda(v)\Phi_\lambda(v)\, dv = 1$.

We will now prove that for all $p \in \mathbb{R}^n$, there exists only one $\lambda := \mathcal{H}(p)$ such that $\mu_\lambda = 1$. For this purpose, we study the function $\mu : \lambda \mapsto \mu_\lambda$ on the set $]\lambda^*, +\infty[$.

First, let us prove that μ is decreasing. To prove this point, we use the adjoint eigenvalue problem, see [179, Chapter 4] for another example of utilization in the study of size-structured models via the relative entropy method. Differentiating the first one with respect to λ, and taking the duality product with Ψ_λ on the left, we obtain

$$\left\langle \Psi_\lambda \middle| \frac{dT}{d\lambda}(\Phi_\lambda) \right\rangle + \left\langle \Psi_\lambda \middle| T\left(\frac{d\Phi_\lambda}{d\lambda}\right) \right\rangle = \frac{d\mu_\lambda}{d\lambda} \left\langle \Psi_\lambda \middle| \Phi_\lambda \right\rangle + \mu_\lambda \left\langle \Psi_\lambda \middle| \frac{d\Phi_\lambda}{d\lambda} \right\rangle,$$

from what we deduce, using $\left\langle \Psi_\lambda \middle| T\left(\frac{d\Phi_\lambda}{d\lambda}\right) \right\rangle = \left\langle T^*(\Psi_\lambda) \middle| \frac{d\Phi_\lambda}{d\lambda} \right\rangle = \mu_\lambda \left\langle \Psi_\lambda \middle| \frac{d\Phi_\lambda}{d\lambda} \right\rangle$ and recalling the normalization of Ψ_λ,

$$\frac{d\mu_\lambda}{d\lambda} = \left\langle \Psi_\lambda \middle| \frac{dT}{d\lambda}(\Phi_\lambda) \right\rangle.$$

As a consequence, as

$$\forall \Phi \in \mathcal{C}^0(V), \qquad \frac{dT}{d\lambda}(\Phi) = -\mathcal{P}\left(\frac{\Phi}{(\mathcal{A}_\lambda)^2}\right)$$

is a negative operator, we deduce that μ is decreasing.

We now focus on the limits of μ towards the boundary of $]\lambda^*, +\infty[$. From equation (4.26), we deduce

$$\int_V \frac{\mathcal{P}(\Phi_\lambda)(v')}{\mathcal{A}_\geq(v')} dv' = \mu_\lambda.$$

We have $\left\| (\mathcal{A}(v))^{-1} \right\|_\infty \underset{\lambda\to\infty}{\longrightarrow} 0$, so that necessarily $\lim_{\lambda\to+\infty} \mu_\lambda = 0$.

Using Fatou's lemma, we get, with $\omega = r$ if $r > 0$, $\omega = c$ else :

$$+\infty = \int_V \liminf_{\lambda\to\lambda^*} \left(\frac{\omega M(v')}{\mathcal{A}_\lambda(v')}\right) dv' \leq \int_V \liminf_{\lambda\to\lambda^*} \left(\frac{\mathcal{P}(\Phi_\lambda)(v')}{\mathcal{A}_\lambda(v')}\right) dv'$$
$$\leq \liminf_{\lambda\to\lambda^*} \left(\int_V \frac{\mathcal{P}(\Phi_\lambda)(v')}{\mathcal{A}_\lambda(v')} dv'\right) = \liminf_{\lambda\to\lambda^*} \mu_\lambda.$$

Finally, we obtain the existence and uniqueness of $\mathcal{H}(p)$ for all $p \in \mathbb{R}^n$. One associated eigenvector is given by $Q_p = \Phi_{\mathcal{H}(p)}$.

□

Remark 4.10. *With a supplementary regularization argument, the proof can be adapted replacing $\mathcal{C}^0(V)$ by $L^1(V)$. The assumption concerning the existence of a coercivity constant c when $r = 0$ may be relaxed in some particular cases. These technical points are not our purpose here, so we do not address these issues further.*

Example 4.11. *Proposition 4.9 (and its extension to $L^1(V)$) solves the case of kernel integral operators if one assume some supplementary hypothesis on the positive kernel K which ensures the compactness of the operator $\mathcal{P}$. As an example assuming*

$$\int_V \sup_{v' \in V} \left(K(v,v')\right) dv < +\infty,$$

we ensure the compactness of $\mathcal{P}$, see [71].

In the particular case where L is a BGK operator given by $L(f) := M(v)\left(\int_V f(v')dv'\right) - f$, the kernel of $\mathcal{L}$ is $K(v,v') := (1+r)M(v)$. The compactness holds. Using the scaling property of Proposition 4.14 below with $V = [-1;1]$ and $n = 1$, and the Hamiltonian derived in the one-dimensional case [33], one could find

$$\forall p \in \mathbb{R}^n, \qquad \mathcal{H}(p) = \frac{p}{\tanh\left(\frac{p}{1+r}\right)} - (1+r),$$

We can also notice that in this case, the eigenfunctions are explicit up to the knowledge of the eigenvalue. We have $\mu_\lambda = \int_V \frac{M(v)}{1-\lambda - v\cdot p}dv$, so that $\mu_\lambda = 1$ gives the dispersion relation found in [33] :

$$\int_V \frac{M(v)}{1-\lambda - v\cdot p}dv = 1.$$

The associated eigenvectors are :

$$Q_p(v) = \frac{M(v)}{1+\mathcal{H}(p) - v\cdot p}, \quad W_p(v) = \frac{1}{1+\mathcal{H}(p) - v\cdot p} \cdot \left(\int_V \frac{M(v)}{(1+\mathcal{H}(p) - v\cdot p)^2}dv\right)^{-1},$$

where the latter solves the adjoint problem.

We now prove a similar result in the case of an elliptic operator in a bounded domain.

Proposition 4.12. *Let V be a bounded smooth domain and $D(v)$ is a uniformly positive definite diffusivity matrix. Suppose $P(f) := \nabla_v (D(v)\nabla_v f)$, with Neumann boundary conditions on ∂V. Then the eigenvalue problem* (4.9) *has a solution.*

Proof of Proposition 4.12. The eigenvalue problem to be solved can be written

$$-\nabla_v \cdot (D(v)\nabla_v Q) - r\left(M(v)\int_V Q(v')dv' - Q\right) + (\mathcal{H}(p) - v\cdot p)\,Q = 0.$$

Suppose first that $r = 0$. In this case, the Krein-Rutman theorem [145] on the cone $K = \mathcal{C}^0_+(V)$ gives the result. Indeed, take a sufficiently large $\mathcal{H}(p)$ such that the operator has an inverse.

By the strong maximum principle and Neumann boundary conditions the resolvant is then compact and positive.
Suppose now that $r > 0$. One can assume that $\int_V Q(v')dv' = 1$. One then has to solve the following nonhomogeneous problem

$$-\nabla_v \cdot (D(v)\nabla_v Q) + (r + \mathcal{H}(p) - v \cdot p)\, Q = rM(v). \tag{4.27}$$

But, by the strong maximum principle and the Neumann boundary conditions, and since M is nonnegative, we deduce that for sufficiently large $\mathcal{H}(p)$, there exists a unique positive solution Q_p to the latter equation. We now have to solve, as for Proposition 4.9, the dispersion relation $\int_V Q(v)dv = 1$ to prove that there is only one $\mathcal{H}(p)$ such that the relation holds. For this purpose, similarly to the proof of Proposition 4.9, we define Q_λ solving

$$-\nabla_v \cdot (D(v)\nabla_v Q_\lambda) + (r + \lambda - v \cdot p)\, Q_\lambda = rM(v). \tag{4.28}$$

for some parameter λ sufficiently large. Differentiating (4.28) with respect to λ, one finds

$$-\nabla_v \cdot \left(D(v)\nabla_v \frac{dQ_\lambda}{d\lambda} \right) + (r + \lambda - v \cdot p)\, \frac{dQ_\lambda}{d\lambda} = -Q_\lambda. \tag{4.29}$$

As a consequence, $\frac{dQ_\lambda}{d\lambda} < 0$, and thus the application $\lambda \mapsto \int_V Q_\lambda dv$ is decreasing. Now integrating (4.28) with respect to v, we deduce that

$$\int_V Q_\lambda(v)dv \leq \frac{r}{\lambda + r - V_{\max}|p|} \longrightarrow 0, \tag{4.30}$$

as λ goes to $+\infty$. Dividing (4.28) by $r + \lambda - v \cdot p$, and integrating over V, we find

$$\int_V Q_\lambda(v)dv = \int_V \frac{rM(v)}{r + \lambda - v \cdot p} dv + \int_V \left[\frac{p \cdot \nabla D}{(r + \lambda - v \cdot p)^2} + \frac{2|p|^2 D(v)}{(r + \lambda - v \cdot p)^3} \right] Q_\lambda dv, \tag{4.31}$$

so that as λ tends to $V_{\max}|p| - r$ by larger values, $\int_V Q_\lambda(v)dv$ tends to $+\infty$ (since the last integral of the r.h.s is positive for sufficiently small values of λ). By a monotonicity argument, we are able to conclude that for all $p \in \mathbb{R}^n$, the dispersion relation $\int_V Q_\lambda(v)dv = 1$ has only one solution, that is called $\mathcal{H}(p)$. □

Example 4.13. *In the simple case given by $P(f) = \alpha \Delta f$, the solution of the eigenvalue problem* (4.9) *can be written down with Airy functions. It appears in some reaction-diffusion-mutation models without maximum principle.*

We finish this section investigating some relevant properties of the Hamiltonian $\mathcal{H}$.

Proposition 4.14. *Assume that (H4) holds. Then the Hamiltonian $\mathcal{H}$ is a Lipschitz continuous. It satisfies $\mathcal{H}(0) = 0$ and $\nabla_p \mathcal{H}(0) = 0$. Finally, it also satisfies the scaling property*

$$\forall \mu \in \mathbb{R}^*, \qquad \mathcal{H}_{\mu L} = \mu \mathcal{H}_L \left(\frac{\cdot}{\mu} \right),$$

where we denote by $\mathcal{H}_{\mu L}$ the Hamiltonian associated to some operator L.

Proof of Proposition 4.14. We get that $\mathcal{H}(0) = 0$ as a byproduct of the integration of (4.9) over V :

$$\forall p \in \mathbb{R}^n, \qquad |\mathcal{H}(p)| = \left| \left(\int_V v Q_p(v) dv \right) \cdot p \right| \leq V_{max} |p|.$$

This latter inequality prove the sublinear behavior of the Hamiltonian. To prove the Lipschitz character of the Hamiltonian, we again use the adjoint formulation of (4.9). Indeed, we can solve it as for the direct problem, so that there exists W_p such that

$$\mathcal{P}\left(Q_p\right)(v) = \left(\Sigma(v) + \mathcal{H}(p) - v \cdot p\right) Q_p(v), \quad \mathcal{P}^*(W_p)(v) = \left(\Sigma(v) + \mathcal{H}(p) - v \cdot p\right) W_p(v).$$

Differentiating these two equalities with respect to p, we get

$$\left(\Sigma(v) + \mathcal{H}(p) - v \cdot p\right) \frac{dQ_p}{dp} + Q_p \left(\nabla_p \mathcal{H} - v\right) = \mathcal{P}\left(\frac{dQ_p}{dp}\right),$$

As previously performed, we integrate against W_p,

$$\left\langle (\Sigma(v) + \mathcal{H}(p) - v \cdot p) W_p \middle| \frac{dQ_p}{dp} \right\rangle + \left\langle W_p | Q_p \left(\nabla_p \mathcal{H} - v\right)\right\rangle = \left\langle W_p \middle| \mathcal{P}\left(\frac{dQ_p}{dp}\right)\right\rangle$$

so that

$$\left\langle W_p | Q_p \left(\nabla_p \mathcal{H} - v\right)\right\rangle = 0 \iff \nabla_p \mathcal{H} = \frac{\langle W_p | v Q_p \rangle}{\langle W_p | Q_p \rangle} \iff |\nabla_p \mathcal{H}| \leq V_{max}, \tag{4.32}$$

and this gives that $\mathcal{H}$ is Lipschitz. Moreover, we always have $Q_0(v) = M(v)$ and $W_0 = 1$, the last one coming from the conservation property. Thus,

$$\nabla_p \mathcal{H}(0) = \langle W_0 | v Q_0 \rangle = \int_V v M(v) dv = 0.$$

The last point follows from the uniqueness of the solution $\mathcal{H}$ of the eigenvalue problem (4.9). Indeed, we have for all $\mu \in \mathbb{R}^*$,

$$\forall v \in V, \qquad \mu \mathcal{H}\left(\frac{p}{\mu}\right) = v \cdot p + \frac{\mu \mathcal{L}(\hat{Q}_p)}{\hat{Q}_p}(v),$$

with $\hat{Q}_p = Q_{\mu p}$, where Q_p is an eigenvector for $\mathcal{H}(p)$. □

Remark 4.15.

1. *Here appears one of the most striking conclusion of our study. The Hamiltonian of the limiting equation in the large deviation regime is Lipschitz continuous. It differs strongly from the case of the Fisher-KPP equation which is obtained as the dryft-diffusion limit of* (4.1). *This means that the diffusion limit is not compatible with large deviations and thus propagation of fronts.*

2. *A classical attempt in the Hamilton-Jacobi theory is the convexity of the Hamiltonian. In [33], the authors manage to prove that for the simplest BGK case, it is indeed convex. However, it seems not to be an easy issue in general. We were not able to conclude if the Hamiltonian is convex or not.*

3. *Thanks to the Proposition 4.14, we can replace $\mathcal{L}$ by a more "barycentric" one $\mathcal{L}_r$:*

$$\mathcal{L}_r(f) = \frac{L(f) + r\,(M(v)\rho - f)}{1+r}, \tag{4.33}$$

solve the underlying eigenvalue problem (4.9) *to get an Hamiltonian $\mathcal{H}_{\mathcal{L}_r}$ and deduce the following relation*

$$\forall p \in \mathbb{R}^n, \qquad \mathcal{H}(p) = (1+r)\mathcal{H}_{\mathcal{L}_r}\left(\frac{p}{1+r}\right).$$

The latter identity can be useful for example when L is also a BGK operator, as in [33].

4. *One could want to derive a expression of the total Hamiltonian which only depends on the Hamiltonian associated to L. However, even though the BGK operator and L commute, we cannot generally derive an expression for the Hamiltonian of their sum. Indeed, the construction of solutions of the spectral problem shows that the Hamiltonians appear as spectral radius of operator. Basically, it is not possible to obtain a exact general formula for the spectral radius of the sum of two operators.*

4.5 Asymptotics, numerics and comments.

4.5.1 Further asymptotics.

This subsection aims at proving some convergence results for the total density ρ^ε in both regions $\{\varphi^0 = 0\}$ and $\{\varphi^0 > 0\}$.

Proposition 4.16. *Let φ^ε be the solution of* (4.8)*. Theorem 4.4 says that it converges locally uniformly towards a nonpositive φ^0, the unique viscosity solution of* (7.23)*. Uniformly on compact subsets of Int $\{\varphi^0 > 0\}$, the convergence $\lim_{\varepsilon\to 0} f^\varepsilon = 0$ holds, and is exponentially fast.*

Proposition 4.17. *Let φ^ε be the solution of* (4.8)*. Assume now that $r > 0$. Then, uniformly on compact subsets of Int $\{\varphi^0 = 0\}$,*

$$\lim_{\varepsilon\to 0} \rho^\varepsilon = 1, \qquad \lim_{\varepsilon\to 0} f^\varepsilon\,(\cdot, v) = M(v).$$

Remark 4.18. *Assume that $r = 0$ and $\mathcal{H}$ is convex. Then the mass stays at its initial position :*

$$\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n \mid \varphi^0(t,x) = 0\} = \mathbb{R}^+ \times \{x \in \mathbb{R}^n \mid \varphi_0(x) = 0\}.$$

Indeed, one can write the solution of the standard Hamilton-Jacobi equation (4.10) *with the Hopf-Lax formula :*

$$\varphi^0(t,x) = \inf_{\gamma\in X}\left\{\varphi_0\,(\gamma(0)) + \int_0^t \mathcal{M}\,(\dot\gamma(t))\,dt \,\middle|\, \gamma(t) = x\right\},$$

where $\mathcal{M}$ is the Lagrangian associated to $\mathcal{H}$. Since $\nabla_p\mathcal{H}(0) = 0$ and $\mathcal{H}$ is strictly convex, so does $\mathcal{M}$, and as a consequence $\mathcal{M}$ is positive away from 0*. We deduce that*

$$\varphi^0(t,x) = 0 \quad \iff \quad t \in \mathbb{R}^+ \quad \textit{and} \quad \varphi_0(x) = 0.$$

Proof of Proposition 4.16. Let K be a compact subset of $\text{Int}\left\{\varphi^0 > 0\right\}$. The local uniform convergence of φ^ε towards φ^0 ensures that there exists $\delta > 0$ such that for sufficiently small $\varepsilon > 0$, $\varphi^\varepsilon \geq \delta$ on K. As a consequence,

$$\forall (t,x,v) \in K \times V, \qquad f^\varepsilon(t,x,v) = M(v)\exp\left(-\frac{\varphi^\varepsilon(t,x,v)}{\varepsilon}\right) < M(v)\exp\left(-\frac{\delta}{\varepsilon}\right) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

□

To prove the convergence result for f^ε in the zone $\{\varphi^0 = 0\}$ we shall assume some regularity for the solutions of the spectral problem (H4). We state this as an hypothesis since it has to be checked on the spectral problem case by case.

(H4') The solution of (H4) satisfies

$$\lim_{p\to 0} \frac{Q_p}{M} = 1\,, \quad \textbf{\textit{uniformly}}\ \textbf{in}\ V.$$

This is however a reasonable hypothesis which will be satisfied by our typical examples. Basically, an elliptic operator provides sufficient smoothness for $(p,v) \mapsto Q_p(v)$. As an example of a kernel operator, let us use the simplest BGK operator, namely $L(f)(\cdot) = M(\cdot)\int_V f(v)dv - f(\cdot)$. We know from [33] and Example 4.11 that

$$\frac{Q_p}{M}(v) = \frac{1+r}{1+r-\mathcal{H}(p) - v\cdot p} \to 1,$$

uniformly in v when $p \to 0$, independently of the value of M.

Proposition 4.19. *Let φ^ε be the solution of* (4.8)*. Assume that $r > 0$ and (H4') above holds. Then, uniformly on compact subsets of Int $\left\{\varphi^0 = 0\right\}$,*

$$\lim_{\varepsilon\to 0} \rho^\varepsilon = 1, \qquad \lim_{\varepsilon\to 0} f^\varepsilon(\cdot, v) = M(v).$$

Proof of Proposition 4.19. We develop similar arguments as in [88]. Note that it suffices to prove the result when K is a cylinder. Let $(t_0, x_0) \in \text{Int}\,(K)$ and the test function

$$\forall (t,x) \in K, \qquad \psi^0(t,x) = |x - x_0|^2 + (t-t_0)^2\,.$$

We can define the same corrected test function ψ^ε as in the viscosity procedure of Section 4.3. We recall that it needs a corrector, given by $\eta := -\ln\left(\frac{Q_{\nabla_x \psi^0(t,x)}(v)}{M(v)}\right)$.

Since $\varphi^0 = 0$ on K, the function $\varphi^0 - \psi^0$ admits a strict maximum in (t_0, x_0). The locally uniform convergence of $\varphi^\varepsilon - \psi^\varepsilon$ gives a sequence $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$ of maximum points with $(t^\varepsilon, x^\varepsilon) \to (t^0, x^0)$ and a bounded sequence v^ε such that at the point $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$ one has (see (4.21)) :

$$\partial_t \psi^\varepsilon + \mathcal{H}\left(\nabla_x \psi^\varepsilon\right) + r \leq \frac{\mathcal{P}\left(Q_{[\nabla_x \psi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon)]}\right)}{Q_{[\nabla_x \psi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon)]}}(v^\varepsilon) - \frac{\mathcal{P}\left(Q_{\nabla_x \psi^0(t^\varepsilon, x^\varepsilon)}\right)}{Q_{[\nabla_x \psi^0(t^\varepsilon, x^\varepsilon)]}}(v^\varepsilon) + r\rho^\varepsilon(t^\varepsilon, x^\varepsilon).$$

Moreover, we can compute the values of the derivatives of φ^ε and ψ^ε at the point $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$ and see that they vanish when $\varepsilon \to 0$:

$$\partial_t \varphi^\varepsilon = \partial_t \psi^\varepsilon = \partial_t \psi^0 \quad \text{and} \quad \nabla_x \varphi^\varepsilon = \nabla_x \psi^\varepsilon = \nabla_x \psi^0 + \varepsilon \nabla_x \eta.$$

As a consequence, one has, since $r > 0$,

$$\rho^\varepsilon(t^\varepsilon, x^\varepsilon) \geq 1 + o(1), \qquad \text{as } \varepsilon \to 0, \tag{4.34}$$

and then $\lim_{\varepsilon\to 0} \rho^\varepsilon(t^\varepsilon, x^\varepsilon) = 1$ if one recalls $\rho^\varepsilon \leq 1$ (which, again, is a consequence of the maximum principle).

However, we need an extra argument to get $\lim_{\varepsilon\to 0} \rho^\varepsilon(t_0, x_0) = 1$. Since $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$ maximizes $\varphi^\varepsilon - \psi^\varepsilon$, we deduce that for all $v \in V$, we have

$$\varphi^\varepsilon\left(t^\varepsilon, x^\varepsilon, v^\varepsilon\right) - \psi^0(t^\varepsilon, x^\varepsilon) - \varepsilon\eta\left(t^\varepsilon, x^\varepsilon, v^\varepsilon\right) \geq \varphi^\varepsilon\left(t_0, x_0, v\right) - \psi^0(t_0, x_0) - \varepsilon\eta\left(t_0, x_0, v\right).$$

Since $\psi^0(t^\varepsilon, x^\varepsilon) \geq 0$, $\psi^0(t_0, x_0) = 0$, $\eta\left(t_0, x_0, v\right) = 0$, we find

$$f^\varepsilon(t_0, x_0, v) = M(v) e^{-\frac{\varphi^\varepsilon(t_0, x_0, v)}{\varepsilon}} \geq M(v) \left(\frac{M(v^\varepsilon)}{Q_{\nabla_x \psi^0(t^\varepsilon, x^\varepsilon)}(v^\varepsilon)} \right) e^{-\frac{\varphi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon)}{\varepsilon}}. \tag{4.35}$$

We shall now prove that $\lim_{\varepsilon\to 0} \varepsilon^{-1}\varphi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon) = 0$. Note that it is not a direct consequence of $\lim_{\varepsilon\to 0} \rho^\varepsilon(t^\varepsilon, x^\varepsilon) = 1$ since this gives only an *a.e.* convergence of $\varepsilon^{-1}\varphi^\varepsilon(t^\varepsilon, x^\varepsilon, \cdot)$ which might not be pointwise at first glance. In this step we will use (H4') for the first time. We set $s^\varepsilon = \nabla_x \psi^0(t^\varepsilon, x^\varepsilon)$ for legibility. Let us rewrite (4.7) at the point $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$ on the form

$$r\rho^\varepsilon(t^\varepsilon, x^\varepsilon)\left(e^{\frac{\varphi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon)}{\varepsilon}} - 1\right) = -\frac{L\left(M e^{-\frac{\varphi^\varepsilon(t^\varepsilon, x^\varepsilon, \cdot)}{\varepsilon}}\right)(v^\varepsilon)}{M(v^\varepsilon) e^{-\frac{\varphi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon)}{\varepsilon}}} - \left(\partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon\right)(t^\varepsilon, x^\varepsilon, v^\varepsilon)$$

From the maximum principle satisfied by P, we get

$$-\frac{L\left(M e^{-\frac{\varphi^\varepsilon(t^\varepsilon, x^\varepsilon, \cdot)}{\varepsilon}}\right)(v^\varepsilon)}{M(v^\varepsilon) e^{-\frac{\varphi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon)}{\varepsilon}}} \leq -\frac{L\left(M e^{-\eta(t^\varepsilon, x^\varepsilon, \cdot)}\right)(v^\varepsilon)}{M(v^\varepsilon) e^{-\eta(t^\varepsilon, x^\varepsilon, v^\varepsilon)}} = -\frac{L\left(Q_{s^\varepsilon}\right)}{Q_{s^\varepsilon}}(v^\varepsilon).$$

Recalling the spectral problem (H4), and using (H4'), we find

$$\frac{L\left(Q_{s^\varepsilon}\right)}{Q_{s^\varepsilon}}(v^\varepsilon) = \frac{\mathcal{L}\left(Q_{s^\varepsilon}\right)}{Q_{s^\varepsilon}}(v^\varepsilon) - r\left(\frac{M(v^\varepsilon)}{Q_{s^\varepsilon}(v^\varepsilon)} - 1\right) = \mathcal{H}(s^\varepsilon) - v \cdot s^\varepsilon - r\left(\frac{M(v^\varepsilon)}{Q_{s^\varepsilon}(v^\varepsilon)} - 1\right) = o_{\varepsilon\to 0}(1),$$

We finally deduce

$$0 \leq r\rho^\varepsilon(t^\varepsilon, x^\varepsilon)\left(e^{\frac{\varphi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon)}{\varepsilon}} - 1\right) \leq o_{\varepsilon\to 0}(1)$$

and thus $\lim_{\varepsilon\to 0} \varepsilon^{-1}\varphi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon) = 0$. We take again advantage of (H4') in (4.35) to obtain $\lim_{\varepsilon\to 0} f^\varepsilon(t_0, x_0, v) \geq M(v)$, and this implies $\lim_{\varepsilon\to 0} f^\varepsilon(t, x, v) = M(v)$ locally uniformly on $K \times V$. □

4.5.2 Study of the viscosity solution and of the speed of propagation.

To be self-contained, we recall here how to study the propagation of the front after deriving the limit variational equation, in the case $r > 0$. From Evans and Souganidis [88], we are able to identify the solution of the variational Hamilton-Jacobi equation (7.23) using the Lagrangian duality. We recall the equation :

$$\begin{cases} \min\left\{\partial_t \varphi^0 + \mathcal{H}\left(\nabla_x \varphi^0\right) + r, \varphi^0\right\} = 0, & \forall (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n, \\ \varphi^0(0,x) = \varphi_0(x). \end{cases}$$

We will suppose in this Subsection that the hamiltonian $\mathcal{H}$ is convex and is a function of $|p|$. The relevant result in the present context is the following

Proposition 4.20 (Speed of propagation)**.** *Assume that*

$$\varphi_0(x) := \begin{cases} 0 & x = 0 \\ +\infty & else \end{cases},$$

and define $c^* = \inf_{p>0}\left(\frac{\mathcal{H}(p)+r}{p}\right)$, see [92, 93]. Then the nullset of φ propagates at speed c^* :

$$\forall t \geq 0, \qquad \{\varphi(t,\cdot) = 0\} = B(0, c^* t).$$

Proof of Proposition 4.20. The Lagrangian associated to $\mathcal{H} + r$ is by definition

$$\mathcal{L}(p) := \sup_{q \in \mathbb{R}^n} \left(p \cdot q - \mathcal{H}(q) - r\right),$$

and one has, since $\mathcal{H}(q) = \mathcal{H}(|q|)$:

$$\mathcal{L}(p) = \sup_{q \in \mathbb{R}^n} \left(|p||q| \left(\frac{p}{|p|} \cdot \frac{q}{|q|}\right) - \mathcal{H}(|q|) - r\right) = \sup_{q \in \mathbb{R}^n} \left(|p||q| - \mathcal{H}(|q|) - r\right).$$

$$\mathcal{L}(p) = 0 \iff |p| = \inf_{u>0}\left(\frac{\mathcal{H}(u) + r}{u}\right) = c^*.$$

To solve the variational Hamilton-Jacobi equation, let us define

$$J(x,t) = \inf_{x \in X}\left\{\int_0^t \left[\mathcal{L}(\dot{x})\right] ds \big| x(0) = x, x(t) = 0\right\}$$

the minimizer of the action associated to the Lagrangian. Thanks to the so-called Freidlin condition, see [88, 103] we deduce that the solution of (7.23) is

$$\varphi(x,t) = \max\left(J(x,t), 0\right).$$

The Lax formula gives

$$J(x,t) = \min_{y \in \mathbb{R}}\left\{t\mathcal{L}\left(\frac{x-y}{t}\right) + \varphi_0(y)\right\} = t\mathcal{L}\left(\frac{x}{t}\right)$$

thanks to the assumption on the initial condition. Finally, as $\mathcal{L}$ is increasing with $|p|$, the nullset of φ is exactly $B(0, c^* t)$.

□

4.5.3 Numerical simulations

We show in Figure 4.5.3 some numerical simulations of the evolution of the nullset of the solution of the variational Hamilton-Jacobi equation to illustrate our study. The speed of the front is easily numerically computable with this approach. When the Hamiltonian is not known explicitly, which is the most frequent case, it is still possible to solve numerically the spectral problem (H4) to obtain a numerical Hamiltonian, which can afterwards be used to compute the whole numerical solution.

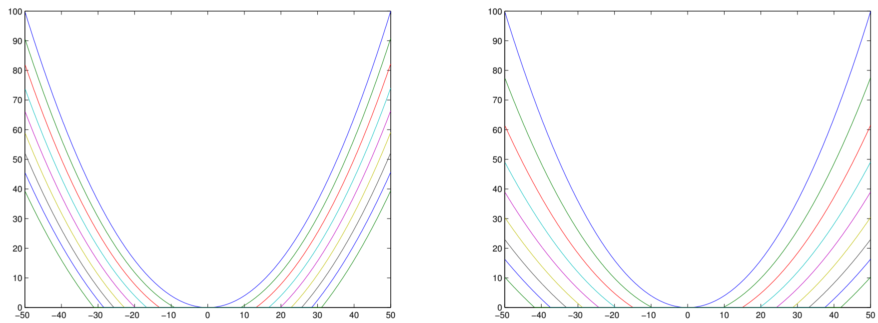

FIGURE 4.1 – Numerical simulations of the variational Hamilton-Jacobi equation with $r = 1$, in the BGK case $\mathcal{H}(p) \equiv \frac{p-\tanh(p)}{\tanh(p)}$ (on the left) and in the "KPP case" $\mathcal{H}(p) = |p|^2$ (on the right). On both figures the linear propagation is noticed. In the "quadratic case" (KPP) the speed is larger than in the "at most linear" case.

4.6 Remarks and perspectives in an unbounded velocity domain (*e.g.* $V = \mathbb{R}^n$).

In the previous Sections, the boundedness of the velocity space V (Hypothesis (H0)) was a central hypothesis. Indeed, it gives immediately the compactness of operators to solve the spectral problem (H4), and facilitates the derivation of the uniform estimates of φ^ε. Moreover, it automatically bounds the sequence v^ε in the viscosity procedure of Lemmas 4.7 and 4.8. This last property appears not to be true in general, see below.

In this last Section, we would like to comment on the case when V is not bounded, and more precisely the case of the full space $V = \mathbb{R}^n$. We expect that, given that (H4) holds (which basically requires stronger assumptions on the operator $\mathcal{L}$ in the full space) the convergence result is still valid despite technicalities due to the unboundedness of the space.

We first discuss the case of the transport-diffusion equation to illustrate the crucial character of (H4) : The spectral problem (H4) does not have any non trivial solution in that case, and we show that the scaling $(t, x, v) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$ is not relevant. We then provide an example - the Vlasov equation - where the problem is compact in the velocity space. However, extending the convergence results in that case will need extra work and this issue will be discussed in a forthcoming work. Since we believe that this paper should be understood through examples, we end this Section with formal computations on a non-local convolution model.

4.6.1 The Laplacian equation in an unbounded velocity domain.

In this Subsection, we want to investigate the asymptotic properties of solutions of the following kinetic-diffusion equation

$$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \qquad \partial_t f + v \cdot \nabla_x f = \sigma \Delta_v f. \tag{4.36}$$

First of all, one can notice that the associated spectral problem of Hypothesis (H4), which writes

$$\forall v \in \mathbb{R}^n, \qquad \sigma \Delta_v Q + (v \cdot p)\, Q = \mathcal{H}(p) Q$$

does not admit any nontrivial positive solutions. It relies on the lack of compactness of the Laplace operator on an unbounded domain. As a consequence, the method we have used before to average the velocity variable in the bounded velocity domain case cannot be applied here. We will now show that the scaling $(t,x,v) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$ is not well adapted and propose a more relevant scaling. In this case, as for the heat equation for example, one can guess this scaling by computing the fundamental solution of the kinetic diffusion operator. We recall this computation for the sake of completeness [142].

Proposition 4.21. *Let $f(t,x,v)$ be the solution of* (4.36) *on $\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n$, associated to the initial data $\delta_x \delta_{v-w}$. Then*

$$\forall (t,x,v) \in \mathbb{R}^{+*} \times \mathbb{R}^n \times \mathbb{R}^n, \qquad f_w(t,x,v) = \frac{\sqrt{3}}{2\pi\sigma t^2} \exp\left(- \frac{|v-w|^2 t^2 + 3|2x - (v+w)t|^2}{4\sigma t^3} \right).$$

Proof of Proposition 4.21. This computation can be done using the Fourier transform $\mathcal{F}$ in space and velocity, since the operator is linear. One obtains

$$\forall (t,k,p) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n,$$
$$\mathcal{F}(f)(t,k,p) = \exp\left(-i(p+kt)w\right) \exp\left(-\sigma t \left(\left| p + \frac{kt}{2} \right|^2 + |k|^2 \frac{t^2}{12} \right) \right),$$

and the inverse Fourier transform can be easily computed using the invariances of Gaussians with respect to the Fourier transformations.

□

We now perform an alternative scaling on this equation, namely $(t,x,v) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{v}{\varepsilon}\right)$. Then the fundamental solution f_0 becomes

$$\forall (t,x,v) \in \mathbb{R}^{+*} \times \mathbb{R}^n \times \mathbb{R}^n, \qquad f_0^\varepsilon(t,x,v) = \frac{\sqrt{3}\varepsilon^2}{2\pi\sigma t^2} \exp\left(-\frac{1}{\varepsilon} \frac{|v|^2 t^2 + 3|2x - vt|^2}{4\sigma t^3} \right).$$

In this framework, and only with this scaling, we recover the sharp front ansatz that we studied in the previous part of the article with a bounded domain, and the associated phase (which now depends on v) $\varphi^0(t,x,v) = \frac{|v|^2 t^2 + 3|2x - vt|^2}{4\sigma t^3}$. It is also possible to obtain this result by performing the Hopf-Cole transformation in (4.36) and then solving the limiting Hamilton-Jacobi equation on the phase φ^0 which reads :

$$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \qquad \partial_t \varphi^0 + v \cdot \nabla_x \varphi^0 + \sigma |\nabla_v \varphi^0|^2 = 0.$$

We obtained an example where the spectral problem has no solution (see also [33] for another fundamental example), and this makes the information propagate as $x \sim t^2$: There is a front acceleration, as noticed for others models, see [34, 35, 31].

4.6.2 The Vlasov-Fokker-Planck equation

We would like now to comment on the Vlasov-Fokker-Planck equation, where the velocity operator provides enough compactness to solve the spectral problem (H4). Our equation reads

$$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \qquad \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot \left(\sigma^2 \nabla_v f + v f \right). \tag{4.37}$$

The normalized stationary density is given by the Gaussian equilibrium $M(v) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{v^2}{2\sigma^2}\right)$. After performing the hyperbolic scaling and the kinetic WKB ansatz (4.6), it yields

$$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \qquad \partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon = \sigma^2 \left(\frac{1}{\varepsilon} \Delta \varphi^\varepsilon - \frac{1}{\varepsilon^2} |\nabla_v \varphi^\varepsilon|^2 \right) - \frac{v}{\varepsilon} \cdot \nabla_v \varphi^\varepsilon,$$

By parabolic regularity, one obtains, given an initial condition $\varphi_0(x,v) \in \mathcal{C}_b^2 (\mathbb{R}^n \times \mathbb{R}^n)$, one unique bounded solution φ^ε in $\mathcal{C}^{2,\alpha} (\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n)$ for all $\varepsilon > 0$. The spectral problem associated to (4.37) is :

$$\nabla_v \cdot \left(\sigma^2 \nabla_v Q_p + v Q_p \right) + (v \cdot p)\, Q_p = \mathcal{H}(p) Q_p, \tag{4.38}$$

As a particular feature a the Gaussian case, one can solve (4.38) explicitely using the Fourier transformation. It yields the following eigenelements

$$\mathcal{H}(p) = \sigma^2 |p|^2, \qquad Q_p(v) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(v - \sigma^2 p)^2}{2\sigma^2} \right).$$

Hence, our hypothesis (H4) is fulfilled.

We shall comment here on the complications due to the unboundedness of the space. We cannot perform the same proof as for the proof of Theorem 4.4. Indeed, the sequence of approximated extremas in velocity, namely v^ε defined after (7.32), may not exist in general in a unbounded velocities setting. In particular, in this case, the correction η is given by

$$\eta(t,x,v) = -\ln \left(\frac{Q_{[\nabla_x \psi^0(t,x)]}(v)}{M(v)} \right) = v \cdot \nabla_x \psi^0(t,x) - \frac{\sigma^2}{2} |\nabla_x \psi^0(t,x)|^2, \tag{4.39}$$

which is linear in v, so that the function $\varphi^\varepsilon - \varepsilon\eta$ has no possible extrema in the velocity variable. This indicates that the correction term of order ε converges locally uniformly but not globally towards the corrector η. We postpone the analysis of this case to a forthcoming work.

4.6.3 Formal computations on a confined non-local equation.

We finish this paper with *formal* computations on a case where the diffusive part of the operator is replaced by a nonlocal convolution operator. This is motivated by biological problems where mutations can have large range. We keep the drift part to ensure compactness. We are given a probability kernel K on $\mathbb{R}$, and we set

$$L(f) := (\mathrm{K} \star f - f) + \nabla \cdot (vf)\,.$$

Solving the eigenvalue problem using the Fourier transform in the full space, we obtain that necessarily

$$\mathcal{H}(p) = \hat{\mathrm{K}}(ip) - 1, \qquad F(Q_p)(\xi) = \exp\left(\int_0^{\xi} \frac{\hat{\mathrm{K}}(\xi') - \hat{\mathrm{K}}(ip)}{\xi' - ip} d\xi'\right).$$

As $\hat{\mathrm{K}}(ip) = \int_V \mathrm{K}(x) e^{px} dx$, we observe that this would define an Hamiltonian on the zone where K admits exponential moments. We have

$$Q_p(v) = \int_V \exp\left(\int_0^{\xi} \frac{\hat{\mathrm{K}}(\xi') - \hat{\mathrm{K}}(ip)}{\xi' - ip} d\xi'\right) \exp\left(iv\xi\right) d\xi,$$

where the last integral over the velocities has to be understood in the Fourier-Plancherel L^2 sense. One can easily prove that such a Q_p is well normalized and real. The point which makes this Subsection be only formal is that we were not able to prove that such a Q_p is indeed a positive eigenvector. Let us provide a few examples that strengthen this conjecture.

Example 4.22. *We now specify some convolution kernels.*

1. $K(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. *Then* $\mathcal{H}$ *is well-defined on* $\mathbb{R}$ *and* $H(p) = e^{\frac{p^2}{2}} - 1$.
2. $K(x) = \frac{1}{2} e^{-|x|}$. *Then* $\mathcal{H}$ *is well-defined on* $]-1,1[$ *and* $\mathcal{H}(p) = \frac{p^2}{1-p^2}$. *In this case, we can compute a bit further* $F(Q_p)$ *:*

$$F(Q_p)(\xi) = \frac{1}{(1+|\xi|^2)^{\frac{1}{2(1-p^2)}}} \cdot \exp\left(i \frac{p}{p^2-1} \arctan(\xi)\right).$$

In particular, when $p = 0$, *one has*

$$F(Q_p)(\xi) = \frac{1}{(1+|\xi|^2)^{\frac{1}{2}}},$$

which inverse Fourier transform can be computed with Airy functions and is positive. A numerical plot confirms formally the positivity of Q_p *(result not shown).*

Acknowledgement

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Deuxième partie

Dynamique adaptative de populations structurées en espace et en trait phénotypique

Chapitre 5

Invasion fronts with variable motility : phenotype selection, spatial sorting and wave acceleration

Cette note est en collaboration avec Vincent Calvez, Nicolas Meunier, Sepideh Mirrahimi, Benoît Perthame, Gaël Raoul et Raphaël Voituriez. Les fronts d'invasion en écologie ont été largement étudiés. Cependant peu de résultats mathématiques existent pour le cas d'un coefficient de motilité variable (à cause des mutations). A partir d'un modèle minimal de réaction-diffusion, nous expliquons le phénomène observé d'accélération du front (lorsque la motilité n'est pas bornée), et nous démontrons l'existence d'ondes progressives ainsi que la sélection des individus les plus motiles (lorsque la motilité est bornée). Le point clé pour la construction des fronts est la relation de dispersion qui relie la vitesse de l'onde avec la décroissance en espace. Lorsque la motilité n'est pas bornée nous montrons que la position du front suit une loi d'échelle en $t^{3/2}$. Lorsque le taux de mutation est faible, nous montrons que, dans notre contexte, l'équation canonique pour la dynamique du meilleur trait est une EDP. C'est une équation de type Burgers avec terme source.

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Version française abrégée

Dans cette note nous étudions un modèle simple décrivant des fronts invasifs en écologie, pour lesquels la motilité des individus est sujette à variations. Le modèle, issu de [43], est le suivant,

$$\begin{cases} \partial_t n(t,x,\theta) = \theta \partial^2_{xx} n(t,x,\theta) + rn(t,x,\theta)\left(1-\rho(t,x)\right) + \alpha \partial^2_{\theta\theta} n(t,x,\theta)\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in (-\infty,+\infty),\, \theta > 0\,, \\ \rho(t,x) = \int n(t,x,\theta)\, d\theta\,, \qquad n(t,-\infty,\theta) = \overline{N}(\theta)\,, \qquad n(t,+\infty,\theta) = 0\,. \end{cases} \tag{5.1}$$

Les conditions aux limites sont complétées ci-dessous. Nous étudions la dynamique de cette équation sous différents régimes. Dans un premier temps nous étudions le problème de propagation de front à vitesse constante. Cela circonscrit l'espace des traits $\theta \in (0,\Theta)$, qui doit être borné, $\Theta < +\infty$. Nous montrons formellement que la situation est similaire au cas de l'équation de Fisher-KPP. Il existe une vitesse minimale c^* de propagation des ondes de réaction-diffusion (Résultat 5.1). La relation de dispersion analogue à celle de Fisher-KPP (un trinôme du second degré en l'occurrence) est donnée via la résolution d'un problème spectral. On lit sur la distribution des phénotypes (la fonction propre du problème spectral) que les fortes motilités sont favorisées, conclusion opposée au cas des domaines bornés en espace [80]. Ce même problème spectral intervient lorsqu'on étudie la propagation du front en régime asymptotique hyperbolique $(t,x) \to (t/\varepsilon, x/\varepsilon)$ dans la limite WKB de l'optique géométrique, en suivant l'ansatz $n^\varepsilon(t,x,\theta) = \exp(u^\varepsilon(t,x)/\varepsilon) N^\varepsilon(t,x,\theta)$.

Dans un second temps, nous considérons un espace des traits non borné, $\Theta = +\infty$. Dans ce cas le front accélère sans cesse et nous montrons heuristiquement que la loi de propagation du front est naturellement $\langle x \rangle \sim \left(\alpha^{1/4} r^{3/4}\right) t^{3/2}$. Nous exhibons une solution particulière qui confirme cette heuristique (Résultat 5.2).

Dans un troisième temps, nous étudions le régime de mutations rares, et nous écrivons une équation canonique pour l'évolution du *trait sélectionné localement* à l'avant du front en régime asymptotique. La dérivation formelle de cette équation conduit à une équation de transport de type Burgers, avec terme source (Proposition 5.3). La partie transport est dûe à la progression du front qui "transporte" les individus et donc le trait sélectionné localement. Le terme source est dû à la pression de sélection qui tend à faire augmenter la valeur du trait sélectionné localement.

Introduction

Recently, several works have addressed the issue of front invasion in ecology, where the motility of individuals is subject to variability [60, 9]. It has been postulated that selection of more motile individuals can occur, even if they have no advantage regarding their reproductive rate (*spatial sorting*) [194, 141, 184, 193]. This phenomenon has been described in the invasion of cane toads in northern Australia [181]. It has been shown that the speed of the front increases, coincidentally with significant changes in toads morphology. Up to now, only numerical simulations have been proposed to address this issue. Here we analyse a simple model of Bénichou et al [43] which contains the basic features of this process : spatial mobility, logistic reproduction, and variable motility. It is given by equation (5.1) where the last term on the right hand side accounts for modifications of the dispersal rate θ of individuals due to mutations. We consider that mutations are random and that they act as a diffusion process in the phenotype space. When needed, we impose Neumann boundary conditions in the variable θ and far-field conditions in the variable x.

5.1 Phenotype selection and spatial sorting in the traveling wave

We first consider bounded dispersal rates, say $\theta \in (0, \Theta)$. Following [43], we seek a traveling wave solution of equation (5.1) connecting 0 to the uniform stationary state $\overline{N}(\theta) \equiv \Theta^{-1}$. For x large, we make the ansatz $n(t, x, \theta) = \exp(\lambda(x - ct))Q(\theta)$, where $c > 0$ is the speed of the wave, $\lambda < 0$ is the spatial decay and $Q(\theta)$ denotes the phenotypic distribution of the individuals at the edge of the front. The dispersion relation is equivalent to the following spectral problem : Given a spatial decay rate $\lambda < 0$, find $c(\lambda)$ and a corresponding eigenvector $Q(\theta, \lambda)$ such that

$$\begin{cases} \left(\lambda c(\lambda) + \theta\lambda^2 + r\right) Q(\theta, \lambda) + \alpha \partial^2_{\theta\theta} Q(\theta, \lambda) = 0\,, \\ \partial_\theta Q(0, \lambda) = \partial_\theta Q(\Theta, \lambda) = 0\,, \quad \forall \theta\; Q(\theta, \lambda) \geq 0\,, \quad \int Q(\theta, \lambda)\, d\theta = 1\,. \end{cases} \tag{5.2}$$

The wave speed $c(\lambda)$ is such that 0 is the principal eigenvalue of this spectral problem. Like for the Fisher-KPP equation, there is a minimal speed $c^* > 0$ associated with a critical spatial decay $\lambda^* < 0$.

Formal Result 5.1 (Front propagation and spatial sorting)**.** *For all $c \geq c^*$, there exists formally a traveling front solution $n(t, x, \theta) = N(x - ct, \theta)$. The profile satisfies $N(z, \theta) \sim \exp(\lambda z) Q(\theta, \lambda)$ as $z \to +\infty$. The phenotypic distribution at the edge of the front $Q(\theta, \lambda)$ is unbalanced towards more motile individuals. More precisely, we have*

$$\langle \theta \rangle_{\text{edge}}(\lambda) := \int \theta Q(\theta, \lambda)\, d\theta > \frac{\Theta}{2}\,, \qquad \langle \theta \rangle_{\text{edge}}(\lambda) \underset{\alpha \to 0}{\longrightarrow} \Theta\,. \tag{5.3}$$

This problem is closely related to the issue of front propagation in kinetic equations [33]. There is a natural extension of this result for other mutation operators as integral operators $\alpha \int G(\theta, \theta') n(t, x, \theta')\, d\theta'$. In this case, the solution of the spectral problem is deduced from the Krein-Rutman theorem.

Interestingly, we can measure the asymmetry of the phenotypic distribution $Q(\theta, \lambda)$. The relevant quantity here is the mean diffusion coefficient at the edge of the front $\langle \theta \rangle_{\text{edge}}(\lambda)$. In order to show (5.3), we integrate the spectral problem (5.2) over $(0, \Theta)$. We get $\lambda c + \lambda^2 \langle \theta \rangle_{\text{edge}} + r = 0$. Dividing by $Q(\theta, \lambda)$, and integrating again the spectral problem we get after integration by parts,

$$(\lambda c + r)\Theta + \frac{\Theta^2}{2}\lambda^2 + \alpha \int \left| \frac{\partial_\theta Q(\theta,\lambda)}{Q(\theta,\lambda)} \right|^2 d\theta = 0\,.$$

Hence, the mean diffusion coefficient is given by

$$\langle \theta \rangle_{\text{edge}}(\lambda) = \frac{\Theta}{2} + \frac{\alpha}{\Theta\lambda^2} \int \left| \frac{\partial_\theta Q(\theta,\lambda)}{Q(\theta,\lambda)} \right|^2 d\theta > \frac{\Theta}{2}\,. \tag{5.4}$$

Individuals at the edge of the front are more motile than at the back of the front. There, the population is homogeneous since $N(-\infty, \theta) \equiv \Theta^{-1}$ and the averaged diffusion coefficient is $\langle \theta \rangle_{\text{back}} = \Theta/2$. The estimate (5.4) measures how far the phenotypic distribution differs from the uniform distribution. Finally we find easily that as α vanishes, the distribution $Q(\theta, \lambda)$ concentrates as a Dirac mass on $\theta = \Theta$ hence the second statement in (5.3) : the most motile individuals are selected. The conclusion is exactly the opposite on bounded spatial domains [80].

We can derive additional information for the minimal speed c^*. Differentiating (5.2) we obtain

$$\left(\lambda c'(\lambda) + c(\lambda) + 2\theta\lambda\right) Q(\theta,\lambda) + \left(\lambda c(\lambda) + \theta\lambda^2 + r\right) \frac{\partial}{\partial\lambda} Q(\theta,\lambda) + \alpha \partial^2_{\theta\theta} \frac{\partial}{\partial\lambda} Q(\theta,\lambda) = 0\,.$$

Definition of c^* ensures that $c'(\lambda^*) = 0$. We use the notation $\langle f \rangle := \int f(\theta) Q^*(\theta)\, d\theta$. Since the operator in (5.2) is self-adjoint, we obtain after multiplication by $Q(\theta, \lambda)$ and integration, the relation

$$c^* \langle Q^* \rangle + 2\lambda^* \langle \theta Q^* \rangle = 0\,, \quad c^* = -2\lambda^* \frac{\langle \theta Q^* \rangle}{\langle Q^* \rangle}\,. \tag{5.5}$$

Recalling that $\lambda^* c^* + (\lambda^*)^2 \langle \theta \rangle + r = 0$, we can eliminate λ^* and we get the following expression for c^*,

$$(c^*)^2 = 4r\langle \theta \rangle \left(1 - \left(1 - \frac{\langle \theta \rangle \langle Q^* \rangle}{\langle \theta Q^* \rangle} \right)^2 \right)^{-1} > 4r \langle \theta \rangle\,.$$

In other words, the usual formula for the KPP wave speed underestimates the actual minimal speed.

5.2 Spatial sorting and the invasion front

Next, we focus on the invasion front. It is natural to perform the hyperbolic rescaling $(t, x) \to (t/\varepsilon, x/\varepsilon)$ in order to catch the motion of the front [101, 88]. The new equation writes after rescaling

$$\varepsilon \partial_t n^\varepsilon(t,x,\theta) = \varepsilon^2 \theta \partial^2_{xx} n^\varepsilon(t,x,\theta) + r n^\varepsilon(t,x,\theta) \left(1 - \rho^\varepsilon(t,x)\right) + \alpha \partial^2_{\theta\theta} n^\varepsilon(t,x,\theta)\,.$$

We perform the partial WKB ansatz in the x variable :

$$n^\varepsilon(t,x,\theta) = \exp(u^\varepsilon(t,x)/\varepsilon) N^\varepsilon(t,x,\theta),$$

with the renormalization $\int N^\varepsilon(t,x,\theta)\,d\theta = 1$. As $\varepsilon \to 0$, the first order expansion yields the equation

$$\partial_t u^0(t,x)N^0 = \theta|\partial_x u^0(t,x)|^2 N^0(t,x,\theta) + r\left(1-\rho^0(t,x)\right)N^0(t,x,\theta) + \alpha\partial^2_{\theta\theta}N^0(t,x,\theta)\,.$$

The edge of the front is delimited by the area $\{u^0(t,x) < 0\}$. On this set we have $\rho^0(t,x) = 0$ by construction. Therefore we shall solve again the spectral problem (5.2) for $N^0 \geq 0$. Consequently the motion of the front is driven by the eikonal equation built on the effective speed $c(\lambda)$,

$$\max\left(u^0, \partial_t u^0 + \partial_x u^0 \cdot c(\partial_x u^0)\right) = 0\,.$$

The rigorous derivation of this Hamilton-Jacobi equation requires some work. We need basically refined *a priori* estimates on $(u^\varepsilon)_\varepsilon$. We formally show the main argument leading to establish the viscosity limit u^0 of u^ε in the set $\{u^0(t,x) < 0\}$. We leave the complete proof for future work. Let v^0 be a $\mathcal{C}^2$ test function such that $u^0 - v^0$ has a strict maximum at (t^0, x^0). The function $u^\varepsilon - v^0$ has a maximum at $(t^\varepsilon, x^\varepsilon)$, with $(t^\varepsilon, x^\varepsilon)$ close to (t^0, x^0). Plugging v^0 into the equation satisfied by $(u^\varepsilon, N^\varepsilon)$, namely

$$\left[\partial_t u^\varepsilon(t,x) - \theta|\partial_x u^\varepsilon(t,x)|^2 - \varepsilon\theta\partial^2_{xx}u^\varepsilon(t,x) - r + O(\varepsilon)\right]N^\varepsilon(t,x,\theta) = \alpha\partial^2_{\theta\theta}N^\varepsilon(t,x,\theta)\,,$$

we obtain at $(t^\varepsilon, x^\varepsilon)$:

$$\left[\partial_t v^0(t^\varepsilon,x^\varepsilon) - \theta|\partial_x v^0(t^\varepsilon,x^\varepsilon)|^2 - r + O(\varepsilon)\right]N^\varepsilon(t^\varepsilon,x^\varepsilon,\theta) \leq \alpha\partial^2_{\theta\theta}N^\varepsilon(t^\varepsilon,x^\varepsilon,\theta)\,.$$

Therefore, N^ε is a non-negative, non-trivial subsolution of the spectral problem (5.2). From the characterization of the principal eigenvalue, we have

$$\partial_t v^0(t^\varepsilon,x^\varepsilon) + \partial_x v^0(t^\varepsilon,x^\varepsilon)\cdot c(\partial_x v^0(t^\varepsilon,x^\varepsilon)) + O(\varepsilon) \leq 0\,.$$

Passing to the limit $\varepsilon \to 0$, we obtain that v^0 satisfies $\partial_t v^0 + \partial_x v^0 \cdot c(\partial_x v^0) \leq 0$ at (t^0, x^0). Therefore u^0 is a viscosity sub-solution of the eikonal equation $\partial_t u^0 + \partial_x u^0 \cdot c(\partial_x u^0) = 0$ in the interior of the set $\{u^0(t,x) < 0\}$. The same argument shows that it is also a supersolution and thus a viscosity solution.

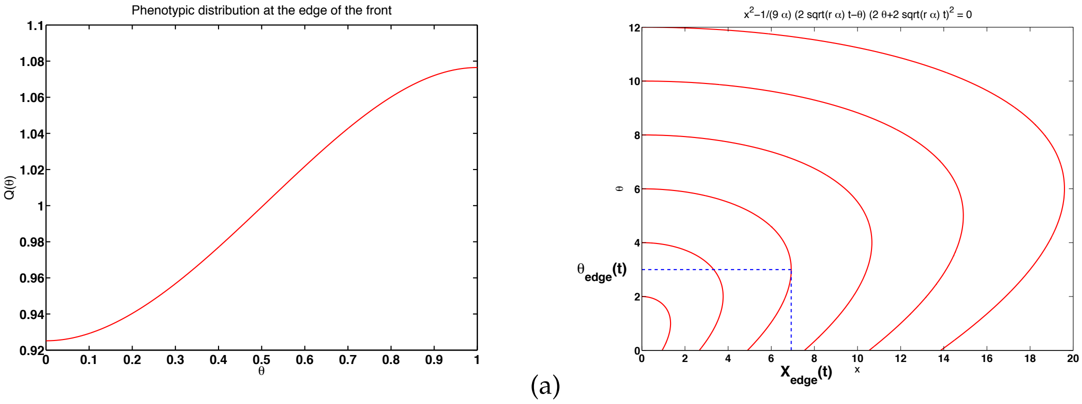
5.3 Front acceleration

In the case where the set of dispersal rates is unbounded, say $\theta \in (0, +\infty)$, then we cannot solve the spectral problem (5.2). There is no intrinsic speed of propagation, and the front is accelerating as time goes on. Heuristically, we expect the averaged diffusion coefficient to grow linearly with time $\langle\theta\rangle \sim \left(\sqrt{\alpha r}\right)t$ (as for the Fisher-KPP equation set in the phenotype space). Hence the invasion front should scale as $\langle x\rangle \sim \left(\alpha^{1/4}r^{3/4}\right)t^{3/2}$ since the speed is given by $c \sim \sqrt{\langle\theta\rangle r}$. Therefore we perform the asymptotic scaling $(t,x,\theta) \to \left(t/\varepsilon, x/\varepsilon^{3/2}, \theta/\varepsilon\right)$, in order to catch the motion of the front asymptotically. The equation writes after rescaling

$$\varepsilon\partial_t n^\varepsilon = \varepsilon^2\theta\partial^2_{xx}n^\varepsilon + rn^\varepsilon\left(1-\rho^\varepsilon(t,x)\right) + \varepsilon^2\alpha\partial^2_{\theta\theta}n^\varepsilon\,. \tag{5.6}$$

We perform the WKB ansatz in both variables (x,θ) : $n^\varepsilon(t,x,\theta) = \exp\left(u^\varepsilon(t,x,\theta)/\varepsilon\right)$. We derive formally the Hamilton-Jacobi equation for u^0 in the limit $\varepsilon \to 0$:

$$\partial_t u^0 - \theta|\partial_x u^0|^2 - \alpha|\partial_\theta u^0|^2 = r\left(1-\rho^0(t,x)\right)\,, \tag{5.7}$$



(a) (b)

FIGURE 5.1 – (a) *Phenotype selection.* At the edge of the front the phenotypic distribution is not uniform : more motile individuals are selected. We solved numerically the relation dispersion for $(r, \alpha, \Theta) = (1, 1, 1)$. (b) *Front acceleration.* The set $\{u^0 = 0\}$ is plotted in the phase space (x, θ), for successive times. The far-right point $X_{\text{edge}}(t)$ determines the location of the front.

where $\rho^0(t, x)$ is the Lagrange multiplier associated with the constraint $\max_\theta u^0 \leq 0$. In particular, when the constraint is inactive, *i.e.* $\max_\theta u^0(t, x, \theta) < 0$, we have $\rho^0(t, x) = 0$. The behavior of the *phase function* $u^0(t, x, \theta)$ is as follows : the nullset $\{u^0(t, x, \theta) = 0\}$ propagates in space and phenotype under the action of space mobility, and mutations, combined with growth of individuals. We are able to compute explicitely the evolution of this set for a particular initial data, using Lagrangian Calculus.

Proposition 5.2 (Front acceleration)**.** *For the particular initial data,*

$$u^0(0, x, \theta) = \begin{cases} 0 & \text{if } (x, \theta) = (0, 0) \\ -\infty & \text{if } (x, \theta) \neq (0, 0) \end{cases}$$

the location of the nullset $\{u^0(t, x, \theta) = 0\}$ is given by the following implicit formula (see Fig. 5.1)

$$x^2 = \frac{1}{9\alpha} \left(\left(2\sqrt{r\alpha}\right) t - \theta \right) \left(2\theta + \left(2\sqrt{r\alpha}\right) t\right)^2 .$$

Sketch of proof. The Hamiltonian is given by $H((x, \theta), (p_x, p_\theta)) = \theta |p_x|^2 + \alpha |p_\theta|^2 + r$, and the corresponding Lagrangian writes $L((x, \theta), (v_x, v_\theta)) = v_x^2/(4\theta) + v_\theta^2/(4\alpha) - r$. The system of characteristics is given by $\dot{X}(t) = (2\theta(t) p_x(t), 2\alpha p_\theta(t))$, and $\dot{P}(t) = -(0, |p_x(t)|^2)$. Using Lagrangian formulation, we deduce after some calculation that, with Z the solution to equation $Z^3 + (12\theta/\alpha)Z + 24x/\alpha = 0$, $u^0(t, x, \theta)$ is given by

$$u^0(t, x, \theta) = -\frac{1}{4\alpha t} \left(\theta + \frac{\alpha}{4} Z^2 \right)^2 .$$

This enables to compute the nullset $\{u^0(t, x, \theta) = 0\}$. □

The far-right point of the curve is attained for $\theta_{\text{edge}} = \left(\sqrt{r\alpha}\right) t$. This determines the location of the front. Hence the position of the front in space is exactly,

$$X_{\text{edge}}(t) = \frac{4}{3} \left(\alpha^{1/4} r^{3/4} \right) t^{3/2} .$$

5.4 Adaptive dynamics at the edge of the front

Interestingly enough, equation (5.7) can be derived also in the context of adaptive dynamics, a theory which studies mutation-selection processes. It is generally assumed that the mutation process is so slow that mutants can replace the resident species before new mutants arise, if they are better adapted to their environment. This yields a *canonical equation* which gives the dynamical evolution of the selected trait in the population [77, 60]. Recently, PDE-based methods have been successfully used to derive such a canonical equation for continuous mutation-selection processes [79, 18, 149]. Here we extend this theory to the case of front invasion coupled with a basic mutation process.

The only difference with (5.1) is that mutations are assumed to be rare ($\alpha \to \varepsilon^2\alpha$) :

$$\partial_t n(t,x,\theta) = \theta\partial^2_{xx} n(t,x,\theta) + rn(t,x,\theta)\left(1-\rho(t,x)\right) + \varepsilon^2\alpha\partial^2_{\theta\theta} n(t,x,\theta)\,.$$

It is natural to perform a long time rescaling $t \to t/\varepsilon$ at the scale of evolutionary changes. Then it is useful to rescale space accordingly $x \to x/\varepsilon$ in order to catch the motion of the front (otherwise it would travel with a speed of order $O(1/\varepsilon)$). With these changes of scales, we end up again with equation (5.6), resp. (5.7) in the WKB limit. We restrict to the edge of the front, namely $\sup_\theta u^0(t,x,\theta) < 0$, $\rho^0(t,x) = 0$. We seek a canonical equation for the *locally selected trait* $\overline{\theta}(t,x)$ such that $u^0(t,x,\overline{\theta}(t,x)) = \sup_\theta u^0(t,x,\theta)$.

Formal Result 5.3 (Derivation of the canonical equation)**.** *The locally selected trait $\overline{\theta}(t,x)$ formally satisfies a Burgers type equation with a source term,*

$$\partial_t\overline{\theta}(t,x) - 2\left(\overline{\theta}(t,x)\partial_x u^0\right)\partial_x\overline{\theta}(t,x) = \frac{|\partial_x u^0|^2}{-\partial^2_{\theta\theta}u^0}\,. \tag{5.8}$$

The speed of the transport equation is $-2\overline{\theta}(t,x)\partial_x u^0$. It coincides with the local minimal speed of the traveling front, see *e.g.* (5.5). The positive source term accounts for the evolutionary drift which pushes the population towards higher motility. This equation can create shock wave singularities (numerical simulations not shown), as for the classical Burgers equation. The explanation is clear : more motile populations, when located behind less motile populations, will invade them.

Proof. We start from the first order condition $\partial_\theta u^0(t,x,\overline{\theta}(t,x)) = 0$. We differentiate this relation with respect to t and x, respectively,

$$\partial^2_{t\theta}u^0(t,x,\overline{\theta}(t,x)) + \left(\partial^2_{\theta\theta}u^0\right)\partial_t\overline{\theta}(t,x) = 0\,, \qquad \partial^2_{x\theta}u^0(t,x,\overline{\theta}(t,x)) + \left(\partial^2_{\theta\theta}u^0\right)\partial_x\overline{\theta}(t,x) = 0\,.$$

On the other hand, we differentiate equation (5.7) with respect to θ,

$$\partial^2_{\theta t}u^0 - |\partial_x u^0|^2 - 2\theta\partial_x u^0\partial^2_{\theta x}u^0 - 2\alpha\partial_\theta u^0\partial^2_{\theta\theta}u^0 = 0\,.$$

Evaluating the latter at $\theta = \overline{\theta}(t,x)$ yields

$$\partial^2_{\theta t}u^0 - 2\overline{\theta}(t,x)\partial_x u^0\partial^2_{\theta x}u^0 = |\partial_x u^0|^2\,.$$

Combining these calculations, we conclude that $\overline{\theta}$ satisfies equation (5.8). □

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Chapitre 6

Ondes progressives pour un modèle non-local de dynamique des populations

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Dans cet article en collaboration avec Vincent Calvez, nous étudions la propagation dans un modèle de reaction-diffusion non-local qui modélise l'invasion des crapauds buffles en Australie [181]. La population de crapauds est structurée par une variable d'espace et une variable de trait phénotypique et la diffusivité spatiale dépend du trait. Nous utilisons un argument de type degré topologique de Schauder pour la construction de solutions en ondes progressives pour ce modèle. La vitesse de l'onde c^* est obtenue après la résolution d'un problème spectral dans la variable de trait. Un vecteur propre obtenu via la résolution de ce problème spectral donne une indication sur la forme du profil à l'avant du front. La difficulté principale est l'obtention de bornes L^∞ uniforme malgré la combinaison de termes non-locaux et de la diffusivité hétérogène.

Contents

6.1 Introduction.

In this paper, we focus on propagation phenomena in a model for the invasion of cane toads in Australia, proposed in [43]. It is a structured population model with two structural variables, the space $x \in \mathbb{R}^n$ and the motility $\theta \in \Theta$ of the toads. The mobility of the toads is the ability to move spontaneously and actively. Here $\Theta := (\theta_{\min}, \theta_{\max})$, with $\theta_{\min} > 0$ denotes the bounded set of traits. One modeling assumption is that the space diffusivity depends only on θ. The mutations are simply modeled by a diffusion process with constant diffusivity α in the variable θ. Each toad is in local competition with all other individuals (independently of their trait) for resources. The free growth rate is r. The resulting reaction term is of monostable type. Denoting $n(t, x, \theta)$ the density of toads having trait $\theta \in \Theta$ in position $x \in \mathbb{R}^n$ at time $t \in \mathbb{R}^+$, the model writes :

$$\begin{cases} \partial_t n - \theta \Delta_x n - \alpha \partial_{\theta\theta} n = rn(1-\rho)\,, & (t,x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \Theta, \\ \partial_\theta n(t,x,\theta_{\min}) = \partial_\theta n(t,x,\theta_{\max}) = 0\,, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n. \end{cases} \tag{6.1}$$

with

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \qquad \rho(t,x) = \int_\Theta n(t,x,\theta)\, d\theta.$$

The Neumann boundary conditions ensure the conservation of individuals through the mutation process.

The invasion of cane toads has interested several field biologists. The data collected [193, 181] show that the speed of invasion has always been increasing during the eighty first years of propagation and that younger individuals at the edge of the invasion front have shown significant changes in their morphology compared to older populations. This example of ecological problem among others (see the expansion of bush crickets in Britain [197]) illustrates the necessity of having models able to describe space-trait interactions. Several works have addressed the issue of front invasion in ecology, where the trait is related to dispersal ability [74, 61]. It has been postulated that selection of more motile individuals can occur, even if they have no advantage regarding their reproductive rate, due to spatial sorting [141, 184, 193, 194].

Recently, some models for populations structured simultaneously by phenotypical traits and a space variable have emerged. A similar model to (6.1) in a discrete trait setting has been studied by Dockery *et al.* in [80]. Interestingly, they prove that in a bounded space domain and with a rate of growth $r(x)$ heterogeneous in space, the only nontrivial Evolutionarily Stable State (ESS) is a population dominated by the slowest diffusing phenotype. This conclusion is precisely the opposite of what is expected at the edge of an invading front. In [4], the authors study propagation in a model close to (6.1), where the trait affects the growth rate r but not the dispersal ability. This latter assumption is made to take into account that the most favorable phenotypical trait may depend on space. The model reads

$$\partial_t n - \Delta_{x,\theta} n = \left(r\left(\theta - Bx\cdot e\right) - \int_{\mathbb{R}} k\left(\theta - Bx\cdot e, \theta' - Bx\cdot e\right) n(t,x,\theta')d\theta' \right) n(t,x,\theta),$$

and the authors prove the existence of travelling wave solutions. A version with local competition in trait of this equation has also been studied in [23]. As compared to [4, 23], the main difficulty here is to obtain a uniform $L^\infty\left(\mathbb{R}\times\Theta\right)$ bound on the density n solution of (6.1). It is worth recalling that this propagation phenomena in reaction diffusion equations, through the theory of travelling waves, has been widely studied since the pioneering work of Aronson and Weinberger [11] on the Fisher-KPP equation [97, 143]. We refer to [168, 171, 25] and the references therein for recent works concerning travelling waves for generalized Fisher-KPP equations in various heterogeneous media, and to [65, 136, 191] for works studying front propagation in models where the non locality appears in the dispersion operator.

Studying propagation phenomena in nonlocal equations can be pretty involved since some qualitative features like Turing instability may occur at the back of the front, see [22, 122], due to lack of comparison principles. Nevertheless, it is sometimes still possible to construct travelling fronts with rather abstract arguments. In this article, we aim to give a complete proof of some formal results that were previously announced in [34]. Namely construct some travelling waves solutions of (6.1) with the expected qualitative features at the edge of the front. Let us now give the definition of spatial travelling waves we seek for (6.1).

Definition 6.1. *We say that a function $n(t,x,\theta)$ is a travelling wave solution of speed $c\in\mathbb{R}^+$ in direction $e\in\mathbb{S}^n$ if it writes*

$$\forall (t,x,\theta)\in\mathbb{R}^+\times\mathbb{R}^n\times\Theta, \qquad n(t,x,\theta) := \mu\left(\xi := x\cdot e - ct, \theta\right),$$

where the profile $\mu\in\mathcal{C}_b^2\left(\mathbb{R}\times\Theta\right)$ is nonnegative, satisfies

$$\liminf_{\xi\to-\infty} \mu\left(\xi,\cdot\right) > 0, \qquad \lim_{\xi\to+\infty}\mu\left(\xi,\cdot\right) = 0,$$

pointwise and solves

$$\begin{cases} -c\partial_\xi\mu = \theta\partial_{\xi\xi}\mu + \alpha\partial_{\theta\theta}\mu + r\mu(1-\nu), & (\xi,\theta)\in\mathbb{R}\times\Theta, \\ \partial_\theta\mu(\xi,\theta_{min}) = \partial_\theta\mu(\xi,\theta_{max}) = 0, & \xi\in\mathbb{R}. \end{cases} \tag{6.2}$$

where ν is the macroscopic density associated to μ, that is $\nu\left(\xi\right) = \int_\Theta \mu\left(\xi,\theta\right)d\theta$.

To state the main existence result we first need to explain which heuristic considerations yield to the derivation of possible speeds for fronts. As for the standard Fisher-KPP equations, we expect that the fronts we build in this work are so-called *pulled fronts* : they are driven by the dynamics of small populations at the edge of the front. In this case, the speed of the front can be obtained through the linearized equation of (6.2) around $\mu << 1$. The resulting equation (which is now a local elliptic equation) writes

$$\begin{cases} -c\partial_\xi \widetilde{\mu} = \theta \partial_{\xi\xi}\widetilde{\mu} + \alpha \partial_{\theta\theta}\widetilde{\mu} + r\widetilde{\mu}, & (\xi,\theta) \in \mathbb{R}\times\Theta, \\ \partial_\theta \widetilde{\mu}(\xi,\theta_{\min}) = \partial_\theta \widetilde{\mu}(\xi,\theta_{\max}) = 0, & \xi \in \mathbb{R}. \end{cases} \tag{6.3}$$

Particular solutions of (6.3) are a combination of an exponential decay in space and a monotonic profile in trait :

$$\forall (\xi,\theta) \in \mathbb{R}\times\Theta, \qquad \widetilde{\mu}(\xi,\theta) = Q_\lambda(\theta)e^{-\lambda\xi},$$

where $\lambda > 0$ represents the spatial decreasing rate and Q_λ the trait profile. The pair $(c(\lambda), Q_\lambda)$ solves the following *spectral problem* :

$$\begin{cases} \alpha Q_\lambda(\theta)'' + \left(-\lambda c(\lambda) + \theta\lambda^2 + r\right) Q_\lambda(\theta) = 0\,, & \theta \in \Theta, \\ \partial_\theta Q_\lambda\left(\theta_{\min}\right) = \partial_\theta Q_\lambda\left(\theta_{\max}\right) = 0, \\ Q_\lambda(\theta) > 0,\ \int_\Theta Q_\lambda(\theta)\, d\theta = 1\,. \end{cases} \tag{6.4}$$

We refer to Section 6.2, Proposition 6.5 for a proof showing that (6.4) has a unique solution $(c(\lambda), Q_\lambda)$ for all $\lambda > 0$. We also prove there that we can define the minimal speed c^* and its associated decreasing rate through the following formula :

$$c^* := c(\lambda^*) = \min_{\lambda>0} c(\lambda). \tag{6.5}$$

Remark 6.2. *We emphasize that this structure of spectral problem giving information about propagation in models of "kinetic" type is quite robust. We refer to [4, 23, 35, 36] for works where this kind of dispersion relations also give the speed of propagation of possible travelling wave solutions, and to [32, 33, 37] for recent works where the same kind of spectral problem appears to find the limiting Hamiltonian in the WKB expansion of hyperbolic limits.*

We are now ready to state the main Theorem of this paper :

Theorem 6.3. *Let $\Theta := (\theta_{min}, \theta_{max})\,, \theta_{min} > 0, \theta_{min} < +\infty$ and c^* be the minimal speed defined after* (6.5)*. Then, there exists a travelling wave solution of* (6.1) *of speed c^* in the sense of Definition 6.1 .*

This Theorem, together with the heuristic argument, has been announced in [34].

Remark 6.4. *As in [4, 11], we expect that waves going with faster speeds $c > c^*$ do exist and are constructible by a technique of sub- and super solutions. Nevertheless, since it does not make much difference with [4], we do not address this issue here.*

The paper is organized as follows. In Section 6.2, we study the spectral problem (6.4) and provide some qualitative properties. In Section 6.3, we elaborate a topological degree argument to solve (6.2) in a bounded slab. Finally in Section 6.4, we construct the profile going with speed c^* which proves the existence of Theorem 6.3.

6.2 The spectral problem.

We discuss the spectral problem naturally associated to (6.1) that we have stated in (6.4). We state and prove some useful properties of Q_λ and some relations between c^* and λ^*.

Proposition 6.5 (Qualitative properties of the spectral problem). *For all $\lambda > 0$, the spectral problem* (6.4) *has a unique solution $(c(\lambda), Q_\lambda)$. Moreover, the function $\lambda \mapsto c(\lambda)$ has a minimum, that we denote by c^* and that we call the minimal speed. We denote by $\lambda^* > 0$ an associated decreasing rate and $Q_{\lambda^*} := Q^*$ the corresponding profile. Then we have the following properties :*

(i) For all $\lambda > 0$, the profile Q_λ is increasing w.r.t θ. There exists θ_0 such that Q_λ is convex on $[\theta_{min}, \theta_0]$ and concave on $[\theta_0, \theta_{max}]$. Moreover, θ_0 satisfies $-\lambda c(\lambda) + \lambda^2\theta_0 + r = 0$

(ii) We define $\langle\theta_\lambda\rangle := \int_\Theta \theta Q_\lambda(\theta) d\theta$, the mean trait associated to the decay rate λ. We also define $\langle\theta^*\rangle := \langle\theta_{\lambda^*}\rangle$. One has

$$\forall\lambda > 0, \qquad -\lambda c(\lambda) + \lambda^2 \langle\theta_\lambda\rangle + r = 0, \qquad \langle\theta_\lambda\rangle > \frac{\theta_{max} + \theta_{min}}{2}. \tag{6.6}$$

(iii) About the special features of the minimal speed, we have

$$c^* > 2\sqrt{r\langle\theta^*\rangle}, \tag{6.7}$$

$$c^* \geq \lambda^* \left(\theta_{max} + \theta_{min}\right). \tag{6.8}$$

Remark 6.6. *Even if it does not play much role in the analysis, let us notice that from the same equation defining θ_0 and $\langle\theta_\lambda\rangle$, one can deduce that Q_λ changes its convexity at the mean trait.*

Proof of Proposition 6.5. We first prove the existence and uniqueness of $(c(\lambda), Q_\lambda)$ for all positive λ. Let $\beta > 0$ and K be the positive cone of nonnegative functions in $\mathcal{C}^{1,\beta}(\Theta)$. We define L on $\mathcal{C}^{1,\beta}(\Theta)$ as below

$$L(u) = -\alpha u''(\theta) - (\theta - \theta_{\max})\lambda^2 u(\theta).$$

The resolvent of L endowed with the Neumann boundary condition is compact from the regularizing effect of the Laplace term. Moreover, the strong maximum principle and the boundedness of Θ gives that it is strongly positive. Using the Krein-Rutman theorem we obtain that there exists a nonnegative eigenvalue $\frac{1}{\gamma(\lambda)}$, corresponding to a positive eigenfunction Q_λ. This eigenvalue is simple and none of the other eigenvalues corresponds to a positive eigenfunction. As a consequence, $\lambda c(\lambda) := r + \lambda^2\theta_{\max} - \gamma(\lambda)$ solves the problem.

We come to the proof of (i). Since $Q_\lambda \in \mathcal{C}^2(\Theta)$ and satisfies Neumann boundary conditions, there exists θ_0 such that $Q_\lambda''(\theta_0) = 0$. Since $-\lambda c(\lambda) + \lambda^2\theta + r$ is increasing with θ, the sign of Q_λ'' and thus the convexity of Q_λ follows. We deduce :

$$\lambda^2\theta_{min} + r \leq \lambda c(\lambda) \leq \lambda^2\theta_{max} + r.$$

This yields

$$c(\lambda) \underset{\lambda\to 0}{\sim} \frac{r}{\lambda}, \qquad \lambda c(\lambda) = \mathcal{O}_{\lambda\to+\infty}(\lambda^2).$$

These latter relations and the continuity of $\lambda \mapsto c(\lambda)$ give the existence of a positive minimal speed c^* and a smallest positive minimizer λ^*.

We now prove *(ii)*. We obtain the first relation of (6.6) after integrating (6.4) over Θ and recalling the Neumann boundary conditions. To get the second one, we divide the spectral problem by Q_λ and then integrate over Θ :

$$\langle \theta_\lambda \rangle = \frac{\theta_{max} + \theta_{min}}{2} + \frac{\alpha}{\lambda^2 |\Theta|} \int_\Theta \left| \frac{Q'_\lambda}{Q_\lambda} \right|^2 d\theta > \frac{\theta_{max} + \theta_{min}}{2}. \tag{6.9}$$

We finish with *(iii)*. For this purpose, we define $W_\lambda = (Q_\lambda)^2$. It satisfies Neumann boundary conditions on $\partial\Theta$ and

$$\forall \theta \in \Theta, \qquad \alpha W'' + 2\left(-\lambda c(\lambda) + \lambda^2 \theta + r\right) W = \alpha \left(\frac{W'}{2\sqrt{W}} \right)^2 \geq 0.$$

We thus deduce that

$$\lambda^2 \int_\Theta \theta W d\theta + (-\lambda c(\lambda) + r) \int_\Theta W d\theta > 0,$$

from which we deduce

$$\frac{\int_\Theta \theta \left(Q^*\right)^2 d\theta}{\int_\Theta \left(Q^*\right)^2 d\theta} > \langle \theta^* \rangle . \tag{6.10}$$

Differentiating (6.4) with respect to λ, we obtain

$$\left(-\lambda c'(\lambda) - c(\lambda) + 2\theta\lambda\right) Q_\lambda + \left(-\lambda c(\lambda) + \theta\lambda^2 + r\right) \frac{\partial Q_\lambda}{\partial \lambda} + \alpha \partial_{\theta\theta} \left(\frac{\partial Q_\lambda}{\partial \lambda} \right) = 0.$$

We do not have any information about $\frac{\partial Q_\lambda}{\partial \lambda}$. Nevertheless, one can overcome this issue by testing directly against Q_λ. We obtain, for $\lambda = \lambda^*$:

$$-c^* \int_\Theta \left(Q^*\right)^2 d\theta + 2\lambda^* \int_\Theta \theta \left(Q^*\right)^2 d\theta = 0,$$

since $c'(\lambda^*) = 0$. As a consequence

$$c^* = 2\lambda^* \frac{\int_\Theta \theta \left(Q^*\right)^2 d\theta}{\int_\Theta \left(Q^*\right)^2 d\theta}. \tag{6.11}$$

Combining (6.11) with $-\lambda^* c^* + (\lambda^*)^2 \langle \theta^* \rangle + r = 0$, one obtains

$$\frac{(c^*)^2}{4r} = \frac{1}{2} \left(\frac{\int_\Theta \theta \left(Q^*\right)^2 d\theta}{\int_\Theta \left(Q^*\right)^2 d\theta} \right)^2 \left(\frac{\int_\Theta \theta \left(Q^*\right)^2 d\theta}{\int_\Theta \left(Q^*\right)^2 d\theta} - \frac{\langle \theta^* \rangle}{2} \right)^{-1}. \tag{6.12}$$

which gives (6.7) since $\frac{1}{2} \left(\frac{\int_\Theta \theta (Q^*)^2 d\theta}{\int_\Theta (Q^*)^2 d\theta} \right)^2 \left(\frac{\int_\Theta \theta (Q^*)^2 d\theta}{\int_\Theta (Q^*)^2 d\theta} - \frac{\langle \theta^* \rangle}{2} \right)^{-1} \geq \langle \theta^* \rangle$ always holds true and (6.10) rules out equality.

Finally, using (6.6) and (6.11), one has

$$c^* > 2\lambda^* \langle \theta^* \rangle \geq 2\lambda^* \frac{\theta_{max} + \theta_{min}}{2} = \lambda^* \left(\theta_{max} + \theta_{min} \right).$$

□

6.3 Solving the problem in a bounded slab.

In this Section, we solve an approximated problem in a bounded slab $(-a, a) \times \Theta$ as a first step to solve (6.2).

Definition 6.7. *For all $\tau \geq 0$, we define*

$$\forall \theta \in \Theta, \qquad g_\tau(\theta) = \theta_{min} + \tau\,(\theta - \theta_{min})\,.$$

Now, for all $a > 0$, the slab problem $P_{\tau,a}$ is defined as follows on $[-a, a] \times \Theta$:

$$[P_{\tau,a}] \begin{cases} -c\partial_\xi \mu^a - g_\tau(\theta)\partial_{\xi\xi}\mu^a - \alpha\partial_{\theta\theta}\mu^a = r\mu^a(1-\nu^a)\,, & \mu^a \geq 0, \quad (\xi,\theta) \in (-a,a)\times\Theta, \\ \partial_\theta \mu^a(\xi, \theta_{\min}) = \partial_\theta \mu^a(\xi,\theta_{\max}) = 0\,, \quad \xi \in (-a,a), \\ \mu^a(-a,\theta) = |\Theta|^{-1}\,, \quad \mu^a(a,\theta) = 0\,, \quad \theta \in \Theta. \end{cases} \tag{6.13}$$

with $\nu^a := \int_\Theta \mu^a(\cdot,\theta)d\theta$ and the supplementary renormalization condition $\nu^a(0) = \varepsilon$. For legibility, we set $P_{1,a} := P_a$.

In this problem, the speed c is an unknown as well as μ^a. Moreover, without the supplementary renormalization condition $\nu^a(0) = \varepsilon$, the problem is underdetermined. Indeed, this additional condition is needed to ensure compactness of the family (c^a, μ^a) when a goes to $+\infty$, since the limit problem (6.2) is translation invariant. The boundary condition in $-a$ is chosen this way since we heuristically expect that the population is uniform in trait at the back of the front as observed in the ecological problem, see [181]. However, although we fix this boundary condition in the slab, let us recall again that in general the behavior at the back of the front for the limit problem is not easy to figure out due to possible Turing instabilities. The non-local character of the source term does not provide any full comparison principle for $P_{\tau,a}$. We will prove the existence of a non-negative solution of (6.13), but we don't claim that all the solutions of this slab problem are non-negative. We follow [4, 26] and shall use the Leray-Schauder theory. For this purpose, some uniform *a priori* estimates (with respect to τ, a) on the solutions of the slab problem are required. The main difference with [4, 26] is that it is more delicate to obtain these uniform L^∞ estimates since it is not possible to write neither a useful equation nor an inequation on ν due to the term $\theta\partial_{\xi\xi}\mu$ (as it is the case in kinetic equations). Our strategy is the following. We first prove in Lemma 6.9 that the speed is uniformly bounded from above. Then, Lemmas 6.10 and 6.11 focus on the case $c = 0$ and prove that there cannot exist any solution to the slab problem in this case, provided that the normalization ε is well chosen. Finally, when the speed is given and uniformly bounded, we can derive a uniform *a priori* estimate on the solutions of the slab problem (6.13). Thanks to these *a priori* estimates, we apply a Leray-Schauder topological degree argument with the parameter τ in Proposition 6.14. This strategy is reliable as the problem corresponding to $\tau = 0$ is easier to solve since it is more or less a standard Fisher-KPP equation. All along Section 6.3, we omit the superscript a in μ^a and ν^a.

6.3.1 A Harnack inequality up to the boundary.

We shall apply several times the following useful Harnack inequality for (6.2), which is true up to the boundary in the direction θ. This is possible thanks to the Neumann boundary conditions in this direction.

Proposition 6.8. *Suppose that μ is a positive solution of* (6.2) *such that the total density ν is locally bounded. Then for all $0 < b < +\infty$, there exists a constant $C(b) < +\infty$ such that the following Harnack inequality holds :*

$$\forall(\xi,\theta,\theta') \in (-b,b)\times\Theta\times\Theta, \qquad \mu(\xi,\theta) \le C(b)\mu(\xi,\theta').$$

Proof of Proposition 6.8. One has to figure out how to obtain the validity of the Harnack inequality up to the boundary in Θ. Indeed, it holds on sub-compacts sets thanks to the standard elliptic regularity, given that the density ν is bounded. To obtain the full Harnack estimate, we consider the equation (6.2) after a reflection with respect to $\theta = \theta_{min}$ and $\theta = \theta_{max}$ and for positive values of θ. One obtains the following equation in the weak sense

$$\forall(\xi,\theta) \in \mathbb{R}\times\left(\mathbb{R}^{+*}\cap(\mathbb{R}\backslash\{\theta_{min}+\Theta\mathbb{Z}\})\right),$$
$$-c\partial_\xi\mu(\xi,\theta) - g(\theta)\partial_{\xi\xi}\mu(\xi,\theta) - \alpha\partial_{\theta\theta}\mu(\xi,\theta) = r\mu(\xi,\theta)(1-\nu(t,\xi)).$$

The crucial point is that this equation is also satisfied on the boundaries $\theta = \mathbb{R}^+\cap\{\theta_{min}+\Theta\mathbb{Z}\}$ thanks to the Neumann boundary conditions. Indeed, no Dirac mass in $\theta = \mathbb{R}^+\cap\{\theta_{min}+\Theta\mathbb{Z}\}$ arises while computing the second derivative $\partial_{\theta\theta}$ in the symmetrized equation.

□

6.3.2 An upper bound for c.

Lemma 6.9. *For any normalization parameter $\varepsilon > 0$, there exists a sufficiently large $a_0(\varepsilon)$ such that any pair (c,μ) solution of the slab problem $P_{\tau,a}$ with $a \ge a_0(\varepsilon)$ (and $\mu \ge 0$) satisfies $c \le c_\tau^* \le c^*$, where c_τ^* is defined after solving* (6.15) *below.*

Proof of Lemma 6.9. We just adapt an argument from [4, 26]. It consists in finding a relevant subsolution for a related problem. As $\mu \ge 0$, one has

$$\forall(\xi,\theta) \in (-a,a)\times\Theta, \qquad -c\partial_\xi\mu \le g_\tau(\theta)\partial_{\xi\xi}\mu + \alpha\partial_{\theta\theta}\mu + r\mu. \tag{6.14}$$

As (6.4), the following pertubated spectral problem has a unique solution associated with a minimal speed c_τ^* :

$$\begin{cases} \alpha Q_\tau^*(\theta)'' + \left(-\lambda_\tau^* c_\tau^* + g_\tau(\theta)\left(\lambda_\tau^*\right)^2 + r\right) Q_\tau^*(\theta) = 0\,, & \theta\in\Theta, \\ \left(Q_\tau^*\right)'(\theta_{\min}) = \left(Q_\tau^*\right)'(\theta_{\max}) = 0, & \\ Q_\tau^*(\theta) > 0,\ \int_\Theta Q_\tau^*(\theta)\,d\theta = 1\,. & \end{cases} \tag{6.15}$$

Let us assume by contradiction that $c > c_\tau^*$, then the family of functions $\psi_A(\xi,\theta) := Ae^{-\lambda_\tau^*\xi}Q_\tau^*(\theta)$ verifies

$$\forall(\xi,\theta) \in (-a,a)\times\Theta, \qquad g_\tau(\theta)\partial_{\xi\xi}\psi_A + \alpha\partial_{\theta\theta}\psi_A + r\psi_A = \lambda_\tau^* c_\tau^*\psi_A < -c\partial_\xi\psi_A, \tag{6.16}$$

As the eigenvector Q^* is positive, and $\mu \in L^\infty(-a,a)$, one has $\mu \le \psi_A$ for A sufficiently large. As a consequence, one can define

$$A_0 = \inf\left\{A \mid \forall(\xi,\theta)\in(-a,a)\times\Theta,\ \psi_A(\xi,\theta) > \mu(\xi,\theta)\right\}.$$

Necessarily, $A_0 > 0$ and there exists a point $(\xi_0, \theta_0) \in [-a, a] \times [\theta_{\min}, \theta_{\max}]$ where ψ_{A_0} touches μ :

$$\mu(\xi_0, \theta_0) = \psi_{A_0}(\xi_0, \theta_0).$$

This point minimizes $\psi_{A_0} - n$ and cannot be in $(-a, a) \times \Theta$. Indeed, combining (6.14) and (6.16), one has in the interior,

$$\forall (\xi, \theta) \in (-a, a) \times \Theta,$$
$$c\partial_\xi \left(\psi_{A_0} - \mu\right) + g_\tau(\theta)\partial_{\xi\xi} \left(\psi_{A_0} - \mu\right) + \alpha\partial_{\theta\theta} \left(\psi_{A_0} - \mu\right) + r\left(\psi_{A_0} - \mu\right) < 0.$$

But, if (ξ_0, θ_0) is in the interior, this latter inequality cannot hold since

$$g_\tau(\theta)\partial_{\xi\xi} \left(\psi_{A_0} - \mu\right) + \alpha\partial_{\theta\theta} \left(\psi_{A_0} - \mu\right) \geq 0.$$

Next we eliminate the boundaries. First, (ξ_0, θ_0) cannot lie in the right boundary $\{x = a\} \times \Theta$ since $\psi_{A_0} > 0$ and $\mu = 0$ there. Moreover, thanks to the Neumann boundary conditions satisfied by both ψ_{A_0} and μ, (ξ_0, θ_0) cannot be in $[-a, a] \times \{\theta_{\min}, \theta_{\max}\}$, thanks to Hopf's Lemma. We now exclude the left boundary by adjusting the normalization. If $\xi_0 = -a$, then $\psi_{A_0}(\xi_0, \theta_0) = |\Theta|^{-1}$ and $A_0 = \frac{e^{-\lambda_\tau^* a}}{|\Theta| Q_\tau^*(\theta_0)}$. Then $\nu(0) \leq \frac{e^{-\lambda_\tau^* a}}{|\Theta| Q_\tau^*(\theta_0)}$ which is smaller than ε for a sufficiently large a. We thus conclude that $c \leq c_\tau^*$. We shall now prove that for all $\tau \in [0, 1]$, one has $c_\tau^* \leq c^*$. Differentiating (6.15) with respect to τ and testing against Q_τ^*, one obtains, similarly as in Proof of Proposition 6.5 (iii),

$$\int_\Theta \left[\frac{d\lambda}{d\tau} \left(2\lambda_\tau^* g_\tau(\theta) - c_\tau^*\right) + \left(\lambda_\tau^*\right)^2 g_\tau'(\theta) - \lambda_\tau^* \frac{dc_\tau^*}{d\tau} \right] \left(Q_\tau^*\right)^2 d\theta = 0.$$

But now recalling (6.11), which writes as follows in the τ-case :

$$c_\tau^* = 2\lambda_\tau^* \frac{\int_\Theta g_\tau(\theta) \left(Q_\tau^*\right)^2 d\theta}{\int_\Theta \left(Q_\tau^*\right)^2 d\theta}, \tag{6.17}$$

one obtains

$$\frac{dc_\tau^*}{d\tau} = \lambda_\tau^* \frac{\int_\Theta g_\tau'(\theta) \left(Q_\tau^*\right)^2 d\theta}{\int_\Theta \left(Q_\tau^*\right)^2 d\theta}.$$

We deduce that c_τ^* is increasing with respect to τ, so that $c_\tau^* \leq c_1^* = c^*$.

□

6.3.3 The special case $c = 0$.

We now focus on the special case $c = 0$. We first show (Lemma 6.10) that the density μ is uniformly bounded (with respect to $a > 0$). From this estimate, we deduce in Lemma 6.11 that there exists a constant ε_0 depending only on the fixed parameters of the problem such that necessarily $\nu(0) \geq \varepsilon_0$. Thus, provided that ε is set sufficiently small, our analysis will conclude that the slab problem does not admit a solution of the form $(c, \mu) = (0, \mu)$ for $\varepsilon < \varepsilon_0$. We emphasize that the key *a priori* estimate, *i.e.* $\nu \in L^\infty\left((-a, a) \times \Theta\right)$, is easier to obtain in the case $c = 0$ than in the case $c \neq 0$ (compare Lemmas 6.10 and 6.12).

A priori estimate for μ when $c = 0$.

Lemma 6.10. (A priori estimates, $c = 0$).
Assume $c = 0$, $b > 0$ and $\tau \in [0,1]$. There exists a constant $C(b)$ such that every solution $(c = 0, \mu)$ of (6.13) *satisfies*

$$\forall (\xi,\theta) \in [-b,b] \times \Theta, \quad \mu(\xi,\theta) \leq \frac{C(b)}{\Theta} \frac{\theta_{max}}{\theta_{min}}.$$

Proof of Lemma 6.10. When $c = 0$, the slab problem (6.13) reduces to

$$[P_{\tau,b}] \begin{cases} -g_\tau(\theta)\partial_{\xi\xi}\mu - \alpha\partial_{\theta\theta}\mu = r\mu(1-\nu), & (\xi,\theta) \in (-b,b) \times \Theta, \\ \partial_\theta\mu(\xi,\theta_{\min}) = \partial_\theta\mu(\xi,\theta_{\max}) = 0, & \xi \in (-b,b), \\ \mu(-b,\theta) = |\Theta|^{-1}, \quad \mu(b,\theta) = 0, & \theta \in \Theta. \end{cases}$$

Integration with respect to the trait variable θ yields

$$\begin{cases} -\partial_{\xi\xi}\left(\int_\Theta g_\tau(\theta)\mu(x,\theta)d\theta\right) = r\nu(\xi)(1-\nu(\xi)), & \xi \in \mathbb{R}, \\ \nu(-b) = 1, \quad \nu(b) = 0. \end{cases}$$

Take a point ξ_0 where $\int_\Theta g_\tau(\theta)\mu(\xi,\theta)d\theta$ attains a maximum. At this point, one has necessarily $\nu(\xi_0) \leq 1$. The following sequence of inequalities holds true for all $\xi \in (-b,b)$:

$$\begin{aligned} \theta_{\min}\nu(\xi) = g_\tau(\theta_{min})\nu(\xi) = g_\tau(\theta_{min})\int_\Theta \mu(\xi,\theta)d\theta \leq \int_\Theta g_\tau(\theta)\mu(\xi,\theta)d\theta \\ \leq \int_\Theta g_\tau(\theta)\mu(\xi_0,\theta)d\theta \leq g_\tau(\theta_{max})\nu(\xi_0) \leq g_\tau(\theta_{max}), \end{aligned}$$

and give

$$\forall \xi \in (-b,b), \quad \nu(\xi) \leq \frac{g_\tau(\theta_{max})}{\theta_{min}} \leq \frac{\theta_{max}}{\theta_{min}}.$$

Now, the Harnack inequality of Proposition 6.8 gives

$$\forall (\xi,\theta) \in (-b,b) \times \Theta, \qquad n(\xi,\theta) \leq \frac{C(b)}{|\Theta|}\nu(\xi) \leq \frac{C(b)}{|\Theta|}\frac{\theta_{max}}{\theta_{min}}.$$

□

Non-existence of solutions of the slab problem when $c = 0$.

Lemma 6.11. (Lower bound for $\nu(0)$ when $c = 0$). *There exists $\varepsilon_0 > 0$ such that if a is large enough, then for all $\tau \in [0,1]$, any (non-negative) solution of the slab problem $(c = 0, \mu)$ satisfies $\nu(0) > \varepsilon_0$.*

Proof of Lemma 6.11. We adapt an argument from [4]. It is a bit simpler here since the trait space is bounded. For $b > 0$, consider the following spectral problem in both variables (ξ,θ) :

$$\begin{cases} g_\tau(\theta)\partial_{\xi\xi}\varphi_b + \alpha\partial_{\theta\theta}\varphi_b + r\varphi_b = \psi_b\varphi_b, & (\xi,\theta) \in (-b,b) \times \Theta, \\ \partial_\theta\varphi_b(\xi,\theta_{\min}) = \partial_\theta\varphi_b(\xi,\theta_{\max}) = 0, & \xi \in (-b,b), \\ \varphi_b(-b,\theta) = 0, \quad \varphi_b(b,\theta) = 0, & \theta \in \Theta. \end{cases} \tag{6.18}$$

Again, by Krein-Rutman theory, ψ_b is the only eigenvalue such that there exists a positive eigenvector φ_b. One can rescale the problem in the space direction setting $\xi = b\zeta$:

$$\begin{cases} \dfrac{g_\tau(\theta)}{b^2}\partial_{\zeta\zeta}\varphi_b + \alpha\partial_{\theta\theta}\varphi_b + r\varphi_b = \psi_b\varphi_b\,, \quad (\zeta,\theta) \in (-1,1)\times\Theta\,, \\ \partial_\theta\varphi_b(\zeta,\theta_{\min}) = \partial_\theta\varphi_b(\zeta,\theta_{\max}) = 0\,, \quad \zeta \in (-1,1)\,, \\ \varphi_b(-1,\theta) = 0\,, \quad \varphi_b(1,\theta) = 0\,, \quad \theta \in \Theta. \end{cases}$$

One can prove that $\lim_{b\to+\infty}\psi_b = r$. We give a sketch of proof for the sake of completeness. We introduce the problem

$$\begin{cases} \alpha V_b'' + \left(-\frac{\pi^2}{4}\dfrac{g_\tau(\theta)}{b^2} - \psi_b + r\right)V_b = 0\,, \quad V_b > 0, \qquad \theta\in\Theta, \\ V_b'(\theta_{\min}) = V_b'(\theta_{\max}) = 0\,. \end{cases}$$

The eigenvector (up to a multiplicative constant) φ_b is then given by

$$\forall(\zeta,\theta)\in(-1,1)\times\Theta, \qquad \varphi_b = \sin\left(\frac{\pi}{2}(\zeta+1)\right)V_b(\theta).$$

Moreover, one has

$$\frac{d\psi_b}{db} = \frac{\pi^2}{2b^3}\frac{\int_\Theta g_\tau(\theta)V_b^2 d\theta}{\int_\Theta V_b^2 d\theta}$$

so that $\lim_{b\to+\infty}\psi_b$ exists and solves

$$\begin{cases} \alpha V'' + (-\lim_{b\to+\infty}\psi_b + r)\,V = 0\,, \quad V > 0, \qquad \theta\in\Theta\,, \\ V'(\theta_{\min}) = V'(\theta_{\max}) = 0\,, \end{cases}$$

and it yields necessarily that V is constant and $\lim_{b\to+\infty}\psi_b = r$. As a consequence, we fix b sufficiently large to have $\psi_b > \frac{r}{2}$.

Thanks to the Harnack inequality (of Proposition 6.8), there exists a constant $C(b)$ which does not depend on $a > b$ such that

$$\forall\theta\in\Theta, \qquad C(b)\mu(0,\theta) \geq C(b)\inf_{(-b,b)\times\Theta}\mu(\xi,\theta) \geq \|\mu\|_{L^\infty((-b,b)\times\Theta)}.$$

To compare (6.13) to (6.18), one has, for all $(\xi,\theta)\in[-b,b]\times\Theta$,

$$g_\tau(\theta)\partial_{\xi\xi}\mu + \partial_{\theta\theta}\mu + r\mu = r\mu\nu \leq r\mu|\Theta|\|\mu\|_{L^\infty((-b,b)\times\Theta)} \leq rC\nu(0)\mu(\xi,\theta).$$

We deduce from this computation that as soon as $\nu(0) \leq \frac{1}{2C(b)}$, one has

$$\forall(\xi,\theta)\in[-b,b]\times\Theta, \quad rC\nu(0)\mu(\xi,\theta) < \psi_b\mu(\xi,\theta),$$

and this means that μ is a subsolution of (6.18). We can now use the same arguments as for the proof of Lemma 6.9. We define

$$A_0 = \max\left\{A \mid \forall(\xi,\theta)\in[-b,b]\times\Theta,\ A\varphi_b(\xi,\theta) < \mu(\xi,\theta)\right\},$$

so that $u_{A_0} := \mu - A_0\varphi_b$ has a zero minimum in (ξ_0, θ_0) and satisfies

$$\begin{cases} -g_\tau(\theta)\partial_{\xi\xi}u_{A_0} - \alpha\partial_{\theta\theta}u_{A_0} - ru_{A_0} > -\psi_b u_{A_0}\,, & (\xi,\theta) \in (-b,b)\times\Theta\,, \\ \partial_\theta u_{A_0}(\xi,\theta_{\min}) = \partial_\theta u_{A_0}(\xi,\theta_{\max}) = 0\,, & \xi \in (-b,b)\,, \\ u_{A_0}(-b,\theta) > 0\,, \quad u_{A_0}(b,\theta) > 0\,, & \theta \in \Theta. \end{cases}$$

For the same reasons as in Lemma 6.9 this cannot hold, so that necessarily $\nu(0) > \varepsilon_0 := \frac{1}{2C(b)}$. □

6.3.4 Uniform bound over the steady states, for $0 \le c \le c^*$.

The previous Subsection is central in our analysis. Indeed, it gives a bounded set of speeds where to apply the Leray-Schauder topological degree argument, namely we can restrict ourselves to speeds $c \in [0, c^*]$. Based on this observation, we are now able to derive a uniform L^∞ estimate (with respect to a and τ) for solutions μ of (6.13). This is done in Lemma 6.12 below.

Lemma 6.12. (A priori estimates, $c \in [0, c^*]$).
Assume $c \in [0, c^*]$, $\tau \in [0,1]$ and $a \ge 1$. Then there exists a constant C_0, depending only on the biological parameters $\theta_{\min}$, $|\Theta|$, r and α, such that any solution (c,μ) (with $\mu \ge 0$) of the slab problem $P_{a,\tau}$ satisfies

$$\|\mu\|_{L^\infty((-a,a)\times\Theta)} \le C_0\,.$$

Proof of Lemma 6.12. We divide the proof into two steps. In the first step, we prove successively that μ and $\partial_\theta\mu$ are bounded uniformly in $H^1\left((-a,a)\times\Theta\right)$. In the second step, we use a suitable trace inequality to deduce a uniform $L^\infty\left((-a,a)\times\Theta\right)$ estimate on μ. We define $K_0(a) = \max_{[-a,a]\times\Theta}\mu$. We want to prove that $K_0(a)$ is in fact bounded uniformly in a.

The argument is inspired from [26]. The principle of the proof goes as follows : The maximum principle applied to (6.13) implies that $\nu(\xi_0) \le 1$ if (ξ_0,θ_0) is a maximum point for μ. This does not imply that $\max\mu \le 1$. However, we can control $\mu(\xi_0,\theta_0)$ by the non local term $\nu(\xi_0)$ provided some regularity of μ in the direction θ. In order to get this additional regularity we use the particular structure of the equation (the nonlocal term does not depend on θ and is non negative).

 # Step 0 : Preliminary observations.

Denote by (ξ_0,θ_0) a point where the maximum of μ is reached. If the maximum is attained on the $\xi-$boundary $\xi_0 = \pm a$ then $K_0(a) \le |\Theta|^{-1}$ by definition. If it is attained on the $\theta-$boundary $\theta_0 \in \{\theta_{\min},\theta_{\max}\}$, then the tangential derivative $\partial_\xi\mu$ necessarily vanishes, and the first derivative $\partial_\theta\mu$ vanishes thanks to the boundary condition. Hence $\partial_{\theta\theta}\mu(\xi_0,\theta_0) \le 0$ and $\partial_{\xi\xi}\mu(\xi_0,\theta_0) \le 0$. The same holds true if (ξ_0,θ_0) is an interior point. Evaluating equation (6.13) at (ξ_0,θ_0) implies

$$K_0(a)(1-\nu(\xi_0)) \ge 0\,,$$

and therefore $\nu_0(\xi_0) \le 1$.

 # Step 1 : Energy estimates on μ.

We derive local energy estimates. We introduce a smooth cut-off function $\chi : \mathbb{R} \to [0,1]$ such that

$$\begin{cases} \chi = 1 & \text{on} \quad J_1 = \left(\xi_0 - \frac{1}{2}, \xi_0 + \frac{1}{2}\right), \\ \chi = 0 & \text{outside} \quad J_2 = [\xi_0 - 1, \xi_0 + 1]. \end{cases}$$

Notice that the support of the cut-off function does not necessarily avoid the ξ−boundary. We also introduce the following linear corrector

$$\forall \xi \in [-a,a], \qquad m(\xi) = \frac{1}{|\Theta|}\frac{a-\xi}{2a},$$

which is defined such that $m(-a) = |\Theta|^{-1}$, $m(a) = 0$, and $0 \le m \le |\Theta|^{-1}$ on $(-a,a)$. Testing against $(\mu - m)\chi$ over $[-a,a]\times\Theta$, we get

$$- c\int_{(-a,a)\times\Theta} (\mu - m)\chi \partial_\xi \mu \, d\xi d\theta - \int_{(-a,a)\times\Theta} g_\tau(\theta)\partial_{\xi\xi}(\mu - m)(\mu - m)\chi \, d\xi d\theta$$
$$- \int_{(-a,a)\times\Theta} \alpha \partial_{\theta\theta}\mu(\mu - m)\chi \, d\xi d\theta = \int_{(-a,a)\times\Theta} r\mu(1-\nu)(\mu - m)\chi \, d\xi d\theta.$$

We now transform each term of the l.h.s. by integration by parts. We emphasize that the linear correction m ensures that all the boundary terms vanish. We get

$$\int_{(-a,a)\times\Theta} g_\tau(\theta)\left|\partial_\xi(\mu - m)\right|^2 \chi \, d\xi d\theta + \int_{(-a,a)\times\Theta} \alpha \left|\partial_\theta \mu\right|^2 \chi \, d\xi d\theta$$
$$\le \frac{1}{2}\int_{(-a,a)\times\Theta} g_\tau(\theta)(\mu - m)^2\chi'' \, d\xi d\theta + c\frac{|\Theta|^{-1}}{2a}\int_{(-a,a)\times\Theta} \chi(\mu - m) d\xi d\theta$$
$$- c\int_{(-a,a)\times\Theta} \frac{1}{2}(\mu - m)^2\chi' d\xi d\theta + \int_{(-a,a)\times\Theta} r\mu^2\chi \, d\xi d\theta + \int_{(-a,a)\times\Theta} r\mu\nu m\chi \, d\xi d\theta.$$

We use that $\mu \le K_0(a)$, $\nu(\xi) \le |\Theta| K_0(a)$, $g_\tau(\theta) \ge \theta_{\min}$ and $|c| \le c^*$ to get

$$\theta_{\min}\int_{J_1\times\Theta} \left|\partial_\xi \mu - m'\right|^2 \, d\xi d\theta + \int_{J_1\times\Theta} \alpha \left|\partial_\theta \mu\right|^2 \, d\xi d\theta$$
$$\le c^* \frac{|\Theta|^{-1}}{2a} K_0 |J_2 \times \Theta| - c\int_{[-a,a]\times\Theta} \frac{1}{2}(\mu - m)^2\chi' d\xi d\theta$$
$$+ \frac{1}{2}\int_{(-a,a)\times\Theta} g_\tau(\theta)(\mu - m)^2\chi'' \, d\xi d\theta + \int_{J_2\times\Theta} rK_0^2 \, d\xi d\theta + \int_{J_2\times\Theta} rK_0^2 \, d\xi d\theta,$$

Then we use the pointwise inequality $|\partial_\xi \mu - m_\xi|^2 \ge \partial_\xi \mu^2/2 - m_\xi^2$ in the first integral of the l.h.s. :

$$\frac{\theta_{\min}}{2}\int_{J_1} \left|\partial_\xi \mu\right|^2 \, d\xi d\theta + \int_{J_1} \alpha \left|\partial_\theta \mu\right|^2 \, d\xi d\theta \le \frac{K_0 c^*}{a} + \theta_{\min}\int_{J_1} \left|m'\right|^2 \, d\xi d\theta$$
$$+ \int g_\tau(\theta)\left(\mu^2 + m^2\right)\chi'' \, d\xi d\theta + c^*\int \left(\mu^2 + m^2\right)\chi' d\xi d\theta + 4r|\Theta| K_0^2.$$

Thus, we obtain our first energy estimate : $\mu \in H^1\left([-a,a]\times\Theta\right)$ with a uniform bound of order $\mathcal{O}\left(K_0(a)^2\right)$ uniformly :

$$\min\left(\frac{\theta_{\min}}{2}, 1\right)\int_{J_1} \left(\left|\partial_\xi \mu\right|^2 + \left|\partial_\theta \mu\right|^2\right) d\xi d\theta \le C(|\Theta|, \theta_{\min}, \chi)\left(1 + K_0(a)^2\right), \tag{6.19}$$

as soon as $a \geq \frac{1}{2}$.

We now come to the proof that $\partial_\theta \mu$ is also in $H^1((-a,a)\times\Theta)$. We differentiate (6.13) with respect to θ for this purpose. Here, we use crucially that ν is a function of the variable ξ only. Note that we cannot expect that $\mu \in H^2([-a,a]\times\Theta)$ with a bound of order $\mathcal{O}\left(K_0(a)^2\right)$ at this stage. But we need additional elliptic regularity in the variable θ only.

$$\forall(\xi,\theta)\in(-a,a)\times\Theta, \qquad -c\partial_{\xi\theta}\mu - \tau\partial_{\xi\xi}\partial\mu - g_\tau(\theta)\partial_{\xi\xi\theta}\mu - \alpha\partial_{\theta\theta\theta}\mu = r\partial_\theta\mu(1-\nu)\,. \tag{6.20}$$

We use the cut-off function $\widetilde{\chi}(\xi) = \chi(\xi_0 + 2(\xi-\xi_0))$, for which $\operatorname{Supp}\widetilde{\chi} \subset J_1$, and $\chi(\xi) = 1$ on $J_{1/2} = (\xi_0 - 1/4, \xi_0 + 1/4)$. Multiplying (6.20) by $\widetilde{\chi}\partial_\theta\mu$, we get after integration by parts

$$\begin{aligned}\int_{J_1} \tau\partial_\xi\mu\partial_{\theta\xi}\mu\widetilde{\chi}\,d\xi d\theta + \int_{J_1}\tau\partial_\xi\mu\partial_\theta\mu\widetilde{\chi}'\,d\xi d\theta + \int_{J_1} g_\tau(\theta)\partial_{\xi\theta}\mu\partial_\theta\mu\widetilde{\chi}'\,d\xi d\theta \\ + \int_{J_1} g_\tau(\theta)\left|\partial_{\xi\theta}\mu\right|^2\widetilde{\chi}\,d\xi d\theta + \alpha\int_{J_1}\left|\partial_{\theta\theta}\mu\right|^2\widetilde{\chi}\,d\xi d\theta \leq r\int_{J_1}\left|\partial_\theta\mu\right|^2\widetilde{\chi}\,d\xi d\theta + c\int_{J_1}\widetilde{\chi}'\frac{|\partial_\theta\mu|^2}{2}d\xi d\theta.\end{aligned}$$

Notice that all the boundary terms vanish since $\partial_\theta\mu = 0$ on all segments of the boundary. Using the H^1 estimate (6.19) obtained previously for μ, we deduce

$$\begin{aligned}\frac{\theta_{\min}}{2}\int_{J_{1/2}}\left|\partial_{\theta\xi}\mu\right|^2\,d\xi d\theta + \alpha\int_{J_{1/2}}\left|\partial_{\theta\theta}\mu\right|^2\,d\xi d\theta \\ \leq \left(r + \frac{c^*}{2}\|\widetilde{\chi}'\|_\infty\right)\int_{J_1}\left|\partial_\theta\mu\right|^2\,d\xi d\theta + \frac{1}{2\theta_{\min}}\int_{J_1}\left|\partial_\xi\mu\right|^2 d\xi d\theta \\ + \frac{1}{2}\int_{J_1}\left(\left|\partial_\xi\mu\right|^2 + \left|\partial_\theta\mu\right|^2\right)\left|\widetilde{\chi}'\right|\,d\xi d\theta + \frac{1}{2}\int\theta\left|\partial_\theta\mu\right|^2\widetilde{\chi}''\,d\xi d\theta\end{aligned}$$

from which we conclude

$$\min\left(\frac{\theta_{\min}}{2},1\right)\int_{J_1}\left(\left|\partial_{\xi\theta}\mu\right|^2 + \left|\partial_{\theta\theta}\mu\right|^2\right)\,d\xi d\theta \leq \overline{C}(\Theta,\theta_{\min},\chi)\left(1 + K_0(a)^2\right)\,. \tag{6.21}$$

This crucial computation proves that $\partial_\theta\mu$ also belongs to $H^1((-a,a)\times\Theta)$.

 # Step 2 : Improved regularity of the trace $\mu(\xi,\cdot)$.

We aim to control the regularity of the partial function $\theta \mapsto \mu(\xi_0,\theta)$. For this purpose we use a trace embedding inequality with higher derivatives, namely if both μ and $\partial_\theta\mu$ belongs to $H^1((-a,a)\times\Theta)$, then the trace function $\mu(\xi_0,\cdot)$ belongs to $H^{3/2}_\theta$. More precisely, there exists a constant C_{tr} such that

$$\|\mu(\xi_0,\cdot)\|^2_{H^{3/2}_\theta} \leq C_{tr}\left(\|\partial_\theta\mu\|^2_{H^1_{x,\theta}} + \|\mu\|^2_{H^1_{x,\theta}}\right)\,.$$

Combining the previous inequality with estimates (6.19) and (6.21) of # Step 1, we deduce that

$$\|\mu(\xi_0,\cdot)\|^2_{H^{3/2}_\theta} \leq C\left(1 + K_0(a)^2\right)\,.$$

On the other hand, the interpolation inequality [2, Theorem 5.9, p.141] gives a constant C_{int} such that

$$\|\mu(\xi_0,\cdot)\|_{L^\infty_\theta} \leq C_{int}\|\mu(\xi_0,\cdot)\|^{1/2}_{L^1_\theta}\|\mu(\xi_0,\cdot)\|^{1/2}_{H^{3/2}_\theta}$$

Recall from # Step 0, that $\nu(\xi_0) = \|\mu(\xi_0, \cdot)\|_{L^1_\theta} \leq 1$. As a consequence, we obtain

$$K_0(a)^4 = \|\mu(\xi_0, \cdot)\|^4_{L^\infty_\theta} \leq C\left(1 + K_0(a)^2\right),$$

for some constant C, depending only on Θ, $\theta_{\min}$, and χ. Therefore, $K_0(a)$ is bounded uniformly with respect to $a > 0$. This concludes the proof of Lemma 6.12.

□

6.3.5 Resolution of the problem in the slab.

We now finish the proof of the existence of solutions of (6.13). As previously explained, it consists in a Leray-Schauder topological degree argument. All uniform estimates derived in the previous Sections are key points to obtain *a priori* estimates on steady states of suitable operators. We then simplify the problem with homotopy invariances. We begin with a very classical problem : the construction of KPP travelling waves for the Fisher-KPP equation in a slab.

Lemma 6.13. *Let us consider the following Fisher-KPP problem in the slab $(-a, a)$:*

$$\begin{cases} -c\partial_\xi \nu - \theta_{min}\partial_{\xi\xi}\nu = r\nu(1-\nu), & \xi \in (-a, a), \\ \nu(-a) = 1, \quad \nu(a) = 0. \end{cases}$$

One has the following properties :

1. *For a given c, there exists a unique decreasing solution $\nu^c \in [0,1]$. Moreover, the function $c \to \nu^c$ is continuous and decreasing.*
2. *There exists $\varepsilon^* > 0$ (independent of a) such that the solution with $c = 0$ satisfies $\nu_{c=0}(0) > \varepsilon^*$.*
3. *For all $\varepsilon > 0$, there exists $a(\varepsilon)$ such that for all $c > 2\sqrt{r\theta_{min}}$, $\nu(0) < \varepsilon$ for $a \geq a(\varepsilon)$.*
4. *As a corollary of 2 and 3, for all $\varepsilon < \varepsilon^*$, there exists a unique $c_0 \in [0, 2\sqrt{r\theta_{min}}]$ such that $\nu_{c_0}(0) = \varepsilon$ for $a \geq a(\varepsilon)$.*

Proof of Lemma 6.13. The existence and uniqueness of solutions follows from [11]. By classical maximum principle arguments, $\nu \leq 1$. The inequality $\nu \geq 0$ is not as easily obtained. One needs to truncate the non-linearity replacing $\nu(1-\nu)$ by $\nu_+(1-\nu)$. We refer to Lemma 6.15 were the same argument is exposed.

The solution is necessarily decreasing since

$$\forall \xi \in (-a, a), \qquad \partial_\xi\left(e^{\frac{c}{\theta_{min}}\xi}\partial_\xi \nu\right) \leq 0,$$

and $\partial_\xi \nu(-a) \leq 0$. By classical arguments, the application $c \to \nu^c$ is continuous. For the decreasing character, we write, for $c_1 < c_2$ and $v := \nu_2 - \nu_1$:

$$-c_2\partial_\xi v - \theta_{min}\partial_{\xi\xi} v = (1 - (\nu_1 + \nu_2))\, v + (c_2 - c_1)\, \partial_\xi \nu_1,$$

so that v satisfies

$$\begin{cases} -c_2\partial_\xi v - \theta_{min}\partial_{\xi\xi} v \leq (1 - (\nu_1 + \nu_2))\, v, & \xi \in (-a, a), \\ v(-a) = 0, \quad v(a) = 0. \end{cases}$$

The comparison principle then yields that $v \leq 0$, that is $\nu_2 \leq \nu_1$. The proofs of Lemmas 6.9 and 6.11 can be adapted to prove the remainder of the Lemma.

□

With this ε^* in hand, we can state the main Proposition :

Proposition 6.14. (Solution in the slab). *Let $\varepsilon < \min(\varepsilon_0, \varepsilon^*)$. There exists $C_0 > 0$ and $a_0(\varepsilon) > 0$ such that for all $a \geq a_0$, the slab problem* (6.13) *with the normalization condition $\nu(0) = \varepsilon$ has a solution (c, μ) such that*

$$\|\mu\|_{L^\infty([-a,a]\times\Theta)} \leq C_0, \qquad c \in]\,0, c^*\,].$$

Proof of Proposition 6.14. Given a non negative function $\mu(\xi,\theta)$ satisfying the boundary conditions

$$\forall(\xi,\theta) \in [-a,a]\times\Theta, \qquad \partial_\theta\mu(\xi,\theta_{\min}) = \partial_\theta\mu(\xi,\theta_{\max}) = 0,$$
$$\mu(-a,\theta) = |\Theta|^{-1}, \qquad \mu(a,\theta) = 0\,, \tag{6.22}$$

we consider the one-parameter family of problems on $(-a,a)\times\Theta$:

$$\begin{cases} -c\partial_\xi Z^\tau - g_\tau(\theta)\partial_{\xi\xi}Z^\tau - \alpha\partial_{\theta\theta}Z^\tau = r\mu_+(1-\nu_\mu)\,, & (\xi,\theta)\in(-a,a)\times\Theta, \\ \partial_\theta Z^\tau(\xi,\theta_{\min}) = \partial_\theta Z^\tau(\xi,\theta_{\max}) = 0\,, & \xi\in(-a,a), \\ Z^\tau(-a,\theta) = |\Theta|^{-1}, Z^\tau(a,\theta) = 0, & \theta\in\Theta. \end{cases} \tag{6.23}$$

We have here introduced the notation ν_μ to emphasize that it corresponds to the density associated to μ and not to Z^τ. We have also introduced the function "positive part", defined as

$$\forall x\in\mathbb{R}, \qquad x_+ := x\mathbf{1}_{x\geq 0}.$$

We introduce the map

$$\mathcal{K}_\tau : (c,\mu) \to \left(\varepsilon - \nu_\mu(0) + c, Z^\tau\right),$$

where Z_τ is the solution of the previous linear system (6.23). The ellipticity of the system (6.23) gives that the map $\mathcal{K}_\tau$ is a compact map from

$$\left(X = \mathbb{R}\times\mathcal{C}^{1,\beta}\left((-a,a)\times\Theta\right), \|(c,\mu)\| = \max\left(|c|, \|\mu\|_{\mathcal{C}^{1,\beta}}\right)\right)$$

onto itself ($\forall\beta\in(0,1)$). Moreover, it depends continuously on the parameter $\tau\in[0,1]$. Before going any further, we shall prove that a fixed point (c,μ) of $\mathcal{K}_\tau$ gives a solution of $P_{\tau,a}$. For this purpose, one needs to check that such a fixed point defines a *nonnegative* density μ. We enlighten this property in the next Lemma.

Lemma 6.15. *A fixed point (c,μ) of $\mathcal{K}_\tau$ gives a solution of $P_{\tau,a}$.*

Proof of Lemma 6.15. Such a fixed point solves

$$\begin{cases} -c\partial_\xi\mu - g_\tau(\theta)\partial_{\xi\xi}\mu - \alpha\partial_{\theta\theta}\mu = r\mu_+(1-\nu_\mu)\,, & (\xi,\theta)\in(-a,a)\times\Theta, \\ \partial_\theta\mu(\xi,\theta_{\min}) = \partial_\theta\mu(\xi,\theta_{\max}) = 0\,, & \xi\in(-a,a), \\ \mu(-a,\theta) = |\Theta|^{-1}, \quad \mu(a,\theta) = 0\,, & \theta\in\Theta. \end{cases}$$

with $\nu := \int_\Theta \mu(\cdot,\theta)d\theta$ and the supplementary renormalization condition $\nu(0) = \varepsilon$. It remains to show that μ is then nonnegative. We play with the maximum principle as in [26]. Suppose that μ attains a negative minimum at some point (ξ_0,θ_0). Necessarily, $\xi_0 \neq \pm a$ due to the imposed Dirichlet boundary conditions, and the Neumann boundary condition in θ rules out $\theta_0 \in \partial\Theta$ by the strong maximum principle. Moreover, if $(\xi_0, \theta_0) \in (-a, a) \times \Theta$, then by continuity of μ, one can find an open set $\mathcal{V} \subset (-a, a) \times \Theta$ containing (ξ_0, θ_0) such that one has,

$$\forall (\xi, \theta) \in \mathcal{V}, \qquad -c\partial_\xi \mu - g_\tau(\theta)\partial_{\xi\xi}\mu - \alpha\partial_{\theta\theta}\mu = 0.$$

By the strong maximum principle, this would imply that μ is a negative constant, which is impossible. □

We emphasize that all the estimates done previously are not perturbed. Solving the problem P_a (6.13) is equivalent to proving that the kernel of $\mathrm{Id} - \mathcal{K}_1$ is non-trivial. We can now apply the Leray-Schauder theory.

We define the open set for $\delta > 0$,

$$\mathcal{B} = \left\{ (c, \mu) \mid 0 < c < c^* + \delta, \ \|\mu\|_{\mathcal{C}^{1,\beta}((-a,a)\times\Theta)} < C_0 + \delta \right\}.$$

The different a priori estimates of Lemmas 6.9, 6.10, 6.11, 6.12 give that for all $\tau \in [0, 1]$ and sufficiently large a, the operator $\mathrm{Id} - \mathcal{K}_\tau$ cannot vanish on the boundary of $\mathcal{B}$. Indeed, if it vanishes on $\partial\mathcal{B}$, there exists a solution (c, μ) of (6.13) which also satisfies $c \in \{0, c^* + \delta\}$ or $\|\mu\|_{\mathcal{C}^{1,\beta}((-a,a)\times\Theta)} = C_0 + \delta$ and $\nu(0) = \varepsilon$. But this is ruled out by the condition $\varepsilon < \varepsilon_0$, due to Lemmas 6.9, 6.10, 6.11, 6.12. It yields by the homotopy invariance that

$$\forall \tau \in [0, 1], \quad \deg(\mathrm{Id} - \mathcal{K}_1, \mathcal{B}, 0) = \deg(\mathrm{Id} - \mathcal{K}_\tau, \mathcal{B}, 0) = \deg(\mathrm{Id} - \mathcal{K}_0, \mathcal{B}, 0).$$

We now need to compute $\deg(\mathrm{Id} - \mathcal{K}_0, \mathcal{B}, 0)$. This will be done with two supplementary homotopies. We need these two homotopies to write $\mathrm{Id} - \mathcal{K}_0$ as a tensor of two applications whose degree with respect to $\mathcal{B}$ and 0 are computable. We first define, with $\nu_{Z^0}(\cdot) = \int_\Theta Z^0(\cdot, \theta)d\theta$:

$$\mathcal{M}_\tau : (c, v) \to \left(c - (1-\tau)\nu_v(0) - \tau\nu_{Z^0}(0) + \varepsilon, Z^0\right)$$

If there exists $(c, \mu) \in \partial\mathcal{B}$ such that $\mathcal{M}_\tau(c, \mu) = (c, \mu)$, then (c, μ) is such that $Z^0 = \mu$ and $\nu_{Z^0}(0) = \varepsilon$. However, such a fixed point (c, μ) then satisfies

$$\begin{cases} -c\partial_\xi \mu - \theta_{min}\partial_{\xi\xi}\mu - \partial_{\theta\theta}\mu = r\mu(1-\nu), & \xi \in (-a, a) \times \Theta, \\ \partial_\theta \mu(\xi, \theta_{\min}) = \partial_\theta \mu(\xi, \theta_{\max}) = 0, & \xi \in (-a, a), \\ \mu(-a, \theta) = |\Theta|^{-1}, \quad \mu(a, \theta) = 0, & \theta \in \Theta, \end{cases} \tag{6.24}$$

which is now closely linked to the standard Fisher-KPP equation. Indeed, after integration w.r.t θ, ν satisfies

$$\begin{cases} -c\partial_\xi \nu - \theta_{min}\partial_{\xi\xi}\nu = r\nu(1-\nu), & \xi \in (-a, a), \\ \nu(-a) = 1, \quad \nu(a) = 0, \end{cases} \tag{6.25}$$

and $\nu(0) = \varepsilon$. Given a (unique) solution ν of (6.25) after Lemma 6.13, we can solve the equation for v. The solution of (6.24) is then unique thanks to the maximum principle, and reads $\mu(\xi, \theta) = \frac{\nu(\xi)}{|\Theta|}$. As a consequence, such a fixed point cannot belong to $\partial\mathcal{B}$ after all *a priori* estimates of Lemma 6.13. Thus, by the homotopy invariance and $\mathcal{K}_0 = \mathcal{M}_0$, we have

$$\deg(\mathrm{Id} - \mathcal{K}_0, \mathcal{B}, 0) = \deg(\mathrm{Id} - \mathcal{M}_1, \mathcal{B}, 0).$$

The concluding arguments are now the same as in [26]. Up to the end of the proof, we shall exhibit the dependency of Z^0 in c : $Z^0 = Z_c$. We now define our last homotopy by the formula

$$\mathcal{N}_\tau : (c, \mu) \to \left(c + \varepsilon - \nu_{Z_c}(0), \tau Z_c + (1-\tau) Z_{c_0}\right),$$

where c_0 is the unique $c \in [0, 2\sqrt{r\theta_{\min}}]$ such that $\nu_{Z_c}(0) = \varepsilon$, for $\varepsilon < \varepsilon^*$ and $a(\varepsilon)$ sufficiently large (see again Lemma 6.13). If $\mathcal{N}_\tau$ has a fixed point, then necessarily $\varepsilon = \nu_{Z_c}(0)$ and $\mu = \tau Z_c + (1-\tau) Z_{c_0}$. This gives $\mu = Z_{c_0}$ by uniqueness of the speed c_0. Again, such a μ cannot belong to $\partial\mathcal{B}$ (we recall that $c_0 < 2\sqrt{r\theta_{min}} < c^*$ after (6.7)). By homotopy invariance and $\mathcal{M}_1 = \mathcal{N}_1$:

$$\deg\left(\mathrm{Id} - \mathcal{K}_1, \mathcal{B}, 0\right) = \deg\left(\mathrm{Id} - \mathcal{K}_0, \mathcal{B}, 0\right) = \deg\left(\mathrm{Id} - \mathcal{M}_1, \mathcal{B}, 0\right) = \deg\left(\mathrm{Id} - \mathcal{N}_0, \mathcal{B}, 0\right).$$

Finally, the operator $\left(\mathrm{Id} - \mathcal{N}_0\right)(c, \mu) = \left(\nu_{Z_c}(0) - \varepsilon, \mu - Z_{c_0}\right)$ is such that

$$\deg\left(\mathrm{Id} - \mathcal{N}_0, \mathcal{B}, 0\right) = -1.$$

Indeed, the degree of the first component is -1 as it is a decreasing function of c, and the degree of the second one is 1.

We conclude that $\deg\left(\mathrm{Id} - \mathcal{K}_1, \mathcal{B}, 0\right) = -1$. Therefore it has a non-trivial kernel whose elements are solution of the slab problem. This proves the Proposition.

□

6.4 Construction of spatial travelling waves with minimal speed c^*.

In this Section, we now use the solution of the slab problem (6.13) given by Proposition 6.14 to construct a wave solution with minimal speed c^*. For this purpose, we first pass to the limit in the slab to obtain a profile in the whole space $\mathbb{R} \times \Theta$. Then we prove that this profile necessarily travels with speed c^*.

6.4.1 Construction of a spatial travelling wave in the full space.

Lemma 6.16. *Let $\varepsilon < \min(\varepsilon_0, \varepsilon^*)$. There exists $c_0 \in [0, c^*]$ such that the system*

$$\begin{cases} -c_0 \partial_\xi \mu - \theta \partial_{\xi\xi} \mu - \alpha \partial_{\theta\theta} \mu = r\mu(1-\nu), & (\xi, \theta) \in \mathbb{R} \times \Theta, \\ \partial_\theta \mu(\xi, \theta_{\min}) = \partial_\theta \mu(\xi, \theta_{\max}) = 0, & \xi \in \mathbb{R}, \end{cases} \tag{6.26}$$

has a non-negative solution $\mu \in \mathcal{C}_b^2(\mathbb{R} \times \Theta)$ satisfying $\nu(0) = \varepsilon$.

Proof of Lemma 6.16. For sufficiently large $a > a_0(\varepsilon)$, Proposition 6.14 gives a solution (c^a, μ^a) of (6.13) which satisfies $c^a \in [0, c^*]$, $\|\mu^a\|_{L^\infty((-a,a)\times\Theta)} \le K_0$ and $\nu^a(0) = \varepsilon$. As a consequence,

$$\|\nu^a\|_{L^\infty((-a,a))} \le |\Theta| K_0.$$

The elliptic regularity [112] implies that for all $\beta > 0$, $\|\mu^a\|_{\mathcal{C}^{1,\beta}((-a,a)\times\Theta)} \le C$ for some $C > 0$ uniform in a. Then, the Ascoli theorem gives that possibly after passing to a subsequence $a_n \to +\infty$, (c^a, μ^a) converges towards $(c_0, \mu) \in [0, c^*] \times \mathcal{C}^{1,\beta}(\mathbb{R} \times \Theta)$ which satisfies (6.26) and $\nu(0) = \varepsilon$.

□

Remark 6.17. *We do not obtain after the proof that* $\sup \nu \le 1$*, and nothing is known about the behaviors at infinity at this stage. Nevertheless, we have an uniform bound $\|\nu\|_{L^\infty(\mathbb{R})} \le |\Theta| K_0$.*

6.4.2 The profile is travelling with the minimal speed c^*.

Lemma 6.18. (Lower bound on the infimum). *There exists $\delta > 0$ such that any solution (c, μ) of*

$$\begin{cases} -\theta\partial_{\xi\xi}\mu - \alpha\partial_{\theta\theta}\mu - c\partial_\xi\mu = r(1-\nu)\mu, & (\xi,\theta) \in \mathbb{R}\times\Theta, \\ \partial_\theta\mu(\xi,\theta_{\min}) = \partial_\theta\mu(\xi,\theta_{\max}) = 0, & \xi \in \mathbb{R}, \end{cases}$$

with $c \in [0, c^*]$, ν bounded and $\inf_{\xi\in\mathbb{R}} \nu(\xi) > 0$ satisfies $\inf_{\xi\in\mathbb{R}} \nu(\xi) > \delta$.

Proof of Lemma 6.18. We again adapt an argument from [4] to our context. By the Harnack inequality of Proposition 6.8, one has

$$\forall\,(\xi,\theta,\theta') \in \mathbb{R}\times\Theta^2, \qquad \mu(\xi,\theta) \le C(\xi)\mu(\xi,\theta'). \tag{6.27}$$

Since (6.2) is invariant by translation in space, and the renormalization $\nu(0) = \varepsilon$ is not used in the proof of the Harnack inequality, we can take a constant $C(\xi)$ which is independent from ξ [111]. This yields

$$\forall\,(\xi,\theta) \in \mathbb{R}\times\Theta, \qquad -\theta\partial_{\xi\xi}\mu(\xi,\theta) - \alpha\partial_{\theta\theta}\mu(\xi,\theta) - c\partial_\xi\mu(\xi,\theta) \ge r(1 - C\Theta\mu(\xi,\theta))\mu(\xi,\theta).$$

Hence, μ is a super solution of some elliptic equation with local terms only. For $\eta > 0$ arbitrarily given, we define the family of functions

$$\psi_m(\xi,\theta) = m\left(1 - \eta\xi^2\right) Q^*(\theta).$$

From the uniform L^∞ estimate on μ, there exists M large enough such that $\psi_M(0,\theta) > \mu(0,\theta)$. Moreover, by assumption we have $\psi_m \le \mu$ for $m = \frac{\inf_{\mathbb{R}} \nu}{C|\Theta|\|Q^*\|_\infty} > 0$. As a consequence, we can define

$$m_0 := \sup\{m > 0, \quad \forall(\xi,\theta) \in \mathbb{R}\times\Theta, \quad \psi_m(\xi,\theta) \le \mu(\xi,\theta)\}.$$

As in previous same ideas, see Lemmas 6.9 and 6.11, there exists (x_0, θ_0) such that $\mu - \psi_{m_0}$ has a zero minimum at this point. We have clearly that $\xi_0 \in \left[-\frac{1}{\sqrt{\eta}}; \frac{1}{\sqrt{\eta}}\right]$ since ψ_m is negative elsewhere. We have, at (ξ_0, θ_0) :

$$\begin{aligned} 0 \;&\ge\; -\theta_0\partial_{\xi\xi}\left(\mu - \psi_{m_0}\right) - \alpha\partial_{\theta\theta}\left(\mu - \psi_{m_0}\right) - c\partial_\xi\left(\mu - \psi_{m_0}\right), \\ &\ge\; r\left(1 - C|\Theta|\mu\right)\mu + \theta_0\partial_{\xi\xi}\left(\psi_{m_0}\right) + \alpha\partial_{\theta\theta}\left(\psi_{m_0}\right) + c\partial_\xi\left(\psi_{m_0}\right), \\ &\ge\; r\left(1 - C|\Theta|\mu\right)\mu - 2\eta m_0\theta_0 Q^*(\theta_0) \\ &\qquad\qquad - \left(-\lambda^* c^* + \theta_0(\lambda^*)^2 + r\right)\psi_{m_0}(\xi_0,\theta_0) - 2c\eta\xi_0 m_0 Q^*(\theta_0), \\ &\ge\; \mu(\xi_0,\theta_0)\left(\lambda^* c^* - \theta_0(\lambda^*)^2 - rC|\Theta|\mu(\xi_0,\theta_0)\right) - 2m_0 Q^*(\theta_0)\left(\eta\theta_0 + \eta\xi_0 c\right). \end{aligned}$$

It follows from $\mu(\xi_0,\theta_0) \ge \frac{\nu(\xi_0)}{C|\Theta|}$ (6.27), from the inequalities $|\xi_0| \le \frac{1}{\sqrt{\eta}}$, $c \le c^*$, $m_0 \le M$ and the fact that for all $\theta_0 \in \Theta$, the quantity $c^* - \theta_0\lambda^* - \theta_{\min}\lambda^*$ is positive (see (6.8)) that

$$\begin{aligned} \mu(\xi_0,\theta_0) \;&\ge\; \frac{\lambda^*\left(c^* - \theta_0\lambda^*\right)}{rC|\Theta|} - \frac{2M\|Q^*\|_\infty\left(\eta\theta_{\max} + \sqrt{\eta}c^*\right)}{r\nu(\xi_0)}, \\ &\ge\; \frac{\theta_{\min}\left(\lambda^*\right)^2}{rC|\Theta|} - \frac{2M\|Q^*\|_\infty\left(\sqrt{\eta}c^* + \eta\theta_{\max}\right)}{r\left(\inf_{\xi\in\mathbb{R}}\nu\right)}. \end{aligned}$$

Recalling $\inf_{\xi\in\mathbb{R}} \nu > 0$ and taking arbitrarily small values of $\eta > 0$, we have necessarily $\mu(\xi_0,\theta_0) \geq \frac{\theta_{min}(\lambda^*)^2}{2Cr|\Theta|}$. Since μ and ψ_{m_0} coincide at (ξ_0,θ_0), we have $m_0 \geq \frac{\theta_{min}(\lambda^*)^2}{2rC|\Theta|\|Q^*\|_\infty}$. The definition of m_0 now gives

$$\forall(\xi,\theta) \in \mathbb{R}\times\Theta, \qquad \mu(\xi,\theta) \geq \frac{\theta_{min}\,(\lambda^*)^2}{2C|\Theta|r\|Q^*\|_\infty}\left(1-\eta\xi^2\right)Q^*(\theta).$$

Since η is arbitrarily small, we have necessarily $\nu(\xi) \geq \delta := \frac{\theta_{min}(\lambda^*)^2}{2C|\Theta|r\|Q^*\|_\infty}$ for all $\xi \in \mathbb{R}$.

□

We deduce from this Lemma that up to choosing $\varepsilon < \delta$, the solution necessarily satisfies $\inf_{\mathbb{R}} \nu(\xi) = 0$. Since this infimum cannot be attained, we have necessarily $\liminf_{\xi\to+\infty} \nu(\xi) = 0$ (up to $\xi \to -\xi$ and $c \to -c$). We now prove that this enforces $c = c^*$ for our wave. For this purpose, we show that a solution going slower than c^* cannot satisfy the lim inf condition by a sliding argument.

Proposition 6.19. *Any solution (c,μ) of the system*

$$\begin{cases} -\theta\partial_{\xi\xi}\mu - \alpha\partial_{\theta\theta}\mu - c\partial_\xi\mu = r\mu(1-\nu), & (\xi,\theta) \in \mathbb{R}\times\Theta, \\ \partial_\theta\mu(\xi,\theta_{\min}) = \partial_\theta\mu(\xi,\theta_{\max}) = 0, & \xi \in \mathbb{R}, \end{cases} \tag{6.28}$$

with $c \geq 0$ and $\inf_{\xi\in\mathbb{R}} \nu(\xi) = 0$ satisfies necessarily $c \geq c^*$.

As a consequence, the solution given after Lemma 6.16 goes with the speed c^*. This latter speed appears to be the minimal speed of existence of nonnegative travelling waves, similarly as for the Fisher KPP equation.

Proof of Proposition 6.19. We again play with subsolutions. By analogy with the Fisher-KPP equation, we shall use oscillating fronts associated with speed $c < c^*$ to "push" solutions of (6.28) up to the speed c^*. We now proceed like in [35].

Let us now consider the following spectral problem :

$$\begin{cases} \alpha Q_\lambda(\theta)'' + \left(-\lambda c + \theta\lambda^2 + r - s\right)Q_\lambda(\theta) = 0\,, \\ Q_\lambda'(\theta_{min}) = Q_\lambda'(\theta_{max}) = 0\,. \end{cases} \tag{6.29}$$

When $s = 0$ we know from Proposition 6.5 that for $c = c^*$ there exists some real $\lambda^* > 0$ such that the spectral problem is solvable with a positive eigenvector. Moreover, the minimal speed is increasing with respect to r. Indeed, for all $r_1 < r_2$ and $\lambda > 0$, one has

$$\lambda c_{r_1}(\lambda) = r_1 + \lambda^2\theta_{\max} - \gamma(\lambda) < r_2 + \lambda^2\theta_{\max} - \gamma(\lambda) = \lambda c_{r_2}(\lambda)$$

and thus $c^*_{r_1} < c^*_{r_2}$.

Now suppose by contradiction that $c < c^*$. Take $c < \bar{c} < c^*$, $s > 0$. One can choose $s = s(\bar{c}) > 0$ such that $\bar{c}$ is the minimal speed of the spectral problem (6.29).

Let us now consider (6.29) for complex values of λ. The analytic perturbation theory, see [137, Chapter 7, §1, §2, §3], yields that the eigenvalues are analytic in λ at least in a neighborhood of the real axis. As a consequence, by the Rouché theorem we know that taking $\bar{c}$ sufficiently close to c, there exists $\lambda_c := \lambda_R + i\lambda_I \in \mathbb{C}$ with $\mathrm{Re}(\lambda_c) > 0$ such that there exists

$Q_{\lambda_c} : \Theta \mapsto \mathbb{C}$ which solves the spectral problem (6.29) (with $s = s(\bar{c})$). The local analyticity ensures that $\mathrm{Re}\,(Q_{\lambda_c}) > 0$ when $\bar{c}$ is sufficiently close to c, since $\mathrm{Re}\,(Q_{\lambda_{\bar{c}}}) > 0$.

Let us now define the real function

$$\psi(\xi,\theta) := \mathrm{Re}\left(e^{-\lambda_c\xi}Q_{\lambda_c}(\theta)\right) = e^{-\lambda_R\xi}\left[\mathrm{Re}\,(Q_{\lambda_c}(\theta))\cos(\lambda_I\xi) + \mathrm{Im}\,(Q_{\lambda_c}(\theta))\sin(\lambda_I\xi)\right].$$

By construction, one has

$$-\theta\partial_{\xi\xi}\psi - \alpha\partial_{\theta\theta}\psi - c\partial_\xi\psi - r\psi = -s(\bar{c})\psi.$$

Thus, for all $m \geq 0$, the function $v := \mu - m\psi$ satisfies

$$-\theta\partial_{\xi\xi}v - \alpha\partial_{\theta\theta}v - c\partial_\xi v - rv = ms(\bar{c})\psi - r\nu(\xi)\mu.$$

For all $\theta \in \Theta$, one has $\psi(0,\theta) > 0$ and $\psi\left(\pm\frac{\pi}{\lambda_I},\theta\right) < 0$. As a consequence, there exists an open subdomain $\mathcal{D} \subset \Omega := \left[-\frac{\pi}{\lambda_I},\frac{\pi}{\lambda_I}\right] \times \Theta$ such that $\psi > 0$ on $\mathcal{D}$ and ψ vanishes on the boundary $\partial\mathcal{D}$.

There now exists m_0 such that v attains a zero minimum at $(z_0,\theta_0) \in \mathcal{D}$. If $\theta_0 \in \Theta$, one deduces $\nu(z_0) \geq \frac{s(\bar{c})}{r}$. It could happen that $\theta_0 \in \partial\Theta$ but in this case the latter conclusion remains true thanks to the Neumann boundary conditions satisfied by ψ. From the Harnack estimate of Proposition 6.8, there exists a constant C which depends on $|\mathcal{D}|$ such that one has for all $\xi \in \mathbb{R}$,

$$\forall(z,\theta,\theta') \in \mathcal{D}\times\Theta, \qquad \mu(z+\xi,\theta) \leq C\mu(\xi,\theta')$$

Integrating this estimate over Θ, we conclude that $\nu(0) \geq \frac{s(\bar{c})}{rC}$.

We now want to translate the argument in space. For this purpose, we define, for $\zeta \in \mathbb{R}$, the function $h(\xi,\theta) := \mu(\xi+\zeta,\theta)$. It also satisfies (6.28). As a consequence, for all $\zeta \in \mathbb{R}$, $\nu(\zeta) = \int_\Theta h(0,\theta)d\theta \geq \frac{s(\bar{c})}{rC}$. We emphasize that the renormalization $\nu(0) = \varepsilon$, which is the only reason for which (6.13) is not invariant by translation, is not used here. We then obtain $\inf_{\xi\in\mathbb{R}}\nu(\xi) \geq \frac{s(\bar{c})}{rC}$. This contradicts the property $\inf_{\xi\in\mathbb{R}}\nu(\xi) = 0$.

□

6.4.3 The profile has the required limits at infinity.

Proposition 6.20. *Any solution (c,μ) of the system*

$$\begin{cases} -\theta\partial_{\xi\xi}\mu - \alpha\partial_{\theta\theta}\mu - c\partial_\xi\mu = r\mu(1-\nu), & (\xi,\theta) \in \mathbb{R}\times\Theta, \\ \partial_\theta\mu(\xi,\theta_{\min}) = \partial_\theta\mu(\xi,\theta_{\max}) = 0, & \xi \in \mathbb{R}, \end{cases}$$

with $c \geq 0$ and $\nu(0) = \varepsilon$ satisfies

1. *There exists $m > 0$ such that $\forall\xi \in]-\infty,0]$, $\mu(\xi,\cdot) > mQ(\cdot)$,*
2. $\lim_{\xi\to+\infty}\mu(\xi,\cdot) = 0$.

Proof of Proposition 6.20. We again adapt to our case an argument from [4]. By the Harnack inequality applied on $[-1,0]\times\Theta$, there exists $\widetilde{C}$ such that one has

$$\inf_{(\xi,\theta)\in[-1,0]\times\Theta}\mu(\xi,\theta) \geq \frac{\varepsilon}{\widetilde{C}|\Theta|}, \tag{6.30}$$

recalling $\nu(0) = \varepsilon$. Also recalling

$$\forall \left(\xi, \theta, \theta'\right) \in \mathbb{R} \times \Theta^2, \qquad \mu(\xi, \theta) \leq C\mu(\xi, \theta'),$$

we obtain

$$\forall \left(\xi, \theta\right) \in \mathbb{R} \times \Theta, \qquad -\theta\partial_{\xi\xi}\mu(\xi,\theta) - \alpha\partial_{\theta\theta}\mu(\xi,\theta) - c\partial_{\xi}\mu(\xi,\theta) \geq r(1 - C|\Theta|\mu(\xi,\theta))\mu(\xi,\theta).$$

Let us define, for $m = \frac{1}{2}\min\left(\frac{\varepsilon}{|\Theta|\widetilde{C}\|Q^*\|_\infty}, \frac{\theta_{min}(\lambda^*)^2}{rC\|Q^*\|_\infty|\Theta|}\right)$ and $\eta > 0$ arbitrarily given, the function

$$\psi_\eta(\xi,\theta) = m\left(1 + \eta\xi\right)Q^*(\theta).$$

on $]-\infty, 0] \times \Theta$. We have,

$$\forall (\xi,\theta) \in \,]-\infty,-1] \times \Theta, \qquad \psi_1(\xi,\theta) = m\left(1+\xi\right)Q^*(\theta) \leq 0 \leq \mu(\xi,\theta).$$

Moreover, for $(\xi,\theta) \in]-1,0] \times \Theta$, using (6.30), we have

$$\psi_1(\xi,\theta) = m\left(1+\xi\right)Q^*(\theta) \leq m\|Q^*\|_\infty \leq \frac{1}{2}\frac{\varepsilon\|Q^*\|_\infty}{|\Theta|\widetilde{C}\|Q^*\|_\infty} \leq \inf_{\xi\in[-1,0]\times\Theta}\mu(\xi,\theta) \leq \mu(\xi,\theta).$$

As a consequence we can define

$$\eta_0 := \min\{\eta > 0, \forall(\xi,\theta) \in \,]-\infty,0] \times \Theta, \psi_\eta(\xi,\theta) \leq \mu(\xi,\theta)\} \in [0,1].$$

We will now prove that $\eta_0 = 0$ by contradiction. Suppose that $\eta_0 > 0$. We apply the same technique as in the proofs of Lemmas 6.9 and 6.11 : there exists (ξ_0,θ_0) such that $\mu - \psi_{\eta_0}$ has a zero minimum at this point. Moreover, we have $\xi_0 \in \left[-\frac{1}{\eta_0};0\right]$ since ψ_η is negative elsewhere. Moreover, ξ_0 cannot be 0 since this would give $\mu(0,\theta_0) = mQ^*(\theta_0) \leq \frac{1}{2}\frac{\varepsilon}{|\Theta|\widetilde{C}}$ and this would contradict (6.30). We have, at (ξ_0,θ_0) :

$$\begin{aligned}
0 &\geq -\theta\partial_{\xi\xi}\left(\mu - \psi_{\eta_0}\right) - \alpha\partial_{\theta\theta}\left(\mu - \psi_{\eta_0}\right) - c\partial_{\xi}\left(\mu - \psi_{m_0}\right) \\
&\geq r\left(1 - C\Theta\mu\right)\mu + \theta\partial_{\xi\xi}\psi_{\eta_0} + \alpha\partial_{\theta\theta}\psi_{\eta_0} + c\partial_{\xi}\psi_{m_0} \\
&\geq r\left(1 - C\Theta\mu\right)\mu - \psi_{\eta_0}(\xi_0,\theta_0)\left(-\lambda^*c^* + \theta_0\left(\lambda^*\right)^2 + r\right) + cm_0\eta Q^*(\theta_0) \\
&\geq \mu(\xi_0,\theta_0)\left(\lambda^*c^* - \theta_0(\lambda^*)^2 - rC|\Theta|\mu(\xi_0,\theta_0)\right) + cm_0\eta Q^*(\theta_0) \\
&\geq \mu(\xi_0,\theta_0)\left(\lambda^*c^* - \theta_0(\lambda^*)^2 - rC|\Theta|\mu(\xi_0,\theta_0)\right)
\end{aligned}$$

It yields

$$\frac{\theta_{min}(\lambda^*)^2}{rC|\Theta|} \leq \mu(\xi_0,\theta_0) = \psi_{\eta_0}(\xi_0,\theta_0) \leq m\|Q^*\|_\infty.$$

and this contradicts the very definition of m. As a consequence, $\eta_0 = 0$ and

$$\forall(\xi,\theta) \in \mathbb{R}^- \times \Theta, \qquad \mu(\xi,\theta) \geq mQ^*(\theta)$$

In particular, $\inf_{\mathbb{R}^-}\nu \geq m$ holds.

We now prove that $\lim_{\xi \to +\infty} \mu(\xi, \cdot) = 0$. It is sufficient to prove that $\lim_{\xi \to \infty} \nu(\xi) = 0$. Suppose that there exists δ a subsequence $\xi_n \to +\infty$ such that $\forall n \in \mathbb{N}$, $\nu(\xi_n) \geq \delta$. Adapting the preceding proof we obtain that for all $n \in \mathbb{N}$,

$$\forall (\xi, \theta) \in]-\infty, \xi_n] \times \Theta, \qquad \nu(\xi) \geq \frac{1}{2} \min \left(\frac{\delta}{|\Theta| \widetilde{C} \|Q^*\|_\infty}, \frac{\theta_{min}(\lambda^*)^2}{rC\|Q^*\|_\infty |\Theta|} \right). \tag{6.31}$$

Hence (6.31) is true for all $\xi \in \mathbb{R}$ and Lemma 6.18 gives the contradiction since the normalization ε is well chosen.

□

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Chapitre 7

Formalisme Hamilton-Jacobi pour des équations de réaction-diffusion non locales

Dans cet article en collaboration avec Sepideh Mirrahimi, nous étudions un modèle parabolique non-local de type Lotka-Volterra qui décrit une population structurée par une variable d'espace $x \in \mathbb{R}^d$ et une variable de trait phénotypique $\theta \in \Theta$. En prenant en compte la diffusion spatiale, les mutations génétiques et une compétition entre les individus locale en espace, nous analysons le comportement asymptotique (temps long, espace grand) des queues exponentielles des solutions. En utilisant une transformation de Hopf-Cole, nous prouvons que la propagation de la population en espace peut être décrite par un problème d'obstacle de Hamilton-Jacobi qui est indépendant de la variable de trait. L'Hamiltonien effectif est obtenu à partir d'un problème spectral. Les difficultés principales sont le manque d'estimations *a priori* dans la variable de trait, et l'absence de principe de comparaison du fait du terme non-local.

Contents

7.1 Introduction

It is known that the asymptotic (long-time/long-range) behavior of the solutions of some reaction-diffusion equations, as KPP type equations, can be described by level sets of solutions of some relevant Hamilton-Jacobi equations (see [98, 88, 15, 19, 16, 196]). These equations, which admit traveling fronts as solutions, can be used as models in ecology to describe dynamics of a population structured by a space variable.
A related, but different, method using Hamilton-Jacobi equations with constraint has been developed recently to study populations structured by a phenotypical trait (see [79, 18, 178, 20, 149]). This approach provides an asymptotic study of the solutions in the limit of small mutations and in long time, and shows that the asymptotic solutions concentrate on one or several Dirac masses which evolve in time.
Is it possible to combine these two approaches to study populations structured at the same time by a phenotypical trait and a space variable ?

A challenge in evolutionary ecology is to provide and to analyze models that take into account jointly the evolution and the spatial structure of a population. Most of the existing models either concentrate on the evolution and neglect or simplify significantly the spatial structure, or deal only with the spatial dynamics of a population neglecting the impact of evolution on the dynamics. However, to describe many phenomena in ecology as the local adaptation of species in spatially heterogeneous environments [86], to understand the effect of environmental changes on a population [84] or to estimate the propagation speed of an invasive species [181, 34], it is crucial to consider the interactions between ecology and evolution. We refer also to [147] and the reference therein for general literature on the subject.

In this paper, we study a population that is structured by a continuous phenotypical trait $\theta \in \Theta$, where Θ is a smooth and convex bounded subset of $\mathbb{R}^n$, and a space variable $x \in \mathbb{R}^d$. The individuals having a trait θ at time t and position x are denoted by $n(t, x, \theta)$. We assume that the population moves (in space) with a diffusion process of diffusivity $D > 0$, and that they are subject to mutations, which are also described by a diffusion term with diffusivity $\alpha > 0$. We assume that the individuals in the same position are in competition with all other individuals, independently of their trait, and with a constant competition rate r. Let us notice that the non-locality in the model comes from here. We denote by $ra(x,\theta) \in \mathcal{C}^2\left(\mathbb{R}^d \times \Theta\right)$, the growth rate of trait θ at position x, allowing, in this way, heterogeneity in space. The model reads

$$\begin{cases} \partial_t n = D\Delta_x n + \alpha\Delta_\theta n + rn(a(x,\theta) - \rho), \qquad (t,x,\theta) \in (0,\infty) \times \mathbb{R}^d \times \Theta, \\ \dfrac{\partial n}{\partial \mathbf{n}} = 0 \qquad \text{on } (0,\infty) \times \mathbb{R}^d \times \partial\Theta, \\ n(0,x,\theta) = n^0(x,\theta), \qquad (x,\theta) \in \mathbb{R}^d \times \Theta. \end{cases} \tag{7.1}$$

We assume Neumann boundary conditions in the trait variable, meaning that the available traits are given by the set Θ. Moreover, the initial condition n^0 is given and nonnegative. The variable ρ stands for the total density :

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \qquad \rho(t,x) = \int_\Theta n(t,x,\theta)d\theta.$$

Note that such equations can be derived from stochastic individual based models (see [61]). However, this is not the only way to couple the spatial and trait structures. One could also consider a dependence in θ or x in the spatial diffusivity coefficient, the mutation rate or the competition rate. See for instance [34] for a formal study of a model where the spatial diffusivity rate depends on the trait but the growth rate is homogeneous in space. Although, there have been some attempts to study models structured by trait and space (see for instance [61, 9, 43, 34, 23]), not many rigorous studies seem to have analyzed the dynamics of a population continuously structured by trait and by space, with non-local interactions. However a related model, but for sexual populations and for a particular growth rate $a(x,\theta)$, is studied in [161]. In this case, to avoid the complexity due to the sexual reproduction the authors derive formally an equation on the mean value of the phenotypical trait and prove rigorously existence of traveling wave solutions for this simplified equation. Moreover, a very recent article [4], also studies a model close to (7.1), again with some particular growth rate $a(x,\theta)$, and proves existence of traveling wave solutions. Here, we consider a different approach where we perform an asymptotic analysis. Our objective is to generalize the methods developed recently on models structured only by a phenotypical trait [178, 20, 149] to spatial models, to be able to use the previous results in more general frameworks. Moreover, this approach allows us to study models with general growth rates a, where the speed of propagation is not necessarily constant. See also [160] for another work in this direction, where the Hamilton-Jacobi approach is used to study a population model with a discrete spatial structure.

We expect that the population described by (7.1) propagates in the x-direction and that it attains a certain distribution in θ in the invaded parts. We seek for such behavior by performing an asymptotic analysis of the following rescaled model which corresponds to considering small diffusion in space and long time :

$$\begin{cases} \varepsilon\partial_t n_\varepsilon = \varepsilon^2 D\Delta_x n_\varepsilon + \alpha\Delta_\theta n_\varepsilon + rn_\varepsilon(a(x,\theta) - \rho_\varepsilon), \qquad (t,x,\theta) \in (0,\infty) \times \mathbb{R}^d \times \Theta, \\ \dfrac{\partial n_\varepsilon}{\partial \mathbf{n}} = 0 \qquad \text{on } (0,\infty) \times \mathbb{R}^d \times \partial\Theta, \\ n_\varepsilon(0,x,\theta) = n_\varepsilon^0(x,\theta), \qquad (x,\theta) \in \mathbb{R}^d \times \Theta. \end{cases} \tag{7.2}$$

We expect that, for ε small, n_ε can be approximated by

$$n_\varepsilon \approx e^{\frac{u(t,x)}{\varepsilon}} Q(x,\theta), \quad \text{with} \quad u(t,x) \le 0,$$

such that $n_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} n$, with

$$\text{supp}\, n \subset \{(t,x) | u(t,x) = 0\} \times \Theta.$$

In this way, the propagation of the population would be described by the zero level sets of u. Moreover, the phenotypical distribution of the population at position x would be given by $Q(x,\cdot)$. We will show below that such approximation is possible with Q given by an eigenvalue problem and u the unique solution to a Hamilton-Jacobi equation. These results allow us to describe the propagation and the phenotypical distribution of the population, in terms of the diffusion and mutation rates (D and α) and the fitness $a(x,\theta)$. Note that an important contribution in these computations is the fact that both evolution processes and the movement of the individuals are considered in the model. This is crucial to be able to understand several biological phenomena, as the spatial structure of Drosophila Subobscura, whose wing length increases clinally with latitude [133] or the increasing speed of invasion of cane toads [181]. However, to be able to study quantitatively the invasion of cane toads, one should also introduce a dependence in θ in the spatial diffusion rate. This adds some technical difficulties that we leave for future work.

The purpose of this work is to derive rigorously the limit $\varepsilon \to 0$ in (7.2). Our study is based on the usual *Hopf-Cole transformation* which is used in several works on reaction-diffusion equations (as for front propagation in [98, 88, 15]), in the study of parabolic integro-differential equations modeling populations structured by a phenotypical trait (see e.g. [79, 178]) and also recently in the study of the hyperbolic limit of some kinetic equations [33] :

$$u_\varepsilon := \varepsilon \ln n_\varepsilon, \qquad \text{or equivalently,} \qquad n_\varepsilon = \exp\left(\frac{u_\varepsilon}{\varepsilon}\right). \tag{7.3}$$

Thanks to standard maximum principle arguments, n_ε is nonnegative. The quantity u_ε is then well defined for all $\varepsilon > 0$. By replacing (7.3) in (7.2) we obtain

$$\begin{cases} \partial_t u_\varepsilon = \varepsilon D \Delta_x u_\varepsilon + \frac{\alpha}{\varepsilon}\Delta_\theta u_\varepsilon + D|\nabla_x u_\varepsilon|^2 + \frac{\alpha}{\varepsilon^2}|\nabla_\theta u_\varepsilon|^2 + r(a(x,\theta) - \rho_\varepsilon), \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,x,\theta) \in (0,\infty) \times \mathbb{R}^d \times \Theta, \\ \dfrac{\partial u_\varepsilon}{\partial \mathbf{n}} = 0 \qquad \text{on } (0,\infty) \times \mathbb{R}^d \times \partial\Theta, \\ u_\varepsilon(0,x,\theta) = u_\varepsilon^0(x,\theta) \qquad (x,\theta) \in \mathbb{R}^d \times \Theta. \end{cases} \tag{7.4}$$

Throughout the paper, we will use the following assumptions :

$$\forall \varepsilon > 0, \quad \forall x \in \mathbb{R}^d, \qquad -C_1(x) \le u_\varepsilon^0 \le C. \tag{7.5}$$

$$\lim_{\varepsilon \to 0} u_\varepsilon^0(x,\theta) = u_0(x), \qquad \text{uniformly in } \theta \in \Theta. \tag{7.6}$$

$$\forall (x,\theta) \in \mathbb{R}^d \times \Theta, \qquad \psi(x) = -M|x|^2 + B \le a(x,\theta) - a_\infty < 0, \tag{7.7}$$

for some $a_\infty \in \mathbb{R}$. We also suppose the two following bounds :

$$\|\nabla_\theta a(\cdot,\cdot)\|_\infty = b_\infty. \tag{7.8}$$

$$\forall x \in \mathbb{R}^d, \qquad \rho_\varepsilon^0(x) \le a_\infty. \tag{7.9}$$

To state our results we first need the following lemma :

Lemma 7.1. (Eigenvalue problem).
For all $x \in \mathbb{R}^d$, there exists a unique eigenvalue $H(x)$ corresponding to a strictly positive eigenfunction $Q(x,\cdot) \in \mathcal{C}^0(\Theta)$ which satisfies

$$\begin{cases} \alpha\Delta_\theta Q + ra(x,\cdot)Q = H(x)Q, & \text{in } \Theta, \\ \dfrac{\partial Q(x,\cdot)}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\Theta. \end{cases} \tag{7.10}$$

The eigenfunction is unique under the additional normalization assumption

$$\forall x \in \mathbb{R}^d, \qquad \int_\Theta Q(x,\theta)d\theta = 1. \tag{7.11}$$

Moreover, H and Q are smooth functions.

We note that in this article, we suppose that Θ is bounded to avoid technical difficulties. However, we expect that the results would remain true for unbounded domains Θ under suitable coercivity conditions on $-a$ such that the spectral problem (7.10) has a unique solution.

We can now state our main result :

Theorem 7.2. (Asymptotic behavior). *Assume (7.5)–(7.9). Then*

(i) The family $(u_\varepsilon)_\varepsilon$ converges locally uniformly to $u : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ the unique viscosity solution of

$$\begin{cases} \max(\partial_t u - D|\nabla_x u|^2 - H, u) = 0, & \text{in } (0,\infty) \times \mathbb{R}^d, \\ u(0,\cdot) = u_0(\cdot) & \text{in } \mathbb{R}^d. \end{cases} \tag{7.12}$$

(ii) Uniformly on compact subsets of Int $\{u < 0\} \times \Theta$, $\lim_{\varepsilon\to 0} n^\varepsilon = 0$,

(iii) For every compact subset K of Int $(\{u(t,x) = 0\} \cap \{H(x) > 0\})$, there exists $\overline{C} > 1$ such that,

$$\liminf_{\varepsilon\to 0} \rho_\varepsilon(t,x) \ge \frac{H(x)}{r\overline{C}}, \qquad \text{uniformly in } K. \tag{7.13}$$

We notice that u does not depend on θ and therefore, we do not have any supplementary constraint in (7.12) due to the boundary. The variational equality (7.12) gives indeed the effective propagation behavior of the population ; the zero level-sets of u indicate where the population density is asymptotically positive (see also Lemma 7.6). We recall that the effective Hamiltonian H in (7.12) is defined by the spectral problem (7.10) which hides the information on the trait variability.

To understand Theorem 7.2, it is illuminating to provide the following heuristic argument. We write a *formal* expansion of u_ε :

$$u_\varepsilon(t,x,\theta) = u_0(t,x,\theta) + \varepsilon u_1(t,x,\theta) + \mathcal{O}(\varepsilon^2).$$

Replacing this in (7.4) and keeping the terms of order ε^{-2} we obtain, for all (t,x,θ),

$$|\nabla_\theta u_0(t,x,\theta)|^2 = 0.$$

This suggests that u_0 should be independent of θ : $u_0(t,x,\theta) = u_0(t,x)$. Next, keeping the zero order terms (terms with coefficient ε^0), yields :

$$-\alpha\left(\Delta_\theta u_1 + |\nabla_\theta u_1|^2\right) - ra(x,\theta) = \left[-\partial_t u_0 + D|\nabla_x u_0|^2 - r\rho_0\right](t,x). \tag{7.14}$$

Here, ρ_0 denotes the *formal* limit of ρ_ε when $\varepsilon \to 0$. Moreover, u_1 satisfies Neumann boundary conditions. Since the r.h.s. of (7.14) is independent of θ, Lemma 7.1 implies

$$\left[\partial_t u_0 - |\nabla_x u_0|^2 + r\rho_0\right](t,x) = H(x) \quad \text{and} \quad u_1(t,x,\theta) = \ln Q(x,\theta) + \mu(t,x).$$

We can now write

$$n_\varepsilon(t,x,\theta) \approx e^{\frac{u_0(t,x)}{\varepsilon} + u_1(t,x,\theta)}, \qquad \rho_\varepsilon(t,x) \approx e^{\mu(t,x) + \frac{u_0(t,x)}{\varepsilon}}.$$

As a consequence, ρ_ε uniformly bounded implies that u_0 is nonpositive. Furthermore

$$\rho_\varepsilon > 0 \qquad \Longrightarrow \qquad u_0 = 0.$$

We deduce that

$$\begin{cases} \rho_0(t,x) = 0 \quad \Longrightarrow \quad \partial_t u_0(t,x) - D|\nabla_x u_0|^2(t,x) - H(x) = 0, \\ \rho_0(t,x) > 0 \quad \Longrightarrow \quad u_0(t,x) = 0 \quad \text{and} \quad r\exp(\mu(t,x)) = r\rho_0(t,x) = H(x), \end{cases}$$

and thus

$$\max\left(\partial_t u_0 - D|\nabla_x u_0|^2 - H(x)\,,\, u_0\right) = 0.$$

Moreover the above arguments suggest that

$$n_\varepsilon(t,x,\theta) \underset{\varepsilon\to 0}{\longrightarrow} \begin{cases} \frac{H(x)}{r} Q(x,\theta) & \text{if } u_0(t,x) = 0, \\ 0 & \text{if } u_0(t,x) < 0, \end{cases}$$

with Q and H given by Lemma 7.1. We notice finally that, the roles of the trait variable θ and the spectral problem (7.10) are respectively similar to those of the fast variable and the cell problem in homogenization theory.

Theorem 7.2 does not provide the limits of ρ_ε and n_ε in

$$\mathrm{Int}\left(\{u(t,x) = 0\} \cap \{H(x) > 0\}\right).$$

The determination of such limits in the general case, as was obtained for instance in [88], is beyond the scope of the present paper. The difficulty here is the lack of regularity estimates in the x-direction and the lack of comparison principle for the non-local equation (7.2). This difficulty also appears in the study of propagating wave solutions of (7.1) (see [4]), where it is not clear whether the propagating front is monotone and the density and the distribution of the population at the back of the front is unknown. However, in Section 7.4 (see Proposition 7.7), we prove the convergence of n_ε and ρ_ε in a particular case. The numerical results in Section 7.6.2 suggest that such limits might hold in general.

We emphasize that (7.2) does not admit a comparison principle which leads to technical difficulties. This is not only due to the presence of a non-local term but also due to the structure of the reaction term. We refer to [69, 35] for models admitting comparison principle although the reaction terms contain non-localities.

To prove the convergence of $(u_\varepsilon)_\varepsilon$ in Theorem 7.2, we use some regularity estimates that we state below.

Theorem 7.3. (Regularity results for u_ε). *Assume (7.5), (7.7), (7.8), (7.9). Then the family $(u_\varepsilon)_{\varepsilon>0}$ is uniformly locally bounded in $\mathbb{R}^+ \times \mathbb{R} \times \Theta$. More precisely, the following inequalities hold :*

$$\forall (t,x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^d \times \Theta, \qquad r\psi(x)t - C_1(x) - r\varepsilon DMt^2 \leq u_\varepsilon(t,x,\theta) \leq C + ra_\infty t, \tag{7.15}$$

where $\psi(x) := -Mx^2 + B$ (see 7.7).

Next, let $\gamma > 0$ and for all $\varepsilon > 0$, $v_\varepsilon := \sqrt{C + ra_\infty t + \gamma^2 - u_\varepsilon}$. Then, for all $\varepsilon > 0$, the following bound holds :

$$\forall (t,x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^d \times \Theta, \qquad |\nabla_\theta v_\varepsilon| \leq \frac{\varepsilon}{2\sqrt{\alpha t}} + \left(\frac{r b_\infty \varepsilon^2}{\alpha \gamma}\right)^{\frac{1}{3}} \tag{7.16}$$

In particular, this gives a regularizing effect in trait for all $t > 0$, and the fact that $\nabla_\theta v_\varepsilon$ converges locally uniformly to 0 when ε goes to 0.

We notice from (7.16) that, the limit of $(v_\varepsilon)_\varepsilon$ (and consequently the limit of $(u_\varepsilon)_\varepsilon$) as $\varepsilon \to 0$, is independent of θ for all $t > 0$, while we do not make any regularity assumption on the initial data. To obtain the regularizing effect in θ, we provide a Lipschitz estimate on a well-chosen auxiliary function v_ε instead of u_ε, using the Bernstein method [68]. Note that, we do not have any estimate on the derivative of u_ε with respect to x due to the dependence of ρ_ε on x. Therefore, we cannot prove the convergence of the u_ε's as stated in Theorem 7.2 directly from the regularity estimates above. For this purpose, we use the so called half-relaxed limits method for viscosity solutions, see [17]. Moreover, to prove the convergence to the Hamilton-Jacobi equation (7.12) we are inspired from the method of perturbed test functions in homogenization [90].

Finally, the family $(u_\varepsilon)_\varepsilon$ being locally uniformly bounded from Theorem 7.3, we can introduce its upper and lower semi-continuous envelopes that we will use through the article :

$$\underline{u}(t,x,\theta) := \underset{\substack{\varepsilon\to 0\\ (s,y,\theta')\to(t,x,\theta)}}{\underline{\lim\inf}}\ u_\varepsilon(s,y,\theta'),$$

$$\overline{u}(t,x,\theta) := \underset{\substack{\varepsilon\to 0\\ (s,y,\theta')\to(t,x,\theta)}}{\overline{\lim\sup}}\ u_\varepsilon(s,y,\theta').$$

Thanks to Theorem 7.3, we know that $|\nabla_\theta u_\varepsilon| \to 0$ as $\varepsilon \to 0$, for all $t > 0$. As a conclusion, the previous limits do not depend on the variable θ. We have, for all $\theta \in \Theta$, $x \in \mathbb{R}^d$ and $t > 0$,

$$\underline{u}(t,x,\theta) = \underline{u}(t,x) = \underset{\substack{\varepsilon\to 0\\ (s,y)\to(t,x)}}{\underline{\lim\inf}} u_\varepsilon(s,y,\theta), \tag{7.17}$$

$$\overline{u}(t,x,\theta) = \overline{u}(t,x) = \underset{\substack{\varepsilon\to 0\\ (s,y)\to(t,x)}}{\overline{\lim\sup}} u_\varepsilon(s,y,\theta), \tag{7.18}$$

The remaining part of the article is organized as follows. Section 7.2 is devoted to the proof of Lemma 7.1 and Theorem 7.3. The convergence to the Hamilton-Jacobi equation (the first part of Theorem 7.2) is proved in Section 7.3. In Section 7.4, using the Hamilton-Jacobi description, we study the limits of n_ε and ρ_ε and in particular complete the proof of Theorem 7.2. We also provide some qualitative properties on the effective Hamiltonian H and the corresponding eigenfunction Q in Section 7.5. Finally, in Section 7.6 we give some examples and comments on the spectral problem, and some numerical illustrations for the time-dependent problem.

7.2 Regularity results (The proof of Theorem 7.3)

In this section we prove Theorem 7.3. To this end, we first provide a uniform upper bound on ρ_ε (see Lemma 7.4). Next, using this estimate we give uniform upper and lower bounds on u_ε. Finally we prove a Lipschitz estimate with respect to θ on u_ε.

Lemma 7.4. (Bound on ρ_ε).
Assume (7.7) and (7.9). Then, for all $\varepsilon > 0$, the following a priori bound holds :

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \qquad 0 \le \rho_\varepsilon(t,x) \le a_\infty. \tag{7.19}$$

Proof of Lemma 7.4. The nonnegativity follows directly from the nonnegativity of n_ε. The upper bound can be derived using the maximum principle. We show indeed that ρ_ε is a subsolution of a suitable Fisher-KPP equation. We integrate (7.2) in θ to obtain

$$\varepsilon \partial_t \rho_\varepsilon = \varepsilon^2 D \Delta_x \rho_\varepsilon + r \left(\int_\Theta n_\varepsilon(t,x,\theta) a(x,\theta) d\theta - \rho_\varepsilon^2 \right).$$

Using (7.7) and the non negativity of n_ε, we deduce

$$\varepsilon \partial_t \rho_\varepsilon \le \varepsilon^2 D \Delta_x^2 \rho_\varepsilon + r \rho_\varepsilon \left(a_\infty - \rho_\varepsilon \right),$$

so that the maximum principle and (7.9) ensure

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \qquad \rho_\varepsilon(t,x) \le a_\infty.$$

□

We can now proceed with the proof of Theorem 7.3. For legibility, we divide the proof into several steps as follows.

 # Step 1. Upper bound on u_ε. Define $\tilde u_\varepsilon := u_\varepsilon - r a_\infty t$. Using (7.4), we find

$$\partial_t \tilde u_\varepsilon = \varepsilon D \Delta_x \tilde u_\varepsilon + \frac{\alpha}{\varepsilon} \Delta_\theta \tilde u_\varepsilon + D |\nabla_x \tilde u_\varepsilon|^2 + \frac{\alpha}{\varepsilon^2} |\nabla_\theta \tilde u_\varepsilon|^2 + r(a(x,\theta) - a_\infty) - r\rho_\varepsilon.$$

Then, we conclude from (7.7), (7.5) and the maximum principle that

$$\forall (t,x,\theta) \in \mathbb{R}^+ \times \mathbb{R}^d \times \Theta, \qquad u_\varepsilon(t,x,\theta) \le u_\varepsilon^0(x,\theta) + r a_\infty t \le C + r a_\infty t.$$

 # Step 2. Lower bound on u_ε. From (7.4), (7.7) and Lemma 7.4 we can write

$$\partial_t u_\varepsilon \geq \varepsilon D \Delta_x u_\varepsilon + \frac{\alpha}{\varepsilon} \Delta_\theta u_\varepsilon + r(a(x,\theta) - a_\infty) \geq \varepsilon D \Delta_x u_\varepsilon + \frac{\alpha}{\varepsilon} \Delta_\theta u_\varepsilon + r\psi(x),$$

with $\psi(x) = -M|x|^2 + B$ (see 7.7). Next we rewrite the above inequality, in terms of $q_\varepsilon := u_\varepsilon - r\psi(x)t + \varepsilon r M D t^2$:

$$\partial_t q_\varepsilon \geq \varepsilon D \Delta_x q_\varepsilon + \frac{\alpha}{\varepsilon} \Delta_\theta q_\varepsilon + r\varepsilon\psi''(x) D t + 2r\varepsilon M D t \geq \varepsilon D \Delta_x q_\varepsilon + \frac{\alpha}{\varepsilon} \Delta_\theta q_\varepsilon$$

Finally the maximum principle combined with Neumann boundary conditions and (7.5) imply that

$$u_\varepsilon \geq u_\varepsilon^0 + r\psi(x)t - r\varepsilon D M t^2 \geq -C_1(x) + r\psi(x)t - r\varepsilon D M t^2.$$

 # Step 3. Lipschitz bound. We conclude the proof of Theorem 7.3 by using the Bernstein method [68] to obtain a regularizing effect with respect to the variable θ. The upper bound (7.15) proved above ensures that the function v_ε is well-defined. We then rewrite (7.4) in terms of v_ε :

$$\partial_t v_\varepsilon = \varepsilon D \Delta_x v_\varepsilon + \frac{\alpha}{\varepsilon} \Delta_\theta v_\varepsilon + D\left(\frac{\varepsilon}{v_\varepsilon} - 2v_\varepsilon\right)|\nabla_x v_\varepsilon|^2 + \left(\frac{\alpha}{\varepsilon v_\varepsilon} - \frac{2\alpha v_\varepsilon}{\varepsilon^2}\right)|\nabla_\theta v_\varepsilon|^2 - \frac{1}{2v_\varepsilon} r(a(x,\theta) - a_\infty - \rho_\varepsilon). \quad (7.20)$$

We differentiate the above equation with respect to θ and multiply it by $\frac{\nabla_\theta v_\varepsilon}{|\nabla_\theta v_\varepsilon|}$ to obtain

$$\begin{aligned}\partial_t |\nabla_\theta v_\varepsilon| \leq{}& \varepsilon D \Delta_x |\nabla_\theta v_\varepsilon| + \frac{\alpha}{\varepsilon} \Delta_\theta |\nabla_\theta v_\varepsilon| + 2D\left(\frac{\varepsilon}{v_\varepsilon} - 2v_\varepsilon\right)\nabla_x v_\varepsilon \cdot \nabla_x |\nabla_\theta v_\varepsilon| \\ &+ 2\left(\frac{\alpha}{\varepsilon v_\varepsilon} - \frac{2\alpha v_\varepsilon}{\varepsilon^2}\right)\nabla_\theta v_\varepsilon \cdot \nabla_\theta |\nabla_\theta v_\varepsilon| + D\left(-\frac{\varepsilon}{v_\varepsilon^2} - 2\right)|\nabla_x v_\varepsilon|^2 |\nabla_\theta v_\varepsilon| \\ &+ \left(-\frac{\alpha}{\varepsilon v_\varepsilon^2} - \frac{2\alpha}{\varepsilon^2}\right)|\nabla_\theta v_\varepsilon|^3 + \frac{r|\nabla_\theta a(x,\theta)|}{2v_\varepsilon}, \quad (7.21)\end{aligned}$$

since the last contribution of the r.h.s of (7.20) becomes nonpositive. From (7.8) and (7.15), it follows that $w_\varepsilon := |\nabla_\theta v_\varepsilon|$ is a subsolution of the following equation

$$\begin{aligned}\partial_t w_\varepsilon \leq{}& \varepsilon D \Delta_x w + \frac{\alpha}{\varepsilon} \Delta_\theta w_\varepsilon + 2D\left(\frac{\varepsilon}{v_\varepsilon} - 2v_\varepsilon\right)\nabla_x v_\varepsilon \cdot \nabla_x w_\varepsilon \\ &+ 2\left(\frac{\alpha}{\varepsilon v_\varepsilon} - \frac{2\alpha v_\varepsilon}{\varepsilon^2}\right)\nabla_\theta v_\varepsilon \cdot \nabla_\theta w_\varepsilon - \frac{2\alpha}{\varepsilon^2}|w_\varepsilon|^3 + \frac{r b_\infty}{2\gamma}. \quad (7.22)\end{aligned}$$

The last step is now to prove that $z(t) := \frac{\varepsilon}{2\sqrt{\alpha t}} + \left(\frac{r b_\infty \varepsilon^2}{\alpha\gamma}\right)^{\frac{1}{3}}$ is a supersolution of (7.22). We compute

$$z'(t) + \frac{2\alpha}{\varepsilon^2}(z(t))^3 = \frac{2\alpha}{\varepsilon^2}\left(z(t)^3 - \left(z(t) - \left(\frac{r b_\infty \varepsilon^2}{\alpha\gamma}\right)^{\frac{1}{3}}\right)^3\right) \geq \frac{r b_\infty}{2\gamma}.$$

The Neumann boundary condition for u_ε implies a Dirichlet boundary condition for w_ε. Thus, (7.16) follows from the comparison principle.

7.3 Convergence to the Hamilton-Jacobi equation (The proof of Theorem 7.2–(i))

In this section, we first prove Lemma 7.1. Next, using the regularity estimates obtained above we prove the convergence of $(u_\varepsilon)_\varepsilon$ to the solution of (7.12) (Theorem 7.2 (i)). This will be derived from the following proposition which also provides a partial result, once we relax assumption (7.6) :

Proposition 7.5. (Convergence to the Hamilton-Jacobi equation).

(i) Assume (7.5), (7.7), (7.8), (7.9) such that Theorem 7.3 holds. Let H be the eigenvalue defined in Lemma 7.1. Then, $\overline{u}$ *(respectively* $\underline{u}$*) is a viscosity subsolution (respectively supersolution) of*

$$\max(\partial_t u - D|\nabla_x u|^2 - H, u) = 0, \qquad \text{in } (0,\infty)\times\mathbb{R}^d. \tag{7.23}$$

(ii) If we assume additionally (7.6), then $\overline{u} = \underline{u}$ *and, as ε vanishes,* $(u_\varepsilon)_\varepsilon$ *converges locally uniformly to* $u = \overline{u} = \underline{u}$ *the unique viscosity solution of*

$$\begin{cases} \max(\partial_t u - D|\nabla_x u|^2 - H, u) = 0, & \text{in } (0,\infty)\times\mathbb{R}^d, \\ u(0,x) = u_0(x). \end{cases}$$

Before proving Proposition 7.5, we first give a short proof for Lemma 7.1.

Proof of Lemma 7.1. Let $X = \mathcal{C}^{1,\mu}(\Theta)$ and K be the positive cone of nonnegative functions in X. We define $L : X \to X$ as below

$$L(u) = -\alpha\Delta_\theta u - r\,(a(x,\theta) - a_\infty)\,u.$$

The resolvent of L together with the Neumann boundary condition is compact from the regularizing effect of the Laplace term. Moreover, the strong maximum principle gives that it is also strongly positive. Using the Krein-Rutman theorem we obtain that there exists a nonnegative eigenvalue corresponding to a positive eigenfunction. This eigenvalue is simple and none of the other eigenvalues correspond to a positive eigenfunction. This defines $H(x)$ and $Q(x,\theta)$ in (7.10) in a unique way. The smoothness of H and Q derives from the smoothness of $a(x,\theta)$ and the fact that they are principal eigenelements.

□

Proof of Proposition 7.5. We prove the result in two steps.

a. Semi-relaxed limits (proof of Proposition 7.5 (i)).

To prove the result we need to show that $\overline{u}$ and $\underline{u}$ are respectively sub and supersolutions of (7.23).

a.1. We prove that $\overline{u} \le 0$.

Suppose that there exists a point (t,x) such that $\overline{u}(t,x) > 0$. From (7.18) and (7.16), there exists a sequence $\varepsilon_n \to 0$ and a sequence of points $(t_n, x_n) \underset{n\to\infty}{\to} (t,x)$ such that,

$$u_{\varepsilon_n}(t_n, x_n, \theta) \underset{n\to\infty}{\to} \overline{u}(t,x), \quad \text{uniformly in } \theta.$$

As a consequence, there exists $\delta > 0$ such that for n sufficiently large, $u_{\varepsilon_n}(t_n, x_n, \theta) > \delta$, for all $\theta \in \Theta$. This implies

$$\rho_{\varepsilon_n}(t_n, x_n) = \int_\Theta \exp\left(\frac{u_{\varepsilon_n}(t_n, x_n, \theta)}{\varepsilon_n}\right) d\theta \geq |\Theta| \exp\left(\frac{\delta}{\varepsilon_n}\right) > a_\infty,$$

for sufficiently large n, which is in contradiction with Lemma 7.4.

a.2. We prove that $\partial_t \overline{u} - D|\nabla_x \overline{u}|^2 - H \leq 0$.

Now, assume that $\varphi \in \mathcal{C}^2(\mathbb{R}^+ \times \mathbb{R})$ is a test function such that $\overline{u}(t,x) - \varphi(t,x)$ has a strict local maximum at (t_0, x_0).

Using the eigenfunction Q introduced in Lemma 7.1, we can define a *corrected test function* [90] by $\chi_\varepsilon(t,x,\theta) = \varphi(t,x) + \varepsilon\eta(x,\theta)$, with $\eta(x,\theta) = \ln(Q(x,\theta))$. Using standard arguments in the theory of viscosity solutions (see [14]), there exists a sequence $(t_\varepsilon, x_\varepsilon, \theta_\varepsilon)$ such that the function $u_\varepsilon(t,x,\theta) - \chi_\varepsilon(t,x,\theta)$ takes a local maximum in $(t_\varepsilon, x_\varepsilon, \theta_\varepsilon)$, which is strict in the (t,x) variables, and such that $(t_\varepsilon, x_\varepsilon) \to (t_0, x_0)$ as $\varepsilon \to 0$. Moreover, as θ_ε lies in the compact set Θ, one can extract a converging subsequence. For legibility, we omit the extraction in the sequel.

Let us verify the viscosity subsolution criterion. At the point $(t_\varepsilon, x_\varepsilon, \theta_\varepsilon)$, we have :

$$\begin{aligned}
\partial_t \chi_\varepsilon - D|\nabla_x \chi_\varepsilon|^2 - H(x_\varepsilon) &= \partial_t u_\varepsilon - D|\nabla_x u_\varepsilon|^2 - H(x_\varepsilon), \\
&= \varepsilon D \Delta_x u_\varepsilon + \tfrac{\alpha}{\varepsilon}\Delta_\theta u_\varepsilon + \tfrac{\alpha}{\varepsilon^2}|\nabla_\theta u_\varepsilon|^2 + r(a(x_\varepsilon,\theta) - \rho_\varepsilon) - H(x_\varepsilon), \\
&\leq \varepsilon D \Delta_x \chi_\varepsilon + \tfrac{\alpha}{\varepsilon}\Delta_\theta \chi_\varepsilon + \tfrac{\alpha}{\varepsilon^2}|\nabla_\theta \chi_\varepsilon|^2 + r a(x_\varepsilon,\theta) - H(x_\varepsilon).
\end{aligned}$$

We must emphasize that the Neumann boundary conditions are implicitly used here in case when θ_ε is on the boundary of Θ. Indeed, this ensures that we have $\nabla_\theta \chi_\varepsilon(t_\varepsilon, x_\varepsilon, \theta_\varepsilon) = 0$ in this latter case. As a consequence, we still have $\nabla_\theta u_\varepsilon(t_\varepsilon, x_\varepsilon, \theta_\varepsilon) = \nabla_\theta \chi_\varepsilon(t_\varepsilon, x_\varepsilon, \theta_\varepsilon)$ so that the first order derivative in the trait variable does not add any supplementary difficulty in the r.h.s.. Moreover the second order terms still have the right sign, since again $\Delta_x u_\varepsilon \leq \Delta_x \chi_\varepsilon$ is enforced by the Neumann boundary condition.

We replace the test function by its definition to obtain

$$\begin{aligned}
&\partial_t \varphi - D|\nabla_x(\varphi + \varepsilon\eta)|^2 - H(x_\varepsilon) \\
&\qquad \leq \varepsilon D(\Delta_x \varphi + \varepsilon \Delta_x \eta) + \alpha\left(\Delta_\theta \eta + |\nabla_\theta \eta|^2\right) + r a(x_\varepsilon, \theta) - H(x_\varepsilon).
\end{aligned}$$

Here appears the crucial importance of choosing $\eta = \ln Q$ with Q the solution of the spectral problem (7.10). Coupling the above equation with the spectral problem (7.10), written in terms of η, we deduce that

$$\partial_t \varphi - D|\nabla_x \varphi + \varepsilon\eta|^2 - H(x_\varepsilon) \ \leq\ \varepsilon D(\Delta_x \varphi + \varepsilon \Delta_x \eta).$$

We conclude, by letting ε go to 0, that at point (t_0, x_0) :

$$\partial_t \varphi - D|\nabla_x \varphi|^2 - H \leq 0.$$

a.3. We prove that $\max\left(\partial_t \underline{u} - D|\nabla_x \underline{u}|^2 - H, \underline{u}\right) \geq 0$.

We first notice that $\underline{u}(t,x) \leq \overline{u}(t,x) \leq 0$. Let $\underline{u}(t,x) < 0$. Then there exists some $\delta > 0$ such that along a subsequence $(\varepsilon_n, t_n, x_n)$, $u_{\varepsilon_n}(t_n, x_n, \theta) < -\delta$ for all $\theta \in \Theta$ and for $n \geq N$ with N sufficiently large. It follows that $\rho_{\varepsilon_n}(t_n, x_n) \to 0$, as $n \to \infty$. With the same notations as in the previous point replacing maximum by minimum, we get

$$\partial_t \varphi - D|\nabla_x \varphi + \varepsilon \eta|^2 - H(x_\varepsilon) \;\geq\; \varepsilon D\left(\Delta_x \varphi + \varepsilon \Delta_x \eta\right) - r\rho_\varepsilon,$$

so that taking the limit $\varepsilon \to 0$ along the subsequence $(t_{\varepsilon_n}, x_{\varepsilon_n})$, we obtain that

$$\partial_t \varphi - D|\nabla_x \varphi|^2 - H \;\geq\; 0.$$

holds at point (t_0, x_0).

b. Strong uniqueness (proof of Proposition 7.5 (ii)).

Obviously, one cannot get any uniqueness result for the Hamilton-Jacobi equation (7.23) without imposing any initial condition. Adding (7.6), we now check the initial condition of (7.12) in the viscosity sense.

One has to prove the following

$$\min\left(\max\left(\partial_t \overline{u} - D|\nabla_x \overline{u}|^2 - H, \overline{u}\right), \overline{u} - u_0\right) \leq 0, \qquad \text{in } \{t = 0\} \times \mathbb{R}^d, \tag{7.24}$$

and

$$\max\left(\max\left(\partial_t \underline{u} - D|\nabla_x \underline{u}|^2 - H, \underline{u}\right), \underline{u} - u_0\right) \geq 0, \qquad \text{in } \{t = 0\} \times \mathbb{R}^d, \tag{7.25}$$

in the viscosity sense.

Here we give only the proof of (7.24), since (7.25) can be derived following similar arguments. Let $\varphi \in \mathcal{C}^2\left(\mathbb{R}^+ \times \mathbb{R}\right)$ be a test function such that $\overline{u}(t,x) - \varphi(t,x)$ has a strict local maximum at $(t_0 = 0, x_0)$. We now prove that either

$$\overline{u}(0, x_0) \leq u_0(x_0),$$

or

$$\begin{cases} \partial_t \varphi(0, x_0) - D|\nabla_x \varphi(0, x_0)|^2 - H(x_0) \leq 0, \\ \text{and} \\ \overline{u}(0, x_0) \leq 0. \end{cases}$$

Suppose then that

$$\overline{u}(0, x_0) > u_0(x_0). \tag{7.26}$$

Following the arguments above in **a.1.** but taking $t = 0$ and using (7.6) we obtain

$$\overline{u}(0, x_0) \leq 0.$$

We next prove that

$$\partial_t \varphi(0, x_0) - D|\nabla_x \varphi(0, x_0)|^2 - H(x_0) \leq 0. \tag{7.27}$$

There exists a sequence $(t_\varepsilon, x_\varepsilon, \theta_\varepsilon)_\varepsilon$ such that $(t_\varepsilon, x_\varepsilon)$ tends to $(0, x_0)$ as $\varepsilon \to 0$ and that $u_\varepsilon - \chi_\varepsilon = u_\varepsilon - \varphi - \varepsilon\eta$ takes a local maximum at $(t_\varepsilon, x_\varepsilon, \theta_\varepsilon)$. Here η still denotes the correction $\ln Q$ with Q the eigenfunction introduced in Lemma 7.1 (see **a.1.**). We first claim that there exists a subsequence $(t_n, x_n, \theta_n)_n$ of the above sequence and a subsequence $(\varepsilon_n)_n$, with $\varepsilon_n \to 0$ as $n \to \infty$, such that $t_n > 0$, for all n.

Suppose that this is not true. Then, there exists a sequence $(\varepsilon_{n'}, x_{n'}, \theta_{n'})_{n'}$ such that the sequence $(\varepsilon_{n'}, x_{n'}) \to (0, x_0)$ and that $u_{\varepsilon_{n'}} - \varphi - \varepsilon_{n'}\eta$ has a local maximum at $(0, x_{n'}, \theta_{n'})$. It follows that, for all (t, x, θ) in some neighborhood of $(0, x_{n'}, \theta_{n'})$, we have

$$u_{\varepsilon_{n'}}(0, x_{n'}, \theta_{n'}) - \chi_{\varepsilon_{n'}}(0, x_{n'}, \theta_{n'}) \geq u_{\varepsilon_{n'}}(t, x, \theta) - \chi_{\varepsilon_{n'}}(t, x, \theta).$$

Computing $\overline{\limsup}_{\substack{n' \to \infty \\ (t,x) \to (t_0, x_0)}}$ at the both sides of the inequality, and using (7.6) one obtains

$$u_0(x_0) - \varphi(0, x_0) \geq \overline{u}(0, x_0) - \varphi(0, x_0).$$

However, this is in contradiction with (7.26). We thus proved the existence of subsequences $(t_n, x_n, \theta_n)_n$ and $(\varepsilon_n)_n$ described above with $t_n > 0$, for all n.

Now having in hand that $t_n > 0$, from (7.4) and the fact that $u_{\varepsilon_n} - \varphi - \varepsilon_n\eta$ takes a local maximum at (t_n, x_n, θ_n), we deduce that

$$\partial_t\varphi - D|\nabla_x\varphi + \varepsilon_n\eta|^2 - H(x_{\varepsilon_n}) \;\leq\; \varepsilon_n D\left(\Delta_x\varphi + \varepsilon_n\Delta_x\eta\right)$$

holds in (t_n, x_n, θ_n). Finally, letting $n \to +\infty$, we find (7.27).

We refer to [14, Section 4.4.5] and [88, Theorem B.1] for arguments giving *strong uniqueness* (*i.e.* a comparison principle for semi-continuous sub and supersolutions) for (7.12). As $\overline{u}$ and $\underline{u}$ are respectively sub and supersolutions of (7.12), we then know that $\overline{u} \leq \underline{u}$. From their early definition, we also have $\overline{u} \geq \underline{u}$. Gathering these inequalities, we finally obtain $u = \overline{u} = \underline{u}$ and that $(u_\varepsilon)_\varepsilon$ converges locally uniformly, as $\varepsilon \to 0$, towards u, the unique viscosity solution of (7.12) in $\mathbb{R}^+ \times \mathbb{R}^d \times \Theta$.

□

7.4 Refined asymptotics (The proof of Theorem 7.2–(ii) and (iii))

In this section, we provide some information on the asymptotic population density. Firstly, we prove parts (ii) and (iii) of Theorem 7.2 which state that the zero sets of u correspond to the zones where the population is positive. Secondly, we provide the limit of $(n_\varepsilon)_\varepsilon$, as $\varepsilon \to 0$, in a particular case (see Proposition 7.7).

We first prove the following lemma :

Lemma 7.6. *Let u be the unique viscosity solution of (7.12) and $H(x)$ the eigenvalue given by Lemma 7.1. Then*

$$(t, x) \in Int\,\{u(t, x) = 0\} \quad \Longrightarrow \quad H(x) \geq 0.$$

Proof of Lemma 7.6. Thanks to (7.12),

$$\partial_t u - D|\nabla_x u|^2 \leq H(x),$$

in the viscosity sense. In the zone Int $\{u(t,x)=0\}$, one has

$$\partial_t u - D|\nabla_x u|^2 = 0,$$

in the strong sense. The proof of the lemma follows. □

We are now able to characterize the different zones of the front and complete the proof of Theorem 7.2 :

Proof of Theorem 7.2, (ii). Let K be a compact subset of Int $\{u<0\}$. The local uniform convergence of u_ε towards u ensures that there exists a constant $\delta>0$ such that for sufficiently small $\varepsilon>0$ and for all $(t,x)\in K$ and $\theta\in\Theta$, $u_\varepsilon(t,x,\theta)<-\delta$. As a consequence, $n_\varepsilon=\exp\left(\frac{u_\varepsilon}{\varepsilon}\right)<\exp\left(-\frac{\delta}{\varepsilon}\right)\to 0$, uniformly as $\varepsilon\to 0$ in $K\times\Theta$.

□

Proof of Theorem 7.2, (iii). Take $(t_0,x_0)\in K\subset\subset \text{Int}\left(\{u=0\}\cap\{H(x)>0\}\right)$, and let Q be the normalized eigenvector given by Lemma 7.1. We denote $C_m=C_m(x_0)=\min_\Theta Q(x_0,\theta)$ and $C_M=C_M(x_0)=\max_\Theta Q(x_0,\theta)$. We also define

$$F_\varepsilon(t,x) := \int_\Theta n_\varepsilon(t,x,\theta)Q(x_0,\theta)d\theta, \qquad I_\varepsilon := \varepsilon \ln F_\varepsilon.$$

From the early definition of u_ε, and the positivity of $Q(x_0,\theta)$, one has

$$e^{\frac{\min_\Theta u_\varepsilon(t,x,\cdot)}{\varepsilon}}\int_\Theta Q(x_0,\theta)d\theta \leq F_\varepsilon(t,x) \leq e^{\frac{\max_\Theta u_\varepsilon(t,x,\cdot)}{\varepsilon}}\int_\Theta Q(x_0,\theta)d\theta.$$

Since $Q(x_0,\theta)$ is normalized, we deduce

$$\forall (t,x)\in[0,\infty)\times\mathbb{R}^d, \qquad \min_\Theta u_\varepsilon(t,x,\cdot) \leq I_\varepsilon(t,x) \leq \max_\Theta u_\varepsilon(t,x,\cdot).$$

Thus $I:=\lim_{\varepsilon\to 0} I_\varepsilon$ is well-defined and nonpositive. We also point out that the latter inequalities imply $\{u=0\}=\{I=0\}$. Multiplying equation (7.2) by $Q(x_0,\theta)$ and integrating in θ yields

$$\varepsilon\partial_t F_\varepsilon - \varepsilon^2 D\Delta_x F_\varepsilon - \alpha\int_\Theta n_\varepsilon \Delta_\theta Q(x_0,\theta) = r\int_\Theta a(x,\theta)Q(x_0,\theta)n_\varepsilon(t,x,\theta)d\theta - r\rho_\varepsilon F_\varepsilon.$$

Combining the above equation by (7.10) we deduce that

$$\varepsilon\partial_t F_\varepsilon - \varepsilon^2 D\Delta_x F_\varepsilon = \left(H(x_0)-r\rho_\varepsilon\right)F_\varepsilon + r\int_\Theta Q(x_0,\theta)n_\varepsilon(t,x,\theta)\left[a(x,\theta)-a(x_0,\theta)\right]d\theta.$$

Since $H(x_0)>0$ and a is continuous, for all $\delta>0$, one can choose a constant $r>0$ such that

$$\forall x\in B_r(x_0), \qquad |a(x,\theta)-a(x_0,\theta)|<\delta H(x_0) \quad \text{with } B_r(x_0)\subset K.$$

We finally deduce that for all $x\in B_r(x_0)$,

$$\varepsilon\partial_t F_\varepsilon - \varepsilon^2 D\Delta_x F_\varepsilon \geq \left((1-\delta)H(x_0)-r\rho_\varepsilon\right)F_\varepsilon.$$

Since $F_\varepsilon \geq C_m \rho_\varepsilon$, it follows that for all $x \in B_r(x_0)$,

$$\varepsilon \partial_t F_\varepsilon - \varepsilon^2 D \Delta_x F_\varepsilon \geq \left((1-\delta) H(x_0) - \frac{r F_\varepsilon}{C_m} \right) F_\varepsilon. \tag{7.28}$$

Moreover, since $(t_0, x_0) \subset \text{Int}\{u(t,x) = 0\} = \text{Int}\{I(t,x) = 0\}$, we have $I(t,x) = 0$ in a neighborhood of (t_0, x_0).

We then apply an argument similar to the one used in [88] to prove an analogous statement for the Fisher-KPP equation. To this end, we introduce the following test function

$$\varphi(t,x) = -|x - x_0|^2 - (t - t_0)^2.$$

As $I - \varphi$ attains a strict minimum in (t_0, x_0), there exists a sequence $(t_\varepsilon, x_\varepsilon)$ of points such that and $I_\varepsilon - \varphi$ attains a minimum in $(t_\varepsilon, x_\varepsilon)$, with $(t_\varepsilon, x_\varepsilon) \to (t_0, x_0)$. It follows from (7.28) that

$$\partial_t \varphi - \varepsilon D \Delta_x \varphi - D |\nabla_x \varphi|^2 \geq \partial_t I_\varepsilon - \varepsilon D \Delta_x I_\varepsilon - D |\nabla_x I_\varepsilon|^2 \geq (1-\delta) H(x_0) - \frac{r F_\varepsilon}{C_m}.$$

As a consequence,

$$\liminf_{\varepsilon \to 0} F_\varepsilon(t_0, x_0) \geq \frac{C_m}{r} (1-\delta) H(x_0),$$

uniformly with respect to points $(t_0, x_0) \in K$ and this gives

$$\liminf_{\varepsilon \to 0} \rho_\varepsilon(t_0, x_0) \geq (1-\delta) H(x_0) \frac{C_m}{r C_M}.$$

We then let $\delta \to 0$ and obtain

$$\liminf_{\varepsilon \to 0} \rho_\varepsilon(t_0, x_0) \geq H(x_0) \frac{C_m(x_0)}{r C_M(x_0)},$$

uniformly with respect to points $(t_0, x_0) \in K$.

Let

$$\widetilde{K} = \{x \mid \exists t \geq 0, \text{ such that } (t,x) \in K\}.$$

To conclude the proof, it is enough to prove that there exists a constant

$$\overline{C} = \overline{C}(\alpha, r, a|_{\widetilde{K} \times \Theta}) \geq 1,$$

such that

$$\frac{C_M(x)}{C_m(x)} \leq \overline{C}, \qquad \text{for all } x \in \widetilde{K}.$$

This is indeed a consequence of the Harnack inequality [42] for the solutions of (7.10) in Θ for all $x \in \widetilde{K}$. We point out that here we can use the Harnack inequality on the whole domain Θ thanks to the Neumann boundary condition. □

The above result is not enough to identify the limit of $(n_\varepsilon)_\varepsilon$ as $\varepsilon \to 0$, as was obtained for example for Fisher-KPP type models in [88]. The main difficulties to obtain such limits are the facts that we do not have any regularity estimate in the x direction on n_ε and that there is no comparison principle for this model due to the non-local term. However, we were able to identify the limit of $(n_\varepsilon)_\varepsilon$ in a particular case :

Proposition 7.7. *Suppose that Q, the eigenvector given by* (7.10)*, does not depend on x, i.e. $Q(x,\theta) = Q(\theta)$. Let the initial data be of the following form*

$$n_\varepsilon(t=0,x,\theta) = m_\varepsilon(x)Q(\theta), \qquad m_\varepsilon(x) \geq 0. \tag{7.29}$$

Then :

(i) There exists a function $m_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ such that for all $t > 0$ and $(x,\theta) \in \mathbb{R}^d \times \Theta$, $n_\varepsilon(t,x,\theta) = m_\varepsilon(t,x)Q(\theta)$.

(ii) For all $(t,x,\theta) \in \{u(t,x) = 0\} \times \Theta$, $\lim_{\varepsilon\to 0} n_\varepsilon(t,x,\theta) = \frac{H(x)}{r}Q(\theta)$.

Remark 7.8. *We note that the assumption on Q in Proposition 7.7, is satisfied for $a(x,\theta) = a(\theta) + b(x)$.*

Proof of Proposition 7.7. Let m_ε be the unique solution of the following equation

$$\begin{cases} \varepsilon\partial_t m_\varepsilon - \varepsilon^2 D\Delta_x m_\varepsilon = r m_\varepsilon \left(H(x) - m_\varepsilon\right), \\ m_\varepsilon(0,x) = m_\varepsilon(x). \end{cases}$$

Define

$$\widetilde{n}_\varepsilon(t,x,\theta) := m_\varepsilon(t,x)Q(\theta).$$

We notice from (7.11) that

$$\int \widetilde{n}_\varepsilon(t,x,\theta)d\theta = m_\varepsilon(t,x).$$

Consequently, from (7.29), (7.10), and the definition of m_ε one can easily verify that $\widetilde{n}_\varepsilon$ is a solution of (7.2), and since (7.2) has a unique solution we conclude that

$$n_\varepsilon(t,x,\theta) = m_\varepsilon(t,x)Q(\theta),$$

and

$$\rho_\varepsilon(t,x) = m_\varepsilon(t,x).$$

As a consequence ρ_ε satisfies the following Fisher-KPP equation

$$\varepsilon\partial_t\rho_\varepsilon - \varepsilon^2 D\Delta_x\rho_\varepsilon = r\rho_\varepsilon\left(H(x) - \rho_\varepsilon\right).$$

Let $(t,x) \in \{u = 0\}$. Then from Lemma 7.6, we obtain $H(x) \geq 0$. Hence, from the above equation and (7.13), following similar arguments as in [88, p.157] we obtain that $\rho_\varepsilon(t,x) \to H(x)$ as $\varepsilon \to 0$, and (ii) follows. □

7.5 Qualitative properties

In this section, we provide some estimates on the effective Hamiltonian H and the eigenfunction Q. We note that the spatial propagation of the population can be described using H through (7.12). In particular, if $H(x) = H$ is constant and if initially the population is restricted to a compact set in space, then the population propagates in space with the constant speed $c = 2\sqrt{H}$. Furthermore, the eigenfunction Q is expected to represent the phenotypical distribution of the population (see Proposition 7.7).

We begin by presenting some qualitative estimates on the effective Hamiltonian H.

Lemma 7.9. *The eigenvalue and normalized eigenfunction introduced in Lemma 7.1 satisfy the following estimates :*

$$\forall x \in \mathbb{R}, \quad H(x) = r \int_\Theta a(x,\theta) Q(x,\theta) d\theta, \tag{7.30}$$

$$\forall x \in \mathbb{R}, \quad \frac{r}{\Theta} \int_\Theta a(x,\theta) d\theta \leq H(x) \leq ra(x, \bar{\theta}(x)) \tag{7.31}$$

where $\bar{\theta}$ is a trait which maximizes $Q(x,\cdot)$: $Q(x,\bar{\theta}(x)) = \max_\Theta Q(x,\theta)$.

In particular, the eigenvalue $H(x)$, which more or less represents the speed of the front, is not necessarily given by the most privileged individuals, that is those having the largest fitness a. See Example 7.15 for a case where the inequality is strict. This property confirms that the front may slow down due to very unfavorable traits.

Proof of Lemma 7.9. By integrating (7.10) with respect to θ and using the Neumann boundary condition and (7.11), we find (7.30).

To prove (7.31) we rewrite (7.10) in terms of $\eta = \ln Q$:

$$\forall (x,\theta) \in \mathbb{R} \times \Theta, \qquad H(x) = \alpha \left(\Delta_\theta \eta + |\nabla_\theta \eta|^2 \right) + ra(x,\theta) \tag{7.32}$$

Then, integrating and using the Neumann boundary conditions in the variable θ for η, one obtains

$$H(x) \geq \frac{r}{|\Theta|} \int_\Theta a(x,\theta) d\theta.$$

Let $Q(x,\bar{\theta}(x)) = \max_\Theta Q(x,\theta)$. Then $\nabla_\theta \eta(x,\bar{\theta}(x)) = 0$ and $\Delta_\theta \eta(x,\bar{\theta}(x)) \leq 0$. Evaluating (7.32) in $\bar{\theta}(x)$, we get

$$H(x) \leq ra(x,\bar{\theta}(x)).$$

□

Lemma 7.10. *Let $a(x,\cdot)$ be a strictly concave function on $\Theta := [\theta_m, \theta_M]$ for all $x \in \mathbb{R}$. Then for all $x \in \mathbb{R}$, the maximum of $Q(x,\cdot)$ is attained in only one point $\bar{\theta}(x)$.*

Proof of Lemma 7.10. The concavity hypothesis implies that for all $x \in \mathbb{R}$, the function $H(x) - ra(x,\cdot)$ is strictly convex. Thus, on the interval $[\theta_m, \theta_M]$, it has at most two zeros. The case of no zeros is excluded from (7.31). Let's study the two remaining cases.

Suppose it has only one zero at $\hat{\theta}$ (see Example 7.12), say it is positive on $\left[\theta_m, \hat{\theta}\right[$ and nonpositive on $\left[\hat{\theta}, \theta_M\right[$. Then from the early definition of the spectral problem, $Q(x,\cdot)$ is convex on $\left[\theta_m, \hat{\theta}\right[$ and concave on $\left[\hat{\theta}, \theta_M\right[$. The Neumann boundary conditions enforce that $Q(x,\cdot)$ is increasing on Θ, and attains its maximum at $\bar{\theta} = \theta_M$.

Suppose it has two zeroes, at $\hat{\theta}_1$ and $\hat{\theta}_2$. Then $H(x) - ra(x,\cdot)$ is nonnegative on $\left[\theta_m, \hat{\theta}_1\right[$ and $\left[\hat{\theta}_2, \theta_M\right[$, and negative on $\left[\hat{\theta}_1, \hat{\theta}_2\right[$. As a consequence, by the same convexity analysis as in the previous case, $Q(x,\cdot)$ attains its maximum on $\left[\hat{\theta}_1, \hat{\theta}_2\right[$, where it is strictly concave, which justifies the existence and uniqueness of $\bar{\theta}$.

□

Remark 7.11. (Limit as $\alpha \to 0$).

As the mutation rate α goes to 0, we expect that the eigenfunction Q_α converges towards a sum of Dirac masses. To justify this, we use again a WKB ansatz, setting $\varphi_\alpha = \sqrt{\alpha}\ln(Q_\alpha)$. Rewriting (7.10) in terms of φ_α we obtain

$$\sqrt{\alpha}\Delta_\theta\varphi_\alpha + |\nabla_\theta\varphi_\alpha|^2 + ra(x,\theta) - H_\alpha(x) = 0.$$

It is classical that the family φ_α is equi-Lipschitz and we can extract a subsequence that converges uniformly. We have indeed that as $\alpha \to 0$, $(\varphi_\alpha, H_\alpha)$ converges to (φ, H), with φ a viscosity solution of the following equation

$$\begin{cases} |\nabla_\theta\varphi|^2 + ra(x,\theta) - H(x) = 0, \\ H(x) = \max_{\theta\in\Theta} ra(x,\theta). \end{cases}$$

Moreover from (7.11) we obtain that

$$\max_{\theta\in\Theta} \varphi(x,\theta) = 0.$$

Finally, we conclude from the above equations that as $\alpha \to 0$, $Q_\alpha \rightharpoonup Q$ with Q a measure satisfying

$$\text{supp } Q(x,\cdot) \subset \{\theta \in \Theta \,|\, \varphi(x,\theta) = 0\} \subset \{\overline{\theta} \in \Theta \,|\, H(x) = ra(x,\overline{\theta}) = r\max_{\theta\in\Theta} a(x,\theta)\}.$$

In other terms, in the limit of rare mutations, the population concentrates on the maximum points of the fitness $a(x,\theta)$.

7.6 Examples and numerics

7.6.1 Examples of spectral problems

In this section, we present various spectral problems to discuss the properties of the principal eigenfunction Q depending on the form of the fitness a. The principal eigenfunction Q is expected, at least in some cases, to represent the asymptotic phenotypic distribution of the population (see Proposition 7.7). The examples are illustrated in Tables 1 and 2.

Example 7.12. (A fitness with linear dependence on θ).

This example is taken from [34]. For $\theta \in [\theta_m, \theta_M]$ and $b : \mathbb{R} \to \Theta$ a smooth function, let $a(x,\theta) = \mu\theta - b(x)$. The spectral problem writes, for all $x \in \mathbb{R}$:

$$\begin{cases} \alpha\partial_{\theta\theta}Q + r\theta Q = (H(x) + rb(x))\,Q, \\ \partial_\theta Q(x,\theta_m) = \partial_\theta Q(x,\theta_M) = 0. \end{cases}$$

The solution of this problem is unique up to a multiplicative constant and can be expressed implicitly with special Airy functions. In Table 1, for $\alpha = 1$, $r = 2$ and $\Theta = [0,1]$, we plot the fitness $a(\theta) = \frac{\theta}{2} + \frac{1}{4}$ and the associated eigenvector Q. Beware that this example will be used again in Section 7.6.2.

Example 7.13. (The maxima of a and Q are not always at the same points.)

For Example 2, we consider $a(x,\theta) := 1 - |\theta - \theta_m|$, for different values of $\theta_m \in \Theta := [0,1]$. The parameters for the simulations are $\alpha = 1, r = 1$. We observe that although the fitness a attains its

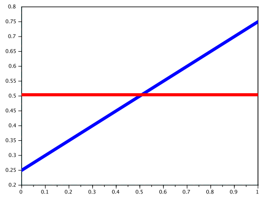
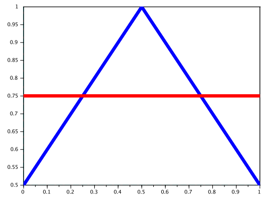
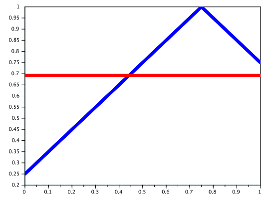
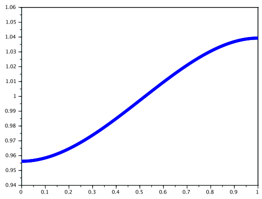
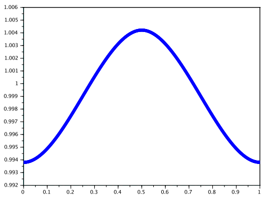
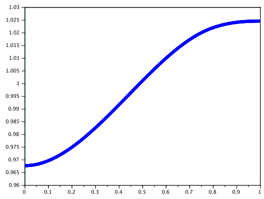
	Example 1	Example 2 ($\theta_m = 0.5$)	Example 2 ($\theta_m = 0.75$)
(1)			
(2)			

TABLE 7.1 – (1) : Fitness a (in blue) and principal eigenvalue H (in red) ; (2) : Renormalized principal eigenfunction Q.

maximum at $\theta = \theta_m$, it is not given that the maximum of the eigenfunction Q is attained at $\theta = \theta_m$. In other words, the trait with the optimal fitness value does not necessarily correspond to the most represented one. Indeed, when $\theta_m = \frac{1}{2}$, the eigenfunction Q is necessarily symmetric with respect to $\theta_m = \frac{1}{2}$, and hence attains a maximum at this point. By contrast, for $\theta_m = \frac{3}{4}$, the most represented trait is not the most favorable one (see Table 1) : The diffusion through the Neumann boundary condition plays a strong role in this case. We observe indeed with this example that, while the fitness a has a non-symmetric profile, the maximum points of Q can be far from the ones of a, due to the diffusion term. However, while α (which equals 1 in this example) takes values close to 0, the maximum points of Q approach the ones of a.

Example 7.14. (An example of a and Q with two maximum points)
For this example, we consider $a(x,\theta) := \varphi_i(\theta)$, for $i = 1,2$, and φ_i a quartic function such that two different traits are equivalently favorable in the population. Nevertheless, Q can still take a single maximum on a different point. First, we consider the following symmetric fitness function :

$$\varphi_1(\theta) := 200\left(\theta - \frac{1}{5}\right)\left(\theta - \frac{2}{5}\right)\left(\theta - \frac{3}{5}\right)\left(\theta - \frac{4}{5}\right),$$

which has two maxima but all traits between the two maxima are also likely to survive. It turns out that the mutation plays a strong role and creates a single peak in Q, which is necessarily $\frac{1}{2}$ by symmetry. In the second case, we consider the fitness function

$$\varphi_2(\theta) := 100\left(\theta - \frac{1}{9}\right)\left(\theta - \frac{1}{3}\right)\left(\theta - \frac{2}{3}\right)\left(\theta - \frac{8}{9}\right),$$

which is still symmetric with respect to the center of Θ. However, since there is a gap between the two traits with the most optimal fitness value, the eigenfunction Q has also two peaks but at different points. See Table 2 for the different plots ($\alpha = 1, r = 1$).

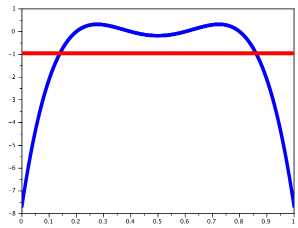
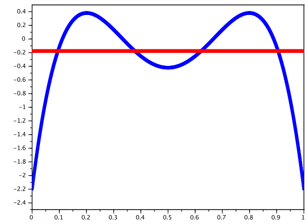
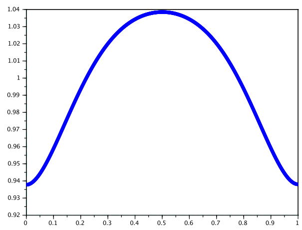
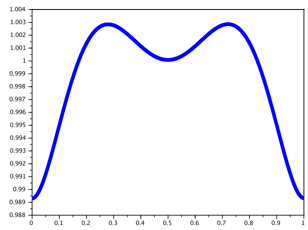
	Exemple 3 (φ_1)	Exemple 3 (φ_2)
(1)		
(2)		

TABLE 7.2 – Fitness a (in blue) and principal eigenvalue H (in red) ; (2) : Renormalized principal eigenfunction Q.

Example 7.15 (An example with Θ unbounded). *Although not within the framework of this article, we expect that under coercivity conditions on $-a$, Theorem 7.5 would be still true with an unbounded domain Θ, and in particular for $\Theta = \mathbb{R}^d$. Here, we give an example with $\Theta = \mathbb{R}$ for which it is easy to compute the eigenelements Q and H. We consider $a(x,\theta) := a_\infty - \frac{b_\infty}{2}(\theta - b(x))^2$ where $b : \mathbb{R} \to \mathbb{R}$ is a smooth function. We can then compute :*

$$Q(x,\theta) = \exp\left(-\frac{1}{2}\sqrt{\frac{rb_\infty}{2\alpha}}(\theta - b(x))^2\right), \qquad H(x) = H = ra_\infty - \sqrt{\frac{r\alpha b_\infty}{2}}.$$

This suggests that, the most represented trait at position x is given by $\theta = b(x)$, and the speed of the propagation of the population is $2\sqrt{H}$.

These solutions are not valid in a bounded domain since they do not satisfy the Neumann boundary conditions.

We note that, with these parameters, the left inequality in (7.31) is strict, i.e. $H(x) < ra(x,\bar{\theta}(x)) = ra_\infty$, but as $b_\infty \to 0$, which corresponds to the limit case where all the traits are equally favorable, we have $H(x) \to ra_\infty$. Finally, it is interesting to notice here that to ensure front expansion, i.e. $H(x) \geq 0$, the fitness must satisfy the following additional condition $\frac{2a_\infty^2}{b_\infty} \geq \frac{\alpha}{r}$.

7.6.2 Numerical illustrations of the dynamics of the front

In this section, we resolve numerically the evolution problem (7.2) for three different values of a. For all the examples, we choose the following initial data

$$\begin{cases} n^\varepsilon(0,x,\theta) = \max\left(1, 2-8\left(\theta-\frac{1}{2}\right)^2\right) & \text{if } x\in[0,0.5], \theta\in\Theta:=[0,1], \\ n^\varepsilon(0,x,\theta)=0 & \text{otherwise.} \end{cases} \tag{7.33}$$

and use the following parameters

$$\alpha = 1, \qquad r=2, \qquad D=1, \qquad \varepsilon = 0.1. \tag{7.34}$$

The numerical simulations have been performed in Matlab. We gather our results in Figures 1 - 2 - 3. For the three different fitness functions, we plot, from left to right :

(+) The density $n^\varepsilon(t,x,\theta)$ for a given final time $t=T$,

(+) The value of $\rho^\varepsilon(x)$ at this same final time (blue line), that we compare to the value of $\max\left(\frac{H(x)}{r},0\right)$ (red line),

(+) The renormalized trait distributions at the edge of the front (red square-shaped line) and at the back (blue star-shaped line) that we compare to the expected renormalized eigenfunctions Q at the same space positions (pink circle-shaped lines).

The fitness functions used in the three figures, are respectively

$$a_1(\theta) = \frac{1}{4}+\frac{\theta}{2},$$

$$a_2(x,\theta) = a_1(\theta)+\left(\sin(x)-\frac{1}{2}\right),$$

and

$$a_3(x,\theta) = a_1(\theta)\left(1+\frac{1}{1+0.05\,x^2}\right).$$

For the three examples, we observe propagation in the x–direction as expected according to Theorem 7.2. We also notice that, in the zones where the front has arrived, i.e. in the set $\mathrm{Int}\{u=0\}$, ρ^ε converges to $\max(\frac{H}{r},0)$. Moreover, for ε small, the renormalized trait distribution of the population at position x, i.e. $\frac{n_\varepsilon(t,x,\cdot)}{\int n_\varepsilon(t,x,\theta' d\theta')}$, is close to $Q(x,\cdot)$. These properties have been proved theoretically for a particular case in Proposition 7.7. We also notice that the convergence of the averaged density ρ^ε seems to be faster than the convergence of the density n^ε.

In Figure 2, we illustrate an example where H is periodic in x and it can take negative values. This corresponds to a case where the population faces some obstacles, i.e. zones where the conditions are not favorable for the population to persist. However, according to the numerical illustrations, the population manages to pass through the obstacles and reach the favorable zones where it can grow up again. Indeed, even if asymptotically as $\varepsilon\to 0$ the density n_ε goes to 0 in these harsh zones, in the ε–level, n_ε is positive but exponentially small. This small density can reach the better zones and grow up. Note that in this case, since we

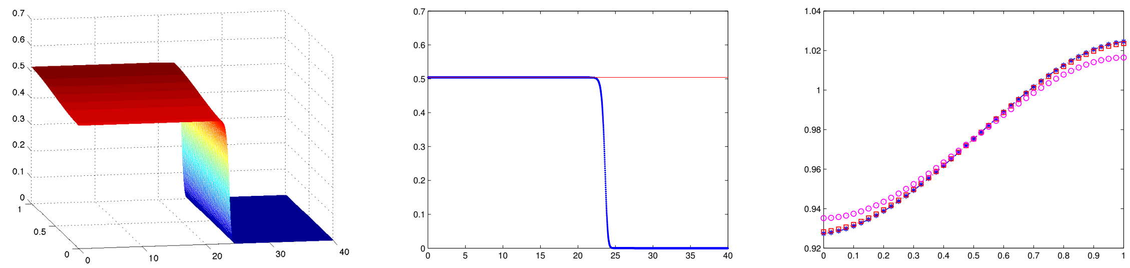


FIGURE 7.1 – We present numerical resolution of (7.2) with the fitness $a_1(\theta) = \frac{1}{4} + \frac{\theta}{2}$ and using the initial data and the parameters given by (7.33) and (7.34). In this case, as expected, ρ^ε converges to $\frac{H}{r}$ in the zone where the front has arrived : in the set $\{u = 0\}$. We also observe that the renormalized trait distribution at the edge (red square-shaped line) and the back of the front (blue star-shaped line), are close to the principal eigenfunction Q (pink circle-shaped line), noting that Q here does not depend on x. These results are in accordance with Proposition 7.7.

consider a periodic growth rate, the solution behaves as a pulsating wave in the x–direction. We refer the interested reader for instance to [24] for a study of pulsating waves. See Figures 1, 2 and 3 for detailed comments.

Acknowledgment

S. M. wishes to thank Gaël Raoul for early discussions and computations on this problem.

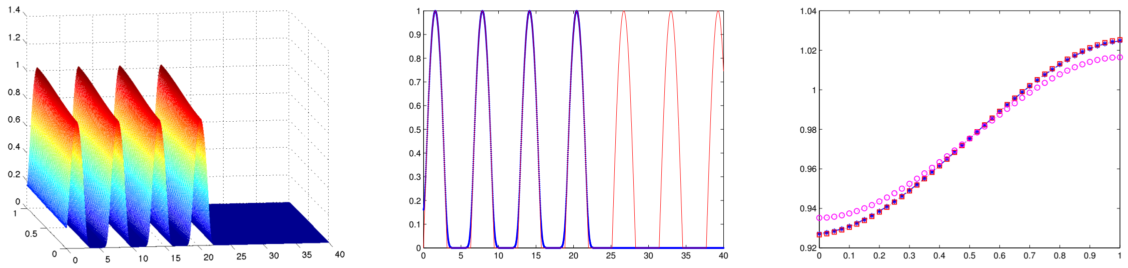


FIGURE 7.2 – This pulsating wave is obtained from the numerical resolution of (7.2) with the fitness $a_2(x,\theta) = a_1(\theta) + \left(\sin(x) - \frac{1}{2}\right)$ and using the initial data and the parameters given by (7.33) and (7.34). The same conclusions as for the fitness a_1 hold. Noticing that H can take negative values in some zones which are unfavorable for the population, we observe that the population can pass through the obstacles and grow up in the favorable zones.

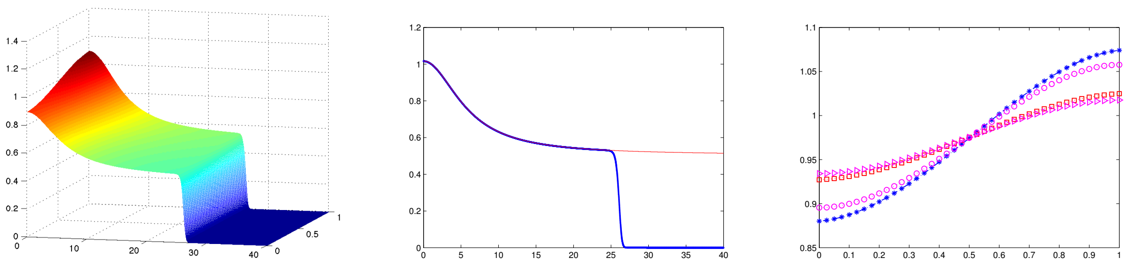


FIGURE 7.3 – We present numerical resolution of (7.2) with the fitness $a_3(x,\theta) = a_1(\theta)\left(1 + \frac{1}{1+0.05x^2}\right)$ and using the initial data and the parameters given by (7.33) and (7.34). In this case, we have numerically obtained the Hamiltonian H, which depends nontrivially on the fitness a_3. We find again that the density ρ_ε converges towards $\frac{H(x)}{r}$. Finally, we also observe an error of $O(\varepsilon)$ between the renormalized trait distributions at the edge and the back of the front with the corresponding eigenfunctions $Q(x,\cdot)$.

Annexe A

Perspective : À propos de la dispersion cinétique en domaine non-borné

Cette annexe présente des éléments d'un travail en cours avec Vincent Calvez, Emmanuel Grenier et Grégoire Nadin. Nous étudions la dispersion dans une équation cinétique de type BGK, dans un domaine de vitesses non-borné, avec une Maxwellienne Gaussienne. Nous mettons en place un nouveau changement d'échelle compatible avec les phénomènes d'accélération constatés dans le Chapitre 2. Nous dérivons le système limite et prouvons un résultat d'unicité de viscosité pour celui-ci. Par ailleurs, nous donnons la forme d'une solution particulière correspondant à une solution fondamentale du système limite. Bien que très partiels, ces éléments permettent de justifier certaines motivations et directions de recherche prises dans dette thèse.

Certaines difficultés subsistent, notamment le passage rigoureux à la limite, l'extension du résultat d'unicité pour des solutions discontinues.

A.1 Introduction

In this Annex, we want to study large deviations for the following (and apparently the simplest) kinetic equation of BGK type :

$$\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R} \times V, \qquad \partial_t f + v \cdot \nabla_x f = M(v)\rho - f. \tag{A.1}$$

We denote the microscopic density by $f(t,x,v)$, and by ρ the macroscopic density

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \qquad \rho(t,x) = \int_V f(t,x,v)dv.$$

Here, the velocity set is the full space $V = \mathbb{R}$, and the density M is a Gaussian

$$\forall v \in V, \qquad M(v) = \frac{1}{\sqrt{\pi}} \exp\left(-v^2\right).$$

In [35], the authors have studied travelling waves for a kinetic-reaction-transport equation with techniques coming from reaction-diffusion. It is shown that the boundedness of the velocity set V is a necessary and sufficient condition for existence of travelling waves. In case of an unbounded velocity set, a new phenomena of front acceleration is discussed. Superlinear spreading is proved and results concerning the associated rate of spreading are given. It appears that the rate of spreading depends strongly on the decay at infinity of the stationary Maxwellian.

Our issue here is to understand more closely the properties of the dispersal operator that are responsible for this front acceleration in the case of a unbounded velocity domain. Our approach relies on performing a scaling limit via an Hopf-Cole transformation. This approach, which has been used for reaction diffusion equations in the last decade, has been recently used for models of kinetic type. In [33] in particular, the large deviations of (A.1) in the case of a bounded velocity case are studied.

Let us recall very briefly the sketch in this case, since it should be compared to what will happen in the unbounded velocity case. If we focus on the large scale hyperbolic limit $(t,x) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$, $\varepsilon \to 0$, the kinetic equation (A.1) reads as follows in the new scaling :

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon}\left(M(v)\rho^\varepsilon - f^\varepsilon\right), \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V. \tag{A.2}$$

Clearly, the velocity distribution relaxes rapidly towards the Maxwellian distribution. This motivates the introduction of the following Hopf-Cole transformation :

$$f^\varepsilon(t,x,v) = M(v) e^{-\frac{\varphi^\varepsilon(t,x,v)}{\varepsilon}}.$$

The equation satisfied by φ^ε reads :

$$\partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon = \int_V M(v') \left(1 - e^{\frac{\varphi^\varepsilon - \varphi^{\varepsilon\prime}}{\varepsilon}}\right) dv', \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V, \tag{A.3}$$

We now recall the main result of [33] :

Theorem A.1. *[33] Let $V \subset \mathbb{R}^n$ be bounded and symmetric, and $M \in L^1(V)$ be nonnegative and symmetric. Then φ^ε converges (locally) uniformly towards φ^0, where φ^0 does not depend on v. Moreover φ^0 is the viscosity solution of the following Hamilton-Jacobi equation :*

$$\int_V \frac{M(v)}{1 - \partial_t \varphi^0(t,x) - v \cdot \nabla_x \varphi^0(t,x)} dv = 1, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{A.4}$$

The denominator of the integrand is positive for all $v \in V$.

We emphasize that in this case, the limit phase is independent of the velocity variable and solves an Hamilton-Jacobi equation. Moreover, the last observation in Theorem A.1 is not compatible with an unbounded velocity set.

It turns out that to understand the dispersion via scaling limits of (A.1) when $V = \mathbb{R}$, one has to find and analyze another type of scaling. The purpose of this work is to introduce and to analyze another change of scaling for (A.1). Inspired by our parallel work [35], we perform the new scaling

$$(t,x,v) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\frac{3}{2}}}, \frac{v}{\varepsilon^{\frac{1}{2}}}\right).$$

This should be relevant as it roughly indicates that along a propagating solution, x should behave as $t^{\frac{3}{2}}$, which is what we expect from [35]. One should now perform in (A.1) the following WKB ansatz

$$f^{\varepsilon}(t,x,v) = \exp\left(-\frac{\varphi^{\varepsilon}(t,x,v)}{\varepsilon}\right).$$

It yields

$$\partial_t \varphi^{\varepsilon} + v \cdot \nabla_x \varphi^{\varepsilon} = 1 - \frac{1}{\sqrt{\pi\varepsilon}} \int_V \exp\left(\frac{\varphi^{\varepsilon}(v) - \varphi^{\varepsilon}(v') - v^2}{\varepsilon}\right) dv', \tag{A.5}$$

This equation is the equivalent of what was (A.3) in the case of bounded velocities. Our aim is to pass rigorously to the limit in (A.5). We can now state the convergence result that identifies the system that φ^{ε} solves in the limit $\varepsilon \to 0$.

Formal Result A.2 (Convergence when $\varepsilon \to 0$)**.** *We define $\mathcal{S}(u)(t,x) = argmin\,(u(t,x,\cdot))$. Let φ^{ε} be the solution of* (A.5)*. Then φ^{ε} converges when $\varepsilon \to 0$ towards φ^0, a viscosity solution of the following system*

$$[S]\begin{cases} \max\left(\partial_t \varphi^0 + v \cdot \nabla_x \varphi^0 - 1, \varphi^0 - \min_V \varphi^0 - v^2\right) = 0, \quad \forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V, \\ \partial_t \left(\min_V \varphi^0\right) \leq 0, \qquad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ \partial_t \left(\min_V \varphi^0\right) = 0 \qquad \textit{if} \qquad \mathcal{S}(\varphi^0)(t,x) = \{0\}, \\ \varphi^0(0,x,v) := \varphi_0(x,v), \qquad \forall (x,v) \in \mathbb{R}^n \times V. \end{cases} \tag{A.6}$$

It is worth making some comments concerning the structure of the limit system. First of all, it is not a standard Hamilton-Jacobi equation as is obtained in [33]. Moreover, we see that the first equation of the system, that we might have thought to be the limiting equation after (A.5), is not sufficient due to the apparition of $\min_V \varphi^0$ for which we need extra information. We now state a comparison principle for Lipschitz continuous viscosity solutions of the limit system $[S]$ (A.6) :

Theorem A.3 (Comparison principle)**.**
Let $T > 0$. Let u_1 (resp. u_2) be a viscosity sub-solution (resp. super-solution) of the limit system $[S]$ on $[0,T] \times \mathbb{R}^n \times V$. Assume that u_1 and u_2 are such that $b_{1,2} := u_{1,2} - v^2$ are two bounded functions on $[0,T] \times \mathbb{R}^n \times V$. Let us define, for $(t,x) \in [0,T] \times \mathbb{R}$, the minima with respect to the velocity variable :

$$m_1(t,x) := \min_V u_1(t,x,\cdot), \qquad m_2(t,x) := \min_V u_2(t,x,\cdot).$$

Suppose finally that b_1 and m_1 are Lipschitz, and that u_1 satisfies the following

Hypothesis A.4. *There exists a function $\varphi : \mathbb{R}^+ \mapsto \mathbb{R}^+$, which satisfies $\lim_{\delta \to 0} \varphi(\delta) = 0$, such that : If for some $(t,x) \in [0,T] \times \mathbb{R}^n$, there exists $\delta > 0$ such that there exists $v \in V$ such that*

$$0 \leq u_1(t,x,v) - m_1(t,x) \leq \delta,$$

then, there exists $\overline{v} \in \mathcal{S}_1(t,x)$ such that $|v - \overline{v}| \leq \varphi(\delta)$.

Then $u_1 \leq u_2$ on $[0,T] \times \mathbb{R}^n \times V$.

This theorem provides uniqueness (and thus well-posedness) of a suitable solution for the limiting system $[S]$. The rest of this note is organized as follows. In the following Section A.2, we indicate some elements of proof of the formal convergence result (Formal Result A.2). The Section A.3 is devoted to the proof of the comparison principle (Theorem A.3). We then give the fundamental solution of the limit system in Section A.4. This is for us the core of this Annex, since it announces the accelerated propagation.

A.2 Towards the limit equation when $\varepsilon \to 0$.

In this Section, we give some elements how to prove the Formal Result A.2. It is not yet a Theorem since the last step remains formal.

About the Formal Result A.2. We obtain a unique solution φ^ε from a fixed point method on the Duhamel formulation of (A.5) :

$$\varphi^\varepsilon(t,x,v) = \varphi_0(x-tv,v) + \frac{1}{\sigma\sqrt{2\pi}}\int_0^t \int_V \left(1 - e^{\frac{\varphi^\varepsilon(t-s,x-sv,v)-\varphi^\varepsilon(t-s,x-sv,v')-v^2}{\varepsilon}}\right) dv'ds, \tag{A.7}$$

We now define b_ε through the following formula $b_\varepsilon := \varphi^\varepsilon - v^2$. It satisfies

$$\partial_t b^\varepsilon + v\cdot\nabla_x b^\varepsilon = 1 - \int_V M_\varepsilon(v')\exp\left(\frac{b^\varepsilon(v)-b^\varepsilon(v')}{\varepsilon}\right)dv',$$

We need to go through the two following steps.

 # Step 1 : Uniform estimates.

We obtain directly from the Duhamel formulation,

$$\forall\varepsilon>0, \qquad b_0(x-tv,v) \le b^\varepsilon(t,x,v) \le b_0(x-tv,v)+t\,.$$

This ensures that b^ε is uniformly bounded on $[0,T]\times\mathbb{R}^n\times V$. To prove the bound (3.9), we define :

$$\psi^\varepsilon_\delta(t,x,v) = b^\varepsilon(t,x,v) - C\delta t - \delta^4|x|^2 - 2\delta|v|.$$

For any $\delta>0$, ψ^ε_δ attains a maximum at point $(t_\delta,x_\delta,v_\delta)$. Suppose that $t_\delta>0$. Then, we have

$$\partial_t b^\varepsilon(t_\delta,x_\delta,v_\delta) \ge C\delta, \qquad \nabla_x b^\varepsilon(t_\delta,x_\delta,v_\delta) = 2\delta^4 x_\delta.$$

As a consequence, we have at the maximum point $(t_\delta,x_\delta,v_\delta)$:

$$\begin{aligned}
C\delta + 2v_\delta\delta^4 x_\delta &\le 1 - \textstyle\int_V M_\varepsilon(v')e^{\frac{2\delta|v_\delta|-2\delta|v'|}{\varepsilon}}e^{\frac{\psi^\varepsilon_\delta(t_\delta,x_\delta,v_\delta)-\psi^\varepsilon_\delta(t_\delta,x_\delta,v')}{\varepsilon}}dv',\\
&\le 1 - e^{\frac{\delta|v_\delta|+\delta^2}{\varepsilon}}\operatorname{erfc}\left(\tfrac{\delta}{\sqrt{\varepsilon}}\right),\\
&\le 1 - e^{\frac{\delta^2}{\varepsilon}}\operatorname{erfc}\left(\tfrac{\delta}{\sqrt{\varepsilon}}\right).
\end{aligned} \tag{A.8}$$

Moreover, the maximal character of $(t_\delta,x_\delta,v_\delta)$ also implies

$$\|b^\varepsilon\|_\infty - \delta^4|x_\delta|^2 - 2\delta|v_\delta| \ge b^\varepsilon(t_\delta,x_\delta,v_\delta) - C\delta t_\delta - \delta^4|x_\delta|^2 - 2\delta|v_\delta| \ge b^\varepsilon(0,0,v_\delta) \ge 0\,.$$

It yields that $|x_\delta| \le \frac{\|b^\varepsilon\|_\infty^{\frac{1}{2}}}{\delta^2}$ and $|v_\delta| \le \frac{\|b^\varepsilon\|_\infty}{2\delta}$. Introducing these last inequalities in (A.8) yields

$$C\delta - \|b^\varepsilon\|_\infty^{\frac{3}{2}}\delta \le 1 - e^{\frac{\delta^2}{\varepsilon}}\operatorname{erfc}\left(\frac{\delta}{\sqrt{\varepsilon}}\right),$$

and thus

$$C - \|b^\varepsilon\|_\infty^{\frac{3}{2}} \le \frac{1}{\delta}\left(1 - e^{\frac{\delta^2}{\varepsilon}}\operatorname{erfc}\left(\frac{\delta}{\sqrt{\varepsilon}}\right)\right).$$

One can choose C such that, for sufficiently small δ, this last inequality is impossible since the r.h.s is $\mathcal{O}(1)$ when $\delta \to 0$. As a consequence $t_\delta = 0$, and we have,

$$\forall (t,x,v) \in [0,T]\times\mathbb{R}^n\times V, \qquad b^\varepsilon(t,x,v) \le b^0(x_\delta, v_\delta) + C\delta t + \delta^4|x|^2 + 2\delta|v| \le \|b_0\|_\infty + \delta t + \delta^4|x|^2.$$

Passing to the limit $\delta \to 0$, we obtain $b^\varepsilon(t,x,v) \le \|b_0\|_\infty$.

To find Lipschitz bounds, we use the same ideas on the difference $b_h^\varepsilon(t,x,v) = b^\varepsilon(t,x+h,v) - b^\varepsilon(t,x,v)$. The equation for b_h^ε reads as follows,

$$\partial_t b_h^\varepsilon + v\cdot\nabla_x b_h^\varepsilon = \int_V M_\varepsilon(v')\exp\left(\frac{b^\varepsilon(v) - b^\varepsilon(v')}{\varepsilon}\right)\left(1 - \exp\left(\frac{b_h^\varepsilon(v) - b_h^\varepsilon(v')}{\varepsilon}\right)\right)dv',$$

Using the same argument as above with a $\delta-$correction, we conclude that

$$\forall (t,x,v) \in [0,T]\times\mathbb{R}^n\times V, \qquad b_h^\varepsilon(t,x,v) \le \sup_{(x,v)\in\mathbb{R}\times V} |b_0(x+h,v) - b_0(x,v)|$$

The same argument applies to $-b_h^\varepsilon$,

$$\partial_t(-b_h^\varepsilon) + v\cdot\nabla_x(-b_h^\varepsilon) = -\int_V M_\varepsilon(v')e^{\frac{b^\varepsilon(v)-b^\varepsilon(v')}{\varepsilon}}\left(1 - \exp\left(-\frac{(-b_h^\varepsilon)(v) - (-b_h^\varepsilon)(v')}{\varepsilon}\right)\right)dv',$$

so that the r.h.s has the right sign when $-b_h^\varepsilon$ attains a maximum. Finally,

$$\forall (t,x,v) \in [0,T]\times\mathbb{R}^n\times V, \quad |b_h^\varepsilon(t,x,v)| \le \sup_{(x,v)\in\mathbb{R}\times V}|b_0(x+h,v) - b_0(x,v)| \le \|\nabla_x b_0\|_\infty |h|.$$

from which the estimate follows.

To obtain regularity in the velocity variable, we differentiate (A.5) with respect to v,

$$(\partial_t + v\cdot\nabla_x)(\nabla_v b^\varepsilon) = -g_\varepsilon(b^\varepsilon)\nabla_v b^\varepsilon - \nabla_x b^\varepsilon,$$

where $g_\varepsilon(b^\varepsilon) = \frac{1}{\varepsilon}\int_V M_\varepsilon(v')e^{\frac{b^\varepsilon - b^{\varepsilon\prime}}{\varepsilon}}dv' \ge 0$. Multiplying by $\frac{\nabla_v b^\varepsilon}{|\nabla_v b^\varepsilon|}$, we obtain

$$\begin{aligned}(\partial_t + v\cdot\nabla_x)(|\nabla_v b^\varepsilon|) &= -g_\varepsilon(b^\varepsilon)|\nabla_v b^\varepsilon| - \left(\nabla_x b^\varepsilon\cdot\frac{\nabla_v b^\varepsilon}{|\nabla_v b^\varepsilon|}\right)\\ &\le \|\nabla_x b_0\|_\infty.\end{aligned}$$

from which we deduce

$$\|\nabla_v b^\varepsilon\|_\infty \le \|\nabla_v b_0\|_\infty + t\|\nabla_x b_0\|_\infty.$$

Finally, a time bound is obtained similarly as the bound on $\nabla_x \varphi^\varepsilon$, using the difference $b_s^\varepsilon(t,x,v) = b^\varepsilon(t+s,x,v) - b^\varepsilon(t,x,v)$. We obtain

$$\forall (t,x,v) \in [0,T] \times \mathbb{R}^n \times V, \qquad |b_s^\varepsilon(t,x,v)| \leq \sup_{(x,v)\in\mathbb{R}\times V} |b^\varepsilon(s,x,v) - b_0(x,v)| \ .$$

We use the Duhamel formulation (A.7) to estimate the last contribution :

$$|b^\varepsilon(s,x,v) - b_0(x,v)| \leq |b_0(x - sv, v) - b_0(x,v)| + o(s) \, ,$$

and thus

$$\forall (t,x,v) \in [0,T] \times \mathbb{R}^n \times V, \qquad \|\partial_t b^\varepsilon\|_\infty \leq \|v \nabla_x b_0\|_\infty .$$

 # Step 2.1 : Viscosity subsolution.

Thanks to the previous step, the sequence φ^ε converges locally uniformly towards φ^0. One wants to prove that φ^0 is a sub solution of (A.6), that is to say that φ^0 solves the following

$$\begin{cases} \partial_t \varphi^0 + v \cdot \nabla_x \varphi^0 - 1 \leq 0, & \forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V, \\ \varphi^0 - \min_V \varphi^0 - v^2 \leq 0, & \forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V, \\ \partial_t \left(\min_V \varphi^0\right) \leq 0, & \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \end{cases}$$

in the viscosity sense. The first inequality comes directly from the fact that

$$\forall \varepsilon > 0, \qquad \forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V, \qquad \partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon - 1 \leq 0,$$

holds and thus one can pass to the limit in the viscosity sense. Now, to prove that the constraint is necessarily unsaturated, assume by contradiction that there exists a point $(t_0, x_0, v_0) \in \mathbb{R}^+ \times \mathbb{R}^n \times V$ such that

$$\left(\varphi^0 - \min_V \varphi^0 - v^2\right)(t_0, x_0, v_0) > 0.$$

Thanks to the local uniform convergence, it turns out that there exists $\delta > 0$ such that for ε sufficiently small, $\left(\varphi^\varepsilon - \min_V \varphi^\varepsilon - v^2\right)(t_0, x_0, v_0) > \delta$. This gives some compact K such that,

$$\forall v' \in K, \qquad \varphi^\varepsilon(t_0, x_0, v_0) - \varphi^\varepsilon(t_0, x_0, v') - v_0^2 > \frac{\delta}{2}.$$

Replacing in (A.5)

$$\begin{aligned} \frac{1}{\sigma\sqrt{2\pi}} \frac{1}{\sqrt{\varepsilon}} \int_V \exp\left(\frac{\varphi^\varepsilon(v_0) - \varphi^\varepsilon(v') - \frac{v_0^2}{2\sigma^2}}{\varepsilon}\right) dv' &> \frac{1}{\sigma\sqrt{2\pi}} \frac{1}{\sqrt{\varepsilon}} \int_K \exp\left(\frac{\varphi^\varepsilon(v_0) - \varphi^\varepsilon(v') - \frac{v_0^2}{2\sigma^2}}{\varepsilon}\right) dv' \\ &> \frac{1}{\sigma\sqrt{2\pi}} \frac{1}{\sqrt{\varepsilon}} |K| \exp\left(\frac{\delta}{\varepsilon}\right) \to_{\varepsilon\to 0} +\infty. \end{aligned}$$

which contradicts the a priori estimates since $(\partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon)(t_0, x_0, v_0)$ is bounded with respect to ε.

Let us finish with the proof of the last inequality. First, notice that φ^0 attains its minimum for $v = 0$ since evaluating $\varphi^0 - \min_V \varphi^0 - v^2 \leq 0$ in $v = 0$ for all (t,x) gives $\varphi^0(t,x,0) = \min_V \varphi^0(t,x)$.

 # Step 2.2 (Formal Step) : Viscosity supersolution.

We now need to prove that φ^0 is a super solution of (A.6) in the viscosity sense

$$\begin{cases} \partial_t \varphi^0 + v \cdot \nabla_x \varphi^0 - 1 = 0 & \text{in the zone} \quad \left\{(t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V \,|\, \varphi^0 - \min_V \varphi^0 - v^2 < 0\right\}, \\ \partial_t \left(\min_V \varphi^0\right) = 0 & \text{if} \quad \mathcal{S}(\varphi^0)(t,x) = \{0\}. \end{cases}$$

We are now looking for the limit when $\varepsilon \to 0$. We have formally from the stationary phase lemma,

$$\begin{aligned} \frac{1}{\sqrt{\pi\varepsilon}} \int_V \exp\left(\frac{\varphi^\varepsilon(v) - \varphi^\varepsilon(v') - v^2}{\varepsilon}\right) dv' \quad &= \quad \exp\left(\frac{\varphi^\varepsilon(v) - v^2}{\varepsilon}\right) \frac{1}{\sqrt{\pi\varepsilon}} \int_V \exp\left(-\frac{\varphi^\varepsilon(v')}{\varepsilon}\right) dv' \\ &\sim_{\varepsilon\to 0} \quad \left(\sum_{v' \in \mathcal{S}(\varphi^\varepsilon)} \frac{1}{\sqrt{|\partial_{vv}\varphi^\varepsilon(v')|}}\right) \exp\left(\frac{\varphi^\varepsilon(v) - \min_V \varphi^\varepsilon - v^2}{\varepsilon}\right) \end{aligned}$$

which would give the result. Unfortunately, some *a priori* regularity is needed to make this argument rigorous, and we don't have it directly.

□

A.3 Uniqueness result for the limit system.

Proof of Theorem A.3. Let us first write the systems which are satisfied by u_1 and u_2 in the viscosity sense :

$$[S_1] \begin{cases} \partial_t u_1 + v \cdot \nabla_x u_1 - 1 \le 0, \quad (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V, \\ u_1 - \min_V u_1 - v^2 \le 0, \quad (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^n \times V, \\ \partial_t \left(\min_V u_1\right) \le 0, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ u_1(0,x,v) \le \varphi_0(x,v), \qquad (x,v) \in \mathbb{R}^n \times V. \end{cases} \tag{A.9}$$

$$[S_2] \begin{cases} \text{In } \left\{(t,x,v) \,|\, u_2 - \min_V u_2 - v^2 < 0\right\}, \\ \qquad\qquad \partial_t u_2 + v \cdot \nabla_x u_2 - 1 \ge 0, \\ \partial_t \left(\min_V u_2\right) \ge 0 \qquad \text{if} \qquad \mathcal{S}(u_2)(t,x) = \{0\}, \\ u_2(0,x,v) \ge \varphi_0(x,v), \qquad (x,v) \in \mathbb{R}^n \times V. \end{cases} \tag{A.10}$$

Note that the coercivity of u_1 and u_2 ensure that the minima in the velocity variable that appear in $[S_1]$ and $[S_2]$ and well defined and attained. For given $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$ we define

$$\mathcal{S}_{1,2}(t,x) := \operatorname{argmin}\left(u_{1,2}\left(t,x,\cdot\right)\right).$$

Note that by assumption, $b_{1,2}$ are bounded. It turns out that

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \qquad \inf b_{1,2} \le m_{1,2}(t,x) \le u_{1,2}(t,x,0) = b_{1,2}(t,x,0) \le \|b_{1,2}\|_\infty$$

and that for all $\overline{v} \in \mathcal{S}_{1,2}(t,x)$,

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \qquad \overline{v}^2 \le m_{1,2}(t,x) - b_{1,2}(t,x,\overline{v}) \le 2\|b_{1,2}\|_\infty$$

As usual in comparison principles for viscosity solutions, we will perform a doubling of variables argument. Let us define

$$\sigma_1 := \sup_{[0,T]\times\mathbb{R}^n} \left(\min u_1 - \min u_2\right), \qquad \sigma_2 := \sup_{[0,T]\times\mathbb{R}^n\times V} \left(u_1 - u_2\right).$$

From previous arguments, both are finite. One has $\sigma_1 \le \sigma_2$. Indeed, since for all (t,x,v),

$$\min u_1(t,x) - u_2(t,x,v) \le u_1(t,x,v) - u_2(t,x,v) \le \sigma_2,$$

the result yields taking the min in v and then the sup in (t,x) in the l.h.s..

We now assume by contradiction that $\sigma_2 > 0$. We now continue the proof with two different steps.

 # Step 1 : Suppose that $\sigma_1 < \sigma_2$.

Since σ_2 is not necessarily attained, take a point $(t_0, x_0, v_0) \in]0,T[\times\mathbb{R}^n \times V$ such that

$$\left(u_1 - u_2\right)(t_0, x_0, v_0) > \frac{\sigma_1 + 3\sigma_2}{4}.$$

We perform a doubling of variables in the (t,x) variables only since there is no derivative with respect to the v-variable in $[S]$. This avoids problems of unboundedness in the v direction. Let us define, for $(t,s,x,y,v) \in [0,T]^2 \times \mathbb{R}^2 \times V$,

$$\begin{aligned}\mathbb{P}(t,s,x,y,v) := \frac{1}{2\varepsilon^2}|x-y|^2 + \frac{\delta^2}{2}\left(|x-x_0|^2 + |y-x_0|^2\right) \\ + \mu^2|v-v_0|^2 + \frac{1}{2\nu^2}|t-s|^2 + \frac{\alpha}{2}(t+s) + \frac{1}{2}\left(\frac{w}{T-t} + \frac{w}{T-s}\right)\end{aligned}$$

and

$$\psi(t,s,x,y,v) := u_1(t,x,v) - u_2(s,y,v) - \mathbb{P}(t,s,x,y,v).$$

Since $u_1(t,x,v) - u_2(s,y,v) = b_1(t,x,v) - b_2(s,y,v)$ and b_1 and b_2 are bounded, ψ is coercive in all variables. It turns out that ψ as a maximum at a point $(\bar{t},\bar{s},\bar{x},\bar{y},\bar{v})_{\varepsilon,\delta,\mu,\nu,\alpha,w}$. Moreover, taking α sufficiently small, namely such that $\alpha T + \frac{w}{T-t_0} < \frac{\sigma_2-\sigma_1}{4}$, the value of the maximum also satisfies

$$\max_{[0,T]^2\times\mathbb{R}^2\times V} \psi \ge \psi(t_0,t_0,x_0,x_0,v_0) = \left(u_1 - u_2\right)(t_0,x_0,v_0) - \alpha t_0 - \frac{w}{T-t_0} > \frac{\sigma_1+\sigma_2}{2}.$$

It is classical from the theory of viscosity solutions to evaluate ψ in various relevant points to find good estimates of quantities as *e.g.* $\frac{1}{2\varepsilon^2}|\bar{x}-\bar{y}|$:

Lemma A.5 (Properties of the point of maximum). *Assume that b_1 and b_2 are bounded and that b_2 is Lipschitz. Then the point of maximum $(\bar{t},\bar{s},\bar{x},\bar{y},\bar{v})_{\varepsilon,\delta,\mu,\nu,\alpha}$ satisfies the following estimates :*

$$|\bar{x}-x_0| \le \frac{2}{\delta}\left(\|b_1\|_\infty + \|b_2\|_\infty\right)^{\frac{1}{2}}, \qquad |\bar{y}-x_0| \le \frac{2}{\delta}\left(\|b_1\|_\infty + \|b_2\|_\infty\right)^{\frac{1}{2}}, \tag{A.11}$$

$$|\bar{x}-\bar{y}| \le 2\left(\mathrm{Lip}(b_1) + 2\delta\left(\|b_1\|_\infty + \|b_2\|_\infty\right)^{\frac{1}{2}}\right)\varepsilon^2, \qquad |\bar{t}-\bar{s}| \le 2\left(\mathrm{Lip}(b_1) + \frac{\alpha}{2}\right)\nu^2. \tag{A.12}$$

Moreover, for sufficiently small ν, both of $\bar{t}$ and $\bar{s}$ are positive.

Proof of Lemma A.5. Evaluating in (t_0, x_0, v_0),

$$(u_1 - u_2)(t_0, x_0, v_0) - \alpha t_0 - \frac{w}{T - t_0} \le u_1(\bar{t}, \bar{x}, \bar{v}) - u_2(\bar{s}, \bar{y}, \bar{v}) - \frac{1}{2\varepsilon^2}|\bar{x} - \bar{y}|^2$$
$$- \frac{\delta^2}{2}\left(|\bar{x} - x_0|^2 + |\bar{y} - x_0|^2\right) - \mu^2|\bar{v} - v_0|^2 - \frac{1}{2\nu^2}|\bar{t} - \bar{s}|^2 - \frac{\alpha}{2}\left(\bar{t} + \bar{s}\right),$$

and this gives

$$\frac{1}{2\varepsilon^2}|\bar{x} - \bar{y}|^2 + \frac{\delta^2}{2}\left(|\bar{x} - x_0|^2 + |\bar{y} - x_0|^2\right)$$
$$+ \mu^2|\bar{v} - v_0|^2 + \frac{1}{2\nu^2}|\bar{t} - \bar{s}|^2 + \frac{\alpha}{2}\left(\bar{t} + \bar{s}\right) \le 2\left(\|b_1\|_\infty + \|b_2\|_\infty\right), \quad \text{(A.13)}$$

and thus (A.11). Simplifying $\psi(\bar{t}, \bar{s}, \bar{y}, \bar{y}, \bar{v}) \le \psi(\bar{t}, \bar{s}, \bar{x}, \bar{y}, \bar{v})$ yields

$$\begin{aligned} \frac{1}{2\varepsilon^2}|\bar{x} - \bar{y}|^2 &\le u_1(\bar{x}, \bar{s}, \bar{v}) - u_1(\bar{y}, \bar{s}, \bar{v}) + \frac{\delta^2}{2}\left(|\bar{x} - x_0|^2 - |\bar{y} - x_0|^2\right), \\ &\le \left(\mathrm{Lip}(b_1) + \frac{\delta^2}{2}|\bar{x} - x_0 + \bar{y} - x_0|\right)|\bar{x} - \bar{y}|. \end{aligned}$$

Combining with (A.11), we get

$$\frac{1}{2\varepsilon^2}|\bar{x} - \bar{y}| \le \mathrm{Lip}(b_1) + 2\delta\left(\|b_1\|_\infty + \|b_2\|_\infty\right)^{\frac{1}{2}}.$$

The same arguments with $(\bar{s}, \bar{s}, \bar{x}, \bar{y}, \bar{v})$ give

$$\frac{1}{2\nu^2}|\bar{t} - \bar{s}| \le \mathrm{Lip}(b_1) + \frac{\alpha}{2},$$

and we obtain (A.12). Let us now show that for a sufficiently small ν, we have $\bar{t}, \bar{s} > 0$. Suppose first that both are 0 simultaneously. Then, the comparison between initial conditions yields

$$\psi(0, 0, x, y, v) \le u_1(0, x, v) - u_2(0, y, v) - \frac{1}{2\varepsilon^2}|x - y|^2 - \frac{\delta^2}{2}\left(|x|^2 + |y|^2\right) - \mu^2|v|^2$$
$$\le (u_1 - u_2)(0, x, v) + \mathrm{Lip}(b_1)|x - y| - \frac{1}{2\varepsilon^2}|x - y|^2 \le \mathcal{O}(\varepsilon^2). \quad \text{(A.14)}$$

and thus a point $(0, 0, x, y, v)$ cannot be a maximum point for sufficiently small ε since $\sup \psi > \sigma_1$. Next, suppose that one of $\bar{t}$ and $\bar{s}$ is positive. Then for sufficiently small ν, the other one is also positive since $|\bar{t} - \bar{s}| = \mathcal{O}\left(\nu^2\right)$. Finally, since ψ tends to $-\infty$ when $t, s \to T$, necessarily $\bar{t}, \bar{s} < T$.

□

We now prove the crucial fact that for suitable parameters $\varepsilon, \delta, \mu, \nu, \alpha$, the inequality

$$u_2(\bar{s}, \bar{y}, \bar{v}) < m_2(\bar{s}, \bar{y}) + (\bar{v})^2$$

holds.

Lemma A.6. *For sufficiently small parameters ε, ν, α, $u_2(\bar{s}, \bar{y}, \bar{v}) < m_2(\bar{s}, \bar{y}) + (\bar{v})^2$ holds.*

Proof of Lemma A.6. We compute

$$u_2(\bar{s},\bar{y},\bar{v}) - m_2(\bar{s},\bar{y}) - (\bar{v})^2 = u_1(\bar{t},\bar{x},\bar{v}) - m_1(\bar{t},\bar{x}) - (\bar{v})^2 \\ - \psi(\bar{t},\bar{s},\bar{x},\bar{y},\bar{v}) - \mathbb{P}(\bar{t},\bar{s},\bar{x},\bar{y},\bar{v}) + \big(m_1(\bar{t},\bar{x}) - m_2(\bar{s},\bar{y})\big)$$

Using again the fact that m_2 is Lipschitz, we can write

$$\begin{aligned} m_1(\bar{t},\bar{x}) - m_2(\bar{s},\bar{y}) &= m_1(\bar{s},\bar{y}) - m_2(\bar{s},\bar{y}) + m_1(\bar{t},\bar{x}) - m_1(\bar{s},\bar{y}), \\ &\leq \sigma_1 + \mathrm{Lip}(m_1)\left(|\bar{t}-\bar{s}| + |\bar{x}-\bar{y}|\right). \end{aligned}$$

Finally, combining with the latter estimate and $u_1(\bar{t},\bar{x},\bar{v}) - m_1(\bar{t},\bar{x}) - (\bar{v})^2 \leq 0$, we obtain,

$$u_2(\bar{s},\bar{y},\bar{v}) - m_2(\bar{s},\bar{y}) - (\bar{v})^2 \leq \sigma_1 - \max_{[0,T]^2\times\mathbb{R}^2\times V} \psi + \mathrm{Lip}(m_1)\left(|\bar{t}-\bar{s}| + |\bar{x}-\bar{y}|\right).$$

Recalling $\max_{[0,T]^2\times\mathbb{R}^2\times V} \psi > \frac{\sigma_1+\sigma_2}{2}$,

$$u_2(\bar{s},\bar{y},\bar{v}) - m_2(\bar{s},\bar{y}) - (\bar{v})^2 < \frac{\sigma_1-\sigma_2}{2} + \mathrm{Lip}(m_1)\left(|\bar{t}-\bar{s}| + |\bar{x}-\bar{y}|\right).$$

We deduce that taking sufficiently small ε, ν, α, the following holds :

$$u_2(\bar{s},\bar{y},\bar{v}) < m_2(\bar{s},\bar{y}) + (\bar{v})^2.$$

□

We now conclude this step with the doubling of variables argument to get a contradiction. We define the following test functions :

$$\psi_1(t,x,v) := u_2(\bar{s},\bar{y},v) + \mathbb{P}(t,\bar{s},x,\bar{y},v), \qquad \psi_2(s,y,v) := u_1(\bar{t},\bar{x},v) - \mathbb{P}(\bar{t},s,\bar{x},y,v).$$

These test functions are defined such that $u_1 - \psi_1$ has a maximum at $(\bar{t},\bar{x},\bar{v})$, and $u_2 - \psi_2$ has a minimum at $(\bar{s},\bar{y},\bar{v})$. Take relevant parameters such that Lemmas A.5 and A.6 hold.

We deduce from the definition of the viscosity sub- and super- solutions and $\bar{t},\bar{s} > 0$ that

$$\frac{1}{2}\frac{w}{(T-\bar{t})^2} + \frac{\bar{t}-\bar{s}}{\nu^2} + \frac{\alpha}{2} + \bar{v}\cdot\left(\frac{\bar{x}-\bar{y}}{\varepsilon^2} + \delta^2(\bar{x}-x_0)\right) - 1 \leq 0, \tag{A.15}$$

and

$$-\frac{1}{2}\frac{w}{(T-\bar{s})^2} + \frac{\bar{t}-\bar{s}}{\nu^2} - \frac{\alpha}{2} + \bar{v}\cdot\left(\frac{\bar{x}-\bar{y}}{\varepsilon^2} - \delta^2(\bar{y}-x_0)\right) - 1 \geq 0,$$
$$\underline{\text{since}} \quad u_2(\bar{s},\bar{y},\bar{v}) < m_2(\bar{s},\bar{y}) + (\bar{v})^2. \tag{A.16}$$

With all this in hand, we are now able to prove that $\sigma_2 > 0$ is impossible. From (A.15) and (A.16), we deduce

$$\frac{1}{2}\frac{w}{(T-\bar{s})^2} + \frac{1}{2}\frac{w}{(T-\bar{t})^2} + \alpha + \delta^2\bar{v}\cdot(\bar{x}-x_0+\bar{y}-x_0) \leq 0, \tag{A.17}$$

But
$$\delta^2|\overline{v}\cdot(\overline{x}-x_0+\overline{y}-x_0)\,| \le \delta^2\frac{4}{\delta}\left(\|b_1\|_\infty+\|b_2\|_\infty\right)^{\frac{1}{2}}|\overline{v}| \to_{\delta\to 0} 0,$$
since $\overline{v}$ is bounded independently from δ (μ is fixed). It yields that for a sufficiently small δ (depending on α), (A.17) is impossible. As a consequence, we have proved by contradiction that $u_1 \le u_2$ on $[0,T]\times\mathbb{R}^n\times V$.

 # Step 2 : Suppose now that $\sigma_1 = \sigma_2 := \sigma$.

This case is now more delicate since we cannot ensure that $u_2(\overline{s},\overline{y},\overline{v}) < m_2(\overline{s},\overline{y}) + (\overline{v})^2$ as previously. Nevertheless, the equality of σ_1 and σ_2 gives us supplementary material that helps to find a contradiction. We will here use crucially the supplementary condition on the time derivative of the minima.

Let us define, for $(t,s,x,y)\in[0,T]^2\times\mathbb{R}^2$, the doubling of variables for $m_1 - m_2$:
$$\chi(t,s,x,y) := m_1(t,x) - m_2(s,y) - \frac{1}{2\nu^2}|t-s|^2 - \frac{1}{2\varepsilon^2}|x-y|^2 - \frac{\mu^2}{2}\left(|x|^2+|y|^2\right) - \frac{\alpha}{2}(t+s) - \frac{w}{T-t} - \frac{w}{T-s}.$$
For legibility, we define
$$\mathbb{P}(t,s,x,y) := \frac{1}{2\nu^2}|t-s|^2 + \frac{1}{2\varepsilon^2}|x-y|^2 + \frac{\mu^2}{2}\left(|x|^2+|y|^2\right) + \frac{\alpha}{2}(t+s) + \frac{w}{T-t} + \frac{w}{T-s}.$$
As m_1 and m_2 are bounded, χ is coercive in the (x,y) variables, and $t,s\in[0,T]$, it turns out that χ has a maximum at a point $(\overline{t},\overline{s},\overline{x},\overline{y})_{\varepsilon,\mu,\nu,\alpha,w}$. We define the same doubling function for $u_1 - u_2$:
$$\tilde{\chi}(t,s,x,y,v) := u_1(t,x,v) - u_2(s,y,v) - \mathbb{P}(t,s,x,y).$$
For all $\delta > 0$, one can take $T(\delta)$ such that $T(\delta) < \frac{\delta}{\alpha}$, and then the parameters ε,μ,ν,w sufficiently small such that $\sup\limits_{[0,T]^2\times\mathbb{R}^2\times V}\tilde{\chi} - \max\limits_{[0,T]^2\times\mathbb{R}^2}\chi \le \delta$ since $\sigma_1=\sigma_2$. We also know that for sufficiently small ν and ε, then $\overline{t},\overline{s}\neq 0,T$. As a consequence, one has
$$\begin{aligned}\max_{[0,T]^2\times\mathbb{R}^2}\chi &= m_1(\overline{t},\overline{x}) - m_2(\overline{s},\overline{y}) - \mathbb{P}(\overline{t},\overline{s},\overline{x},\overline{y})\\ &\le u_1(\overline{t},\overline{x},\mathcal{S}_2(\overline{s},\overline{y})) - u_2(\overline{s},\overline{y},\mathcal{S}_2(\overline{s},\overline{y})) - \mathbb{P}(\overline{t},\overline{s},\overline{x},\overline{y})\\ &\le \sup_{[0,T]^2\times\mathbb{R}^2}\tilde{\chi}\end{aligned}$$
from which we deduce
$$\max_{[0,T]^2\times\mathbb{R}^2}\chi \le u_1(\overline{t},\overline{x},v) + \max_{[0,T]^2\times\mathbb{R}^2}\chi - m_1(\overline{t},\overline{x}) \le \sup_{[0,T]^2\times\mathbb{R}^2}\tilde{\chi}$$
$$0 \le u_1(\overline{t},\overline{x},\mathcal{S}_2(\overline{s},\overline{y})) - u_1(\overline{t},\overline{x},\mathcal{S}_1(\overline{t},\overline{x})) \le \sup_{[0,T]^2\times\mathbb{R}^2}\tilde{\chi} - \max_{[0,T]^2\times\mathbb{R}^2}\chi \le \delta$$
It yields after Hypothesis A.4 that for all $v\in\mathcal{S}_2(\overline{s},\overline{y})$, there exists $\overline{v}\in\mathcal{S}_1(\overline{t},\overline{x})$ such that $|v-w| < \varphi(\delta)$.

Remark A.7. *Note that if we can ensure the same values of the supremums, that is $\delta = 0$ is admissible, then*

$$\mathcal{S}_2(\bar{s}, \bar{y}) \subset \mathcal{S}_1(\bar{t}, \bar{x}).$$

Hypothesis A.4 appears to be a reasonable condition to be able to approximate such property when $\delta > 0$.

We now perform the doubling of variables argument to get a contradiction. We define the following test functions :

$$\chi_1(t,x) := m_2(\bar{s}, \bar{y}) + \mathbb{P}(t, \bar{s}, x, \bar{y}), \qquad \chi_2(s,y) := m_1(\bar{t}, \bar{x}) - \mathbb{P}(\bar{t}, s, \bar{x}, y)$$

These test functions are defined such that $m_1 - \chi_1$ has a maximum at $(\bar{t}, \bar{x})$, and $m_2 - \chi_2$ has a minimum at $(\bar{s}, \bar{y})$. We now discuss on $\mathcal{S}_2(\bar{s}, \bar{y})$.

1. Suppose first that $\mathcal{S}_2(\bar{s}, \bar{y}) = \{0\}$. One then have by the definition of sub- and super-solutions,

$$\partial_s \chi_2(\bar{s}, \bar{y}) \geq 0 \qquad \Longleftrightarrow \qquad -\frac{w}{(T-\bar{s})^2} + \frac{1}{\nu^2}\left(\bar{t} - \bar{s}\right) - \frac{\alpha}{2} \geq 0$$

and

$$\partial_t \chi_1(\bar{t}, \bar{x}) \leq 0 \qquad \Longleftrightarrow \qquad \frac{w}{(T-\bar{t})^2} + \frac{1}{\nu^2}\left(\bar{t} - \bar{s}\right) + \frac{\alpha}{2} \leq 0$$

which combination give $-\alpha \geq 0$, which is a contradiction.

2. Suppose now that there exists $v \in \mathcal{S}_2(\bar{s}, \bar{y})$ such that $v \neq 0$. It turns out that

$$u_2(\bar{s}, \bar{y}, v) - m_2(\bar{s}, \bar{y}) - v^2 < 0,$$

and thus

$$\partial_t m_2(\bar{s}, \bar{y}) + v \cdot \nabla_y m_2(\bar{s}, \bar{y}) - 1 \geq 0,$$

in the viscosity sense, after a chain rule in (A.10) for the Lipschitz function u_2 since v minimizes u_2. This gives

$$-\frac{w}{(T-\bar{s})^2} + \frac{1}{\nu^2}\left(\bar{t} - \bar{s}\right) - \frac{\alpha}{2} + v \cdot \left(\frac{\bar{x} - \bar{y}}{\varepsilon^2} - \mu^2 \bar{y}\right) - 1 \geq 0, \tag{A.18}$$

Now take w such that $\bar{v} \in \mathcal{S}_1(\bar{t}, \bar{x})$ and $|v - \bar{v}| < \delta$. Again, (A.9) and the chain rule give

$$\frac{w}{(T-\bar{t})^2} + \frac{1}{\nu^2}\left(\bar{t} - \bar{s}\right) + \frac{\alpha}{2} + w \cdot \left(\frac{\bar{x} - \bar{y}}{\varepsilon^2} + \mu^2 \bar{x}\right) - 1 \leq 0, \tag{A.19}$$

Combining (A.18) and (A.19), we get

$$-\frac{w}{(T-\bar{s})^2} - \frac{w}{(T-\bar{t})^2} - \alpha + \frac{(\bar{x} - \bar{y})}{\varepsilon^2} \cdot (v - \bar{v}) - \mu^2 \bar{x} \cdot w - \mu^2 \bar{y} \cdot v \geq 0.$$

But,

$$\left|\mu^2 \bar{x} \cdot w\right| \leq 4\left(\|b_1\|_\infty + \|b_2\|_\infty\right)^{\frac{1}{2}} \|b_1\|_\infty \mu, \qquad \left|\mu^2 \bar{y} \cdot v\right| \leq 4\left(\|b_1\|_\infty + \|b_2\|_\infty\right)^{\frac{1}{2}} \|b_2\|_\infty \mu,$$

and

$$\left|\frac{(\bar{x} - \bar{y})}{\varepsilon^2} \cdot (v - w)\right| \leq 2\left(\mathrm{Lip}(b_2) + 2\mu\left(\|b_1\|_\infty + \|b_2\|_\infty\right)^{\frac{1}{2}}\right)|v - w| \leq C\varphi(\delta).$$

Now fixing α and taking δ and μ sufficiently small, we get the wanted contradiction.

In both cases, we obtain a contradiction. The Theorem is now proved.

□

A.4 Derivation of the fundamental solution of the limit system

In this Section, we are interested in deriving the fundamental solution in the Hamilton-Jacobi sense of the limit system. We want to obtain the following Proposition :

Proposition A.8. *Suppose that $\varphi_0(x,v) = v^2$ if $x = 0$ and $+\infty$ else. Then a solution of the system is, for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^+$,*

$$\varphi^0(t,x,v) := \begin{cases} \min\left(v^2 + 3\left(\frac{x}{2}\right)^{\frac{2}{3}}, v^2 + \frac{x}{|v|}\right), & (x,v) \in \left\{x \leq \left(\frac{2}{3}\right)^{\frac{3}{2}} \frac{t^{\frac{3}{2}}}{\sqrt{2}}\right\} \times V, \\ v^2 + 3\left(\frac{x}{2}\right)^{\frac{2}{3}}, & (x,v) \in \left\{\left(\frac{2}{3}\right)^{\frac{3}{2}} \frac{t^{\frac{3}{2}}}{\sqrt{2}} \leq x \leq \frac{t^{\frac{3}{2}}}{\sqrt{2}}\right\} \times \left\{v \leq \frac{x}{t}\right\}, \\ v^2 + \frac{x}{v}, & (x,v) \in \left\{\left(\frac{2}{3}\right)^{\frac{3}{2}} \frac{t^{\frac{3}{2}}}{\sqrt{2}} \leq x\right\} \times \left\{v \geq \frac{x}{t}\right\}, \\ v^2 + \frac{x^2}{t^2} + t, & (x,v) \in \left\{\frac{t^{\frac{3}{2}}}{\sqrt{2}} \leq x\right\} \times \left\{v \leq \frac{x}{t}\right\}. \end{cases} \tag{A.20}$$

The solution on the half part $\mathbb{R}^+ \times \mathbb{R}^- \times V$ is obtained after a central symmetry.

One remaining thing to do to ensure that it is indeed the good solution is to extend the uniqueness result to functions that have a discontinuity at $v = \frac{x}{t}$. This computation is interesting since the acceleration phenomena appears. Moreover, we recover the fact that the large deviation potential is stationary for $x \leq \left(\frac{2}{3}\right)^{\frac{3}{2}} \frac{t^{\frac{3}{2}}}{\sqrt{2}}$, which was announced by physicists in [166].

We have guessed this solution via a reasonable numerical scheme of prediction-correction type, which shall give the solution in the limit $dt \to 0$. We summarize it below :

– **First step. Prediction.** We transport φ^n with velocity v, assuming that φ^n satisfies the constraint :
$$\varphi^{n+\frac{1}{2}} = \varphi^n\left(x - vdt\right).$$
The solution φ^{n+1} has to have the same minimum as $\varphi^{n+\frac{1}{2}}$, as it represents the total mass of particles. We thus need a correction step.

– **Second step. Correction.** We project $\varphi^{n+\frac{1}{2}}$ on the constraint to make sure that φ^{n+1} satisfies the constraint. This corresponds to a redistribution on the Maxwellian.
First micro step : Everybody dies with rate 1 :
$$\varphi^{n+\frac{3}{4}} = \varphi^{n+\frac{1}{2}} + dt.$$
Second micro step : Production :
$$\varphi^{n+1} = \min\left(\min_V\left(\varphi^{n+\frac{1}{2}}\right) + v^2, \varphi^{n+\frac{3}{4}}\right).$$

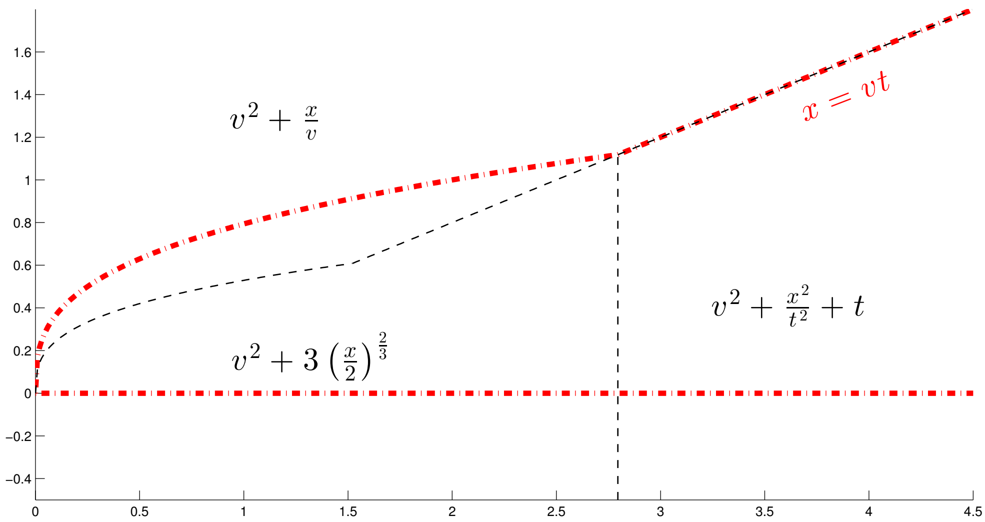


Figure A.1 – Schematic representation of the fundamental solution in the (x, v) plane, for some fixed time t. The dark dashed lines separate the different zones. The red dashed lines are the positions of the minima in velocity for each space position x.

Notice that with this operation we enforce the minimum to be in 0, what was lost after the first step. We can combine the two micro steps writing

$$\varphi^{n+1} = \min\left(\min_V\left(\varphi^{n+\frac{1}{2}}\right) + v^2, \varphi^{n+\frac{1}{2}} + dt\right).$$

We show in Figures A.2 and A.3 numerical results from this numerical scheme.

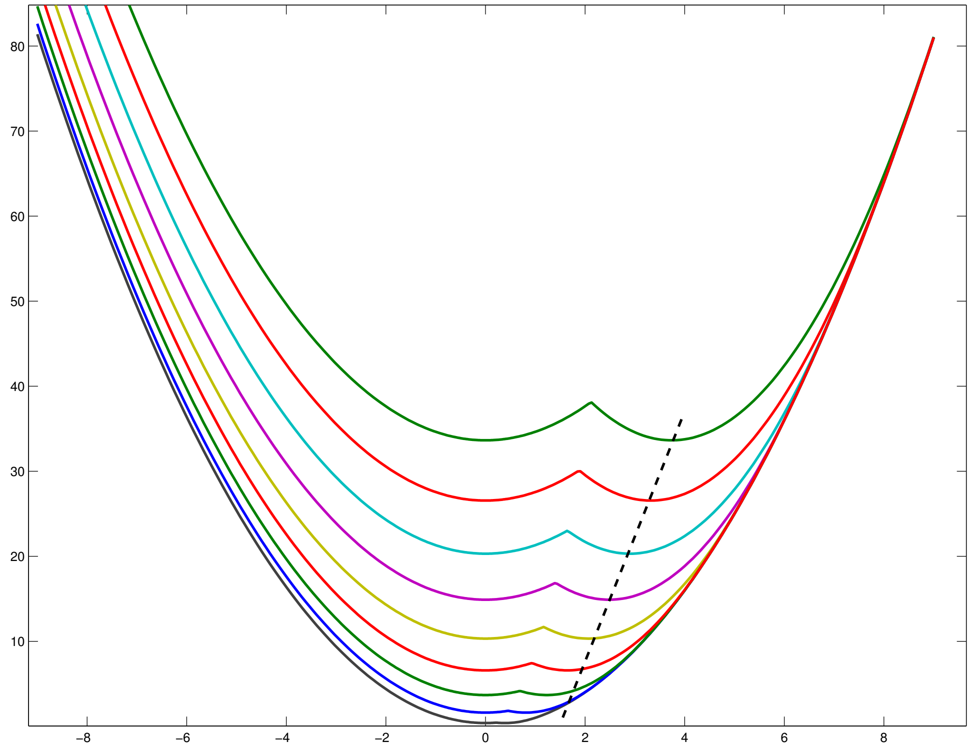


Figure A.2 – Plots of $v \to \varphi(t, x, v)$ for a fixed time t and different values of the space variable x, after computing via the numerical method. The curves are ordered from bottom to top with increasing x. We have added a black dotted line to emphasize that the position of the minimas is linear for large x.

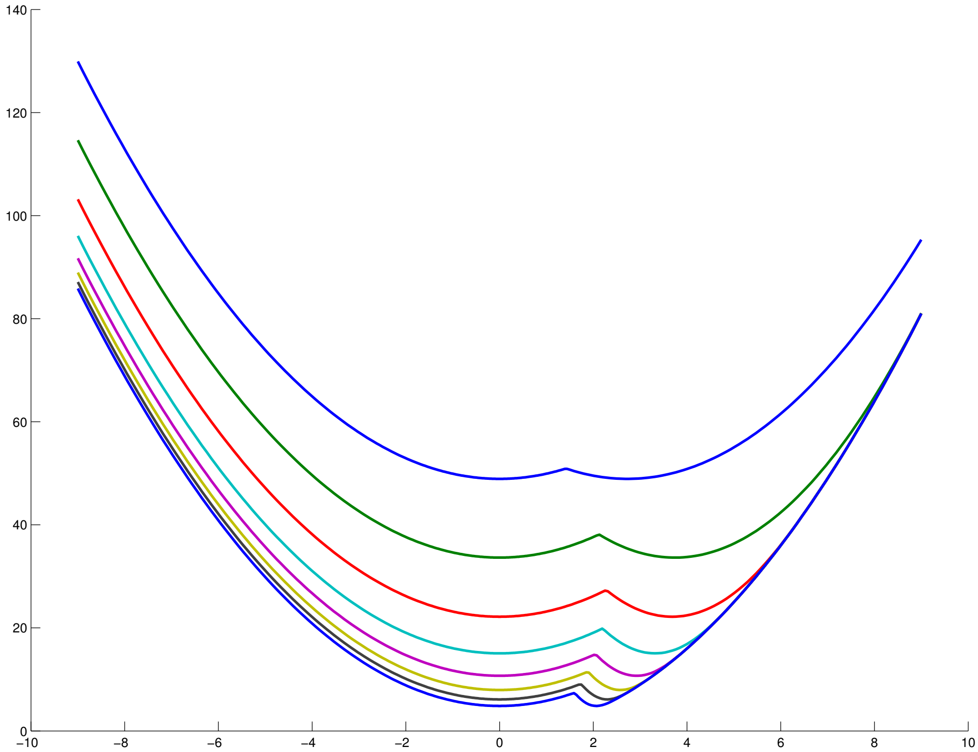


FIGURE A.3 – PLOTS OF $v \to \varphi(t, x, v)$ FOR A FIXED SPACE POSITION x AND DIFFERENT VALUES OF TIME x, AFTER COMPUTING VIA THE NUMERICAL METHOD. THE CURVES ARE ORDERED FROM TOP TO BOTTOM WITH INCREASING t.

Annexe B

Illustrations numériques de modèles de populations avec compétition

Dans cette annexe, on étudie numériquement certains modèles de dynamique adaptative. Ces modèles (intégro-différentiels et EDP) de populations structurées en trait phénotypique cherchent à expliquer comment la sélection d'un trait particulier apparait au sein d'une population. Il est possible d'effectuer une analyse asymptotique en faisant des changements d'échelle adaptés. Les fortes densités de population se concentrent en des masses de Dirac (bien séparées) et des branchements peuvent apparaitre. Ce phénomène est appelé "spéciation".
Le processus via lequel des solutions concentrées apparaissent est subtil. On présente des simulations de type Monte-Carlo, que l'on compare à des simulations déterministes (différences finies).

Contents

B.1 Introduction

Since the 1980's the word 'adaptive evolution' has been coined to describe the mathematical formalisms addressing the selection of a favorable trait in a population structured by a continuous phenotypical trait. Closely related to the concept of 'Evolutionary game theory' [150, 129, 130], the models ingredients are the three principles underlying Darwin's explanation of Evolution

- multiplication of the population,
- selection by competition for resources,
- variability (mutations).

Simple models based on these ingredients explain how the fittest traits can emerge and populations characterized by several well separated traits (also called strategies) can possibly coexist. The theory and numerical simulations show the appearance of clusters and speciation that can be explained simply : the limited resources lead to competition and individuals with close traits use similar resources, therefore competition between them is higher. The question of understanding how, in such a population, a mutant can invade or not a population has been initiated in [128, 157, 109] and a recent survey can be found in [78], see also [158]. In a self-contained population model, the mutations are part of the dynamics and take into account that the newborn may inherit a slightly different trait than its parent.

The formalism for describing selection, in an asexual population, uses integro-differential equations for the population density $n(x,t)$ where x denotes the phenotypical trait and several models have been derived or postulated for mutations, leading to parabolic partial differential equations (PDEs) [187, 48, 49]. In this Chapter, we aim at explaining how speciation occurs in such PDE models. This corresponds to highly concentrated population densities, which means that $n(x,t)$ is close to well separated Dirac masses. Because of their regularizing effects, parabolic PDEs cannot sustain such singular solutions and this phenomenon can only happen asymptotically. With this respect, two typical asymptotic regimes are possible. The first one consists in introducing a small parameter for mutations frequency or size and considers the limiting behavior when this parameter vanishes [79, 18, 178, 20]. The second asymptotic is to consider long times and this leads to singular steady state solutions, very similar to the pure selection case [76, 135]. We present these models in Sections B.2 and B.3 on two different type of competition kernels that we have chosen for their simplicity.

The appearance of these singular solutions is related to an instability mechanism of Turing type. Numerical methods may produce artificially this Turing mechanism in particular because artificial boundary conditions are needed. We discuss this fact in Section B.4 based on finite differences or Monte-Carlo simulations.

B.2 A model with a single nutrient

B.2.1 The chemostat

Following [78, 79], the simplest example to build up a self-contained mathematical model for adaptive evolution is the *chemostat*. Micro-organisms characterized by a parameter $x \in \mathbb{R}$ (it can be thought of as the logarithm of their size) live in a bath containing a nutrient which is continuously renewed with a rate $d > 0$. The nutrient concentration is denoted by $S(t) \geq 0$ (for substrate) and the fresh nutrient $S_{\text{in}} > 0$, the population density of the micro-organism is denoted by $n(x,t)$ and the uptake rate for individuals of trait x is $\eta(x) > 0$.

In such a simple situation, the standard equations for the chemostat is written

$$\begin{cases} \frac{d}{dt}S(t) = d\big(S_{\text{in}} - S(t)\big) - S(t)\int_{-\infty}^{\infty}\eta(x)n(x,t)dx, \\ \frac{\partial}{\partial t}n(x,t) = -dn(x,t) + (1-\mu)S(t)\eta(x)n(x,t) + \mu S(t)\int_{-\infty}^{\infty}M(y,x)\eta(y)n(y,t)dy. \end{cases}$$

The first two principles mentioned earlier from Darwin theory are directly included in the model : the population growth comes from the equation on $n(x,t)$ and the competition comes from the limited amount of nutrients. We assume that initially $S(0) \leq S_{\text{in}}$, then all along the dynamics we have $S(t) \leq S_{\text{in}}$ because $S(t)$ decreases if it attains S_{in}. The term $(1-\mu)\eta(x)n(x,t)$ represents the birth rate without mutations. The parameter $0 < \mu < 1$ represents the proportion of birth undergoing mutations.

Mutations are represented by the probability $M(y,x)$ that a newborn has the trait x when its parent has the trait y. We therefore assume $M(y,x) \geq 0$, $\int_0^\infty M(y,x)dx = 1$.

We may simplify the model in various ways to make it more amenable to analysis. One can suppose that the nutrients reach quickly an equilibrium compared to the evolution time scale for the population. Then one can replace the differential equation on $S(t)$ by the relation

$$S(t) = \frac{dS_{\text{in}}}{d + \int_{-\infty}^{\infty}\eta(x)n(x,t)dx}.$$

One can also replace the mutation term by a mere diffusion leading to

$$\frac{\partial}{\partial t}n(x,t) = -dn(x,t) + S(t)\eta(x)n(x,t) + \lambda\Delta n(x,t).$$

Note however that both representations of mutations by integral terms or by a Laplace term $\lambda\Delta$ can be derived from stochastic individual based models (IBM) depending on the scaling of microscopic mutations, [56, 58, 59]. See also [164].

We can write a general form of the resulting model, that we will keep for the end of this section

$$\begin{cases} \frac{\partial}{\partial t} n(x,t) = n(x,t)\ R\big(x, I(t)\big) + \lambda \Delta n(x,t), \quad x \in \mathbb{R},\ t > 0, \\ I(t) = \int_{-\infty}^{\infty} \eta(x) n(x,t) dx. \end{cases} \tag{B.1}$$

With these notations, the neat growth rate $R(x, I)$ contains both birth and death terms. In the case at hand, it is given by

$$R(x, I) = -d + \frac{dS_{\text{in}}}{d + I} \eta(x).$$

It is natural to handle more general models and then we need some general hypothesis. We assume that R is smooth enough and there are $I_M > I_m > 0$ such that

$$\sup_{x \in R} R_I(x, I) < 0,\ \forall I \geq 0, \qquad \max_{x \in \mathbb{R}} R(x, I_M) = 0, \qquad \min_{x \in \mathbb{R}} R(x, I_m) = 0. \tag{B.2}$$

We also assume that there are positive constants η_m, η_M such that

$$0 < \eta_m \leq \eta(x) \leq \eta_M < \infty, \qquad \eta \in W^{2,\infty}(\mathbb{R}). \tag{B.3}$$

B.2.2 Rescaling

As mentioned earlier, such parabolic models cannot exhibit high concentrations as long as the diffusion coefficient $\mu > 0$ is fixed. This is the reason why we rescale the problem and set $\lambda = \varepsilon^2$. Having in mind that the mutation rate is small we consider the limit $\varepsilon \to 0$. Such a limit only leads to the same equation with $\lambda = 0$, the selection model. This is because the effect of rare mutations on the population can be observed only on a very long time. This leads us naturally to change time and replace t by t/ε so as to consider the evolution on a long time rather than a generation time scale. Then equation (B.1) is changed to

$$\begin{cases} \varepsilon \frac{\partial}{\partial t} n_\varepsilon(x,t) = n_\varepsilon(x,t) R\big(x, I_\varepsilon(t)\big) + \varepsilon^2 \Delta n_\varepsilon(x,t), \quad x \in \mathbb{R},\ t > 0, \\ I_\varepsilon(t) = \int_{-\infty}^{\infty} \eta(x) n_\varepsilon(x,t) dx. \end{cases} \tag{B.4}$$

But we can point out that other scales are also interesting [48].

We are now ready for a possible interpretation of the speciation phenomena

Theorem B.9 ([178, 20])**.** *We assume* (B.2)–(B.3)*, that R is monotonic in x and the initial data is 'well-prepared' (see below). Then, there are two constants $\rho_m > 0$, $\rho_M > 0$ such that*

$$\rho_m \leq \int_{\mathbb{R}} n_\varepsilon(x,t) dx \leq \rho_M \tag{B.5}$$

and $I_\varepsilon(t) \to \bar{I}(t)$ almost everywhere and in the weak sense of measures

$$n_\varepsilon(x,t) \rightharpoonup \bar{\rho}(t) \delta\big(x - \bar{x}(t)\big).$$

The above assumptions, and in particular monotonicity on R in x, can be replaced by strong concavity on R with quadratic behavior at infinity [149].

This Theorem is a mathematical version of the famous *competitive exclusion principle* in ecology. With a single nutrient, a single species will be selected. With N nutrients, we expect in general that N species will co-exist.

It is not easy to characterize the fittest trait $\bar{x}(t)$ and the total population size $\bar{\rho}(t)$. In the situations covered by Theorem B.9, it is proved (see [178, 149]) that

$$R\big(\bar{x}(t), \bar{I}(t)\big) = 0, \qquad \bar{I}(t) = \bar{\rho}(t)\eta\big(\bar{x}(t)\big).$$

Such points appear naturally in the language of evolutionary game theory and are called 'singular points'. Of course this identity only relates $\bar{x}(t)$ and $\bar{I}(t)$. It is possible to go further and establish an analogue of the so-called *canonical equation* [77]

$$\dot{\bar{x}}(t) = \big(-D^2 u\big(\bar{x}(t), t\big)\big)^{-1}.\nabla_x R\big(\bar{x}(t), \bar{I}(t)\big),$$

where $u(x,t)$ is introduced below. Such a differential equation was formally introduced in [79] and it can be established rigorously in a multidimensional framework, see [149].

B.2.3 The constrained Hamilton-Jacobi equation

The proof of Theorem B.9 relies on a WKB approach, as in front propagation [88, 15, 196]. In the context of adaptive dynamics the method was introduced in [79] and yields a new type of Hamilton-Jacobi equation because an algebraic constraint appears. It is based on the real phase defined by the Hopf-Cole transform

$$u_\varepsilon = \varepsilon \ln(n_\varepsilon).$$

This requires that the initial data itself is 'well-prepared', that is 'exponentially' concentrated as $u_\varepsilon^0 = \varepsilon \ln(n_\varepsilon^0)$ with u_ε^0 a function that behaves nicely as $\varepsilon \to 0$ (even though this can be somehow relaxed, see [20]).

The equation on u_ε is written

$$\frac{\partial}{\partial t} u_\varepsilon(x,t) = R\big(x, I_\varepsilon(t)\big) + \varepsilon \Delta u_\varepsilon(x,t) + \big|\nabla u_\varepsilon(x,t)\big|^2.$$

One can prove that u_ε is uniformly lipschitzian (this requires that u_ε^0 is so) and that I_ε is uniformly with bounded variations. This allows us to pass to the limit $\varepsilon \to 0$ and obtain the *constrained Hamilton-Jacobi* equation

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = R\big(x, I(t)\big) + |\nabla u(x,t)|^2. \\ \max\limits_{x \in \mathbb{R}} u(x,t) = 0, \qquad \forall t > 0. \end{cases} \tag{B.6}$$

The algebraic constraint $\max_{x\in\mathbb{R}} u(x,t) = 0$ comes from the uniform a priori bound on the total mass stated in (B.5) together with the definition of u_ε by the Hopf-Cole transform.

Being a parabolic limit, the solution $u(x,t)$ should be understood as a viscosity solution to (B.6), see [68].

As mentioned earlier, the originality of this problem stems from the two unknowns $u(x,t)$ et $I(t)$ which should be solved together. The latter is a Lagrange multiplier associated with the algebraic constraint. This makes the main difference with the standard eikonale equation arising in geometrical optics. A uniqueness result is proved in [178], however under restrictive assumptions. The method of Hopf-Cole transform is very general and, in the present context, it has been extended to systems in [52] (for fronts see [15]).

B.3 Competition models

In a chemostat, the competition between species is global because it arises through the substrate described by $S(t)$. All individuals are equally competing for the ressource. This is not always the case and, in many situations, it is more realistic to assume that there is higher competition between individuals with closer traits. This is the reason why other models have been proposed that implement a trait dependent competition. A class of such models (see [156, 76, 118, 117, 26]) are given by the population dynamics of Lotka-Volterra type

$$\frac{\partial n(x,t)}{\partial t} - \lambda \frac{\partial^2 n(x,t)}{\partial x^2} = n(x,t)\,\big(R(x) - K * n(x,t)\big), \qquad t \geq 0,\ x \in \mathbb{R}. \tag{B.7}$$

The model is completed by an initial data $n(x, t=0) = n^0(x)$ which we take highly concentrated for the numerical simulations presented below in section B.4.

The interpretation of the quantitites arising in this model are
• $n(x,t)$ still denotes the population density at position x and time t,
• $R(x) > 0$ is the intrinsic growth rate of individuals with trait x (if isolated without competition)
• $K \in L^\infty(\mathbb{R})$ is called the competition kernel. It is a probability density : $K \geq 0$, $\int_{\mathbb{R}} K(z)dz = 1$. The convolution $K * n(x) = \int_{\mathbb{R}} K(x-y)n(y,t)dy$ represents the competition for resource,
• λ is the mutation rate that is supposed to be a constant.

When derived from stochastic IBM, as in [187, 58, 59] such models are called *mean field* equations [41, 190]. They arise not only in evolution theory but also in ecology for non-local resources (and x denotes the location then) [30, 116, 200, 100].

The large variety of regimes that can appear in such models can be seen in special cases. Below, we use simple examples to describe two of them, regularly distributed traits, or concentration as a Dirac mass. The main interest of the model (B.7) is mostly from the branching patterns that correspond to multiple concentration points which can either die out or branch again and create new structures (see [108]).

B.3.1 The gaussian case without mutations

Firstly we consider the case

$$\lambda = 0, \qquad R(x) = \frac{1}{\sqrt{2\pi\sigma_1}} e^{-\frac{|x|^2}{2\sigma_1}}, \qquad K(z) = \frac{1}{\sqrt{2\pi\sigma_2}} e^{-\frac{|x|^2}{2\sigma_2}}. \tag{B.8}$$

This corresponds to widely used standard forms of the input parameters because of their statistical meaning.

As usual for pure selection models, $\lambda = 0$, there are Dirac mass stationary solutions $N(x) = \bar{\rho}\delta(x-\bar{x})$ with $R(\bar{x}) = \bar{\rho}K(0)$. But this can be obtained in a long time asymptotic only when

$$R(x) < \bar{\rho}K(x-\bar{x}) \qquad \forall x \neq \bar{x},$$

or, replacing $\bar{\rho}$ from the first condition

$$\frac{R(x)}{R(\bar{x})} < \frac{K(x-\bar{x})}{K(0)} \qquad \forall x \neq \bar{x}.$$

One can deduce from this calculation the

Proposition B.10. *For $\sigma_1 > \sigma_2$ there is a smooth steady state to* (B.7) *given by*

$$N(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{|x|^2}{2\sigma}}, \qquad \sigma = \sigma_1 - \sigma_2,$$

and Dirac masses are not stable steady states.

For $\sigma_1 < \sigma_2$ the Dirac mass $\bar{\rho}\delta(x)$ is a stable steady state (and only the Dirac mass at 0 *is stable).*

The authors in [135] prove that the corresponding stable states are also the long time limits of the dynamics described by equation (B.7). They use a relative entropy method built on the corresponding steady state. The construction of this entropy is rather easy when the positive steady state exists. It is much more difficult in the case where the Dirac masses have to be handled.

B.3.2 The NonLocal-Fisher equation

We now consider the case

$$R \equiv 1. \tag{B.9}$$

Then, the equation (B.7) is called the *NonLocal-Fisher* (NLF) equation. It also arises in mathematical ecology, as an extension of the Fisher/KPP equation. As mentioned earlier, the non-local aspect induced by the convolution represents long range access to resources, see [116, 26, 100] and the references therein.

The positive steady state is simply given by $N \equiv 1$ but a result from [118] states that it can be Turing unstable (i.e. only a bounded set of linearly unstable modes occur). In order to explain this, we may use the Fourier transform of the competition kernel K defined as

$$\widehat{K}(\xi) = \int_{\mathbb{R}} K(x) e^{-ix\xi} dx.$$

Then one has

Proposition B.11 ([118])**.** *Assume there is a ξ_0 such that*

$$\widehat{K}(\xi_0) < 0, \tag{B.10}$$

then for λ small enough the steady state $N \equiv 1$ is linearly unstable.

The result of this statement corresponds qualitatively to the case $\sigma_1 < \sigma_2$ in Proposition B.10 (with mutations neglected).

The Fourier transform also characterizes a nonlinear stability result ; this is the case in the

Theorem B.12 ([26]). *Take $R \equiv 1$ and assume*

$$\widehat{K}(\xi) > 0 \qquad \forall \xi \in \mathbb{R}. \tag{B.11}$$

Then $n \equiv 0$ and $n \equiv 1$ are the only two nonnegative and bounded steady states of (B.7). *Furthermore, there are traveling waves connecting the states $n = 0$ and $n = 1$.*

The result of this Theorem corresponds to the situation $\sigma_1 > \sigma_2$ in Proposition B.10.

In the Turing unstable case it is possible to rescale the problem as we did it in Section B.2.2 and it is observed numerically that, in general, the asymptotic limit leads to Dirac concentrations characterized again by a constrained Hamilton-Jacobi equation [117].

B.4 Numerical methods and branching patterns

In general it is very difficult, in the direct competition model (B.7), to distinguish between the two behaviors : convergence towards a continuous state or speciation. Numerical methods are useful to get an intuition but they can create artifacts and we explain this now.

We present two numerical approaches that allow to simulate solutions to equation (B.7). The first is a standard finite difference scheme, the second one is a Monte-Carlo simulations related to IBM that solves the same equation.

For the sake of simplicity we concentrate on the NonLocal Fisher equation as in Section B.3.2 with a gaussian competition kernel

$$R \equiv 1, \qquad K(x) = \frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{|x|^2}{2\sigma}}. \tag{B.12}$$

Because the Fourier transform of K is positive (a gaussian), we do not expect appearance of concentrations (speciation).

At this stage we insist that the Monte-Carlo algorithms are only seen here as an approximation to (B.7). From this point of view, the closer it is from the PDE, the better it is because one looks only for possible computational cost reduction. Monte-Carlo methods also used as a modeling tool and allow to include further stochastic effects. One of them is 'demographic stochasticity' which makes that too small populations can die out by statistical effects [62, 182]. These effects are not included in the models under consideration here and give quantitatively different answers (in term od evolution speed, branching patterns). It is shown in [108] that the notion of 'survival threshold' in the equations as (B.7) is able to reproduce these effects in great details.

B.4.1 Finite differences

We consider the solution on interval $[-\frac{L}{2}, \frac{L}{2}]$. We use a uniform grid with N points on the segment, with $\Delta x = \frac{L}{N}$ the space step. We denote by $n_i^k \geq 0$ the numerical solution at grid point $x_i = i\triangle x$, $1 \leq i \leq N$, and time $t^k = k\triangle t$ where $\triangle t$ is the time step

$$n(x_i, k\triangle t) \approx n_i^k.$$

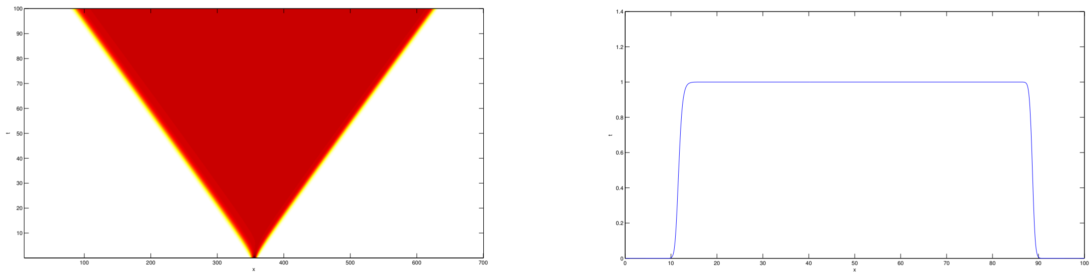


FIGURE B.1 – Left : Numerical population density dynamics obtained for model (B.7)–(B.12) when the initial population is concentrated in the center of the computational domain. Horizontally is x and vertically is t, in gray zone $n \equiv 1$ and the white zone corresponds to $n \equiv 0$. Right : The population density $n(x, T)$ at final time. The deterministic finite difference scheme (B.13)–(B.15) has been used with parameters in (B.16). We observe convergence toward the constant solution in accordance with Theorem B.12.

We use a time splitting algorithm between the growth term and the diffusion that is we solve alternatively the two equations

$$\frac{d}{dt} n(x,t) = n(x,t) \; [1 - K * n(t)\,],$$

$$\frac{\partial n(x,t)}{\partial t} - \lambda \frac{\partial^2 n(x,t)}{\partial x^2} = 0.$$

1. First compute, with a semi-implicit method, the solution to the discrete reaction term

$$\frac{d}{dt} n_i(t) = n_i(t) \; \left[1 - K_d * n_i^k\right].$$

The exact solution is

$$n_i^{k+\frac{1}{2}} = n_i^k \exp\left(\frac{\Delta t}{\lambda}\left(1 - K_d * n_i^k\right)\right), \qquad 1 \le i \le N. \tag{B.13}$$

The discrete convolution is computed according to

$$K_d * n_i^k = \Delta x \cdot \sum_{j=-N}^{N} K_d(j \triangle x)\, n_{i-j}^k, \qquad n_{i-j}^k = 0 \;\text{ for } i - j \notin [1, N]. \tag{B.14}$$

Indeed, as a consequence of the domain truncation, only those terms satisfying $1 \le i - j \le N$ are well defined and the extension by zero amounts to extend n by 0 outside $[-\frac{L}{2}, \frac{L}{2}]$. This is some kind of Dirichlet boundary condition.

2. As for the Laplace term, we use a three points explicit scheme

$$n_i^{k+1} = n_i^{k+\frac{1}{2}} + \frac{\lambda \Delta t}{2\Delta x^2}\left(n_{i+1}^{k+\frac{1}{2}} + n_{i-1}^{k+\frac{1}{2}} - 2\, n_i^{k+\frac{1}{2}}\right), \qquad 1 \le i \le N. \tag{B.15}$$

Because we choose λ small, the explicit scheme is not penalizing in terms of computational time. We use Neumann boundary condition, $n_0^{k+1} = n_1^{k+1}$ and $n_N^{k+1} = n_{N-1}^{k+1}$, but as far as the wave does not reach the boundary, the Dirichlet boundary condition $n_1^{k+1} = n_N^{k+1} = 0$ gives equivalent results.

The stability of the scheme is ensured by the CFL condition $\dfrac{\lambda \Delta t}{2\Delta x^2} \leq 1$, which is verified for

$$\lambda = 0.004, \qquad \sigma = 0.04, \qquad \Delta t = 0.025, \qquad \Delta x = 0.1, \qquad L = 100, \qquad N = 1000. \tag{B.16}$$

We have implemented this method. We choose the initial data concentrated in the center of the domain. The numerical results are depicted in Fig. B.1. We can observe that the population propagates as a traveling wave. For L large enough, for $0 \leq t \leq T$ the front does not reach the numerical boundary and there is almost no mass on the boundary of the interval $[-\frac{L}{2}, \frac{L}{2}]$. This is in accordance to the theory in [26] and the statement in Theorem B.12.

B.4.2 The stochastic individual-based method

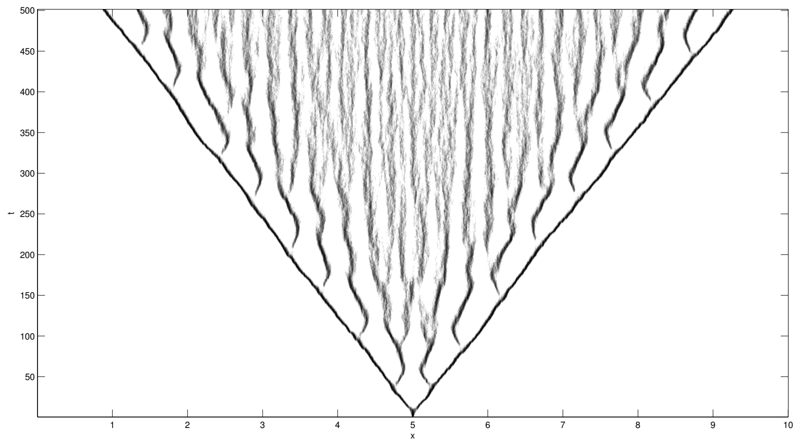


Figure B.2 – Numerical solution with the Monte-Carlo algorithm in section B.4.2. Horizontally is the trait x and vertically is time t. Initially the population is concentrated in one Dirac mass at the center of the domain. We observe that the population distribution converges weakly towards the constant solution as expected (see also Fig. B.1).

We also compare the finite volume simulation with a Monte-Carlo algorithm. Then, the solution is approximated by a sum of Dirac masses

$$n(t) \approx \omega \sum_{j=1}^{N(t)} \delta\big(x - y_j(t)\big).$$

Here the weight ω is taken constant. The simulation starts with a number $N(0)$ of 'individuals located' distributed on an interval of length L. Then $N(0)$ and ω are related by the approximation $n(0) \approx \omega \sum_{j=1}^{N(0)} \delta\big(x - y_j(0)\big)$ in the weak sense of measures.

Several Monte-Carlo algorithms are possible. See for instance [30, 100] for another algorithm motivated by models from ecology.

Here we use the method proposed in [41, 190]. The number of individuals is denoted by $N(k)$ at iteration k. The algorithm uses also a time splitting but not with the same operators as in Section B.4.1. We solve alternatively the two equations

$$\frac{d}{dt} n(x,t) = -n(x,t)\; K * n(t),$$

$$\frac{\partial n(x,t)}{\partial t} - \lambda \frac{\partial^2 n(x,t)}{\partial x^2} = n(x,t).$$

Finally, in the rationale of small mutations and long times, as in section B.2.2, we choose $\triangle t = 1$. Then the algorithm [41, 190] reads as follows.

1. The competition term is now computed as (this makes a difference with [41, 190])

$$C(x) = \frac{\omega}{\sqrt{2\pi\sigma}} \sum_{j=1}^{N(k)} \exp\left(-\frac{|x - y_j|^2}{2\sigma} \right). \tag{B.17}$$

Because the value of $C(x)$ is small, it defines the probability that an individual located at x dies. For a given j, we compute this probability and set $N(k+1) = N(k) - 1$ if this individual dies.

2. If the individual survives, it reproduces. The newborn undergoes a mutation from its parent trait to a new trait given by a Gaussian distribution with variance $\lambda' = 2\lambda$. Then $N(k+1) = N(k) + 1$.

We notice that for n the solution of

$$\partial_t n = \lambda' \triangle n, \qquad n(x, t^k) = n^k(x),$$

we have $n(t^{k+1}) = n^k * \frac{1}{\sqrt{4\pi\lambda'}} e^{\frac{-x^2}{4\lambda'}}$. Hence the choice $\lambda' = 2\lambda$ in the second step of the Monte-Carlo method. We act a gaussian mutation to the new-born only but with twice stronger intensity.

We have used the following parameters values which take into account the small time step in the deterministic algorithm

$$\lambda' = 10^{-6}, \qquad \sigma = 0.04, \qquad L = 10, \qquad N = 3000, \qquad \frac{\omega}{\sqrt{2\pi\sigma}} = 1/18000.$$

These values are such that the mutations are very weak compared to intraspecific competition, again in accordance with the parameters used in the finite difference method. The numerical results are depicted in Fig. B.2. We can observe that the population propagates as a traveling wave as in Fig. B.1 and according to the theoretical prediction in Theorem B.12.

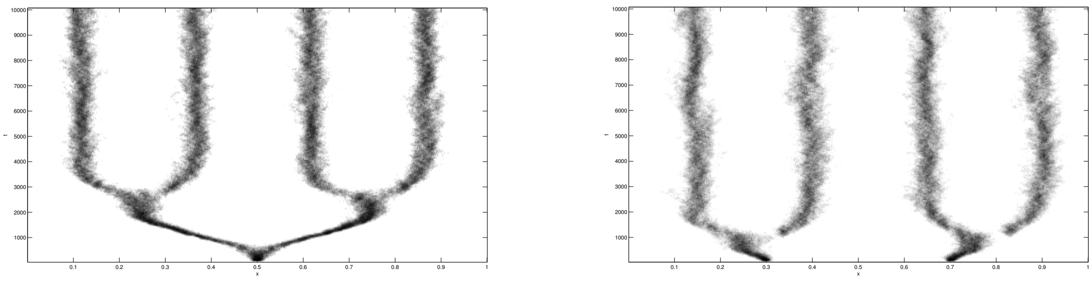


FIGURE B.3 – Dynamics of the concentration points with the Monte-Carlo algorithm in section B.4.3 based on periodizing the convolution. Horizontally is the trait x and vertically is time t. Initially the population is concentrated in one Dirac mass on the left and two Dirac masses on the right.

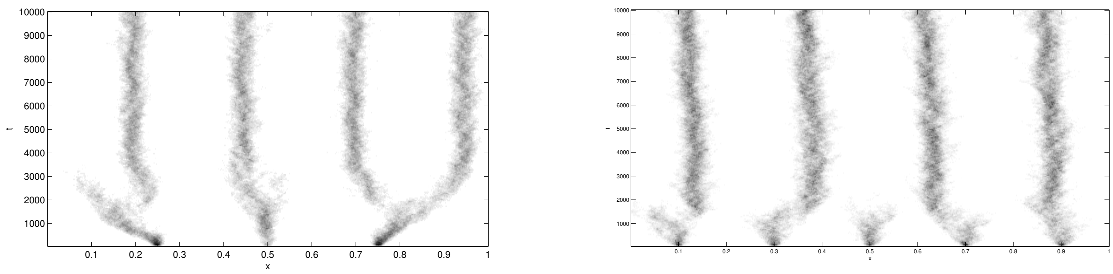


FIGURE B.4 – Dynamics of the concentration points. Same as above but with different initial data. A new phenomena occurs with extinction of branches.

B.4.3 The convolution formula

Surprisingly, in [41, 190] the authors observed that simulations based on this Monte-Carlo method may yield concentration patterns too (clusters). The main difference is that, rather than with B.17, the convolution kernel is computed assuming the y_j are on the circle

$$C(x) = \frac{\omega}{\sqrt{2\pi\sigma}} \sum_{j=1}^{N(k)} \exp\left(-\frac{d(x,y_j)^2}{2\sigma}\right), \tag{B.18}$$

where d is the shortest distance on the circle.

This can be interpreted as periodic boundary conditions rather than extension by zero or as a periodic convolution kernel

$$K_s(x) \propto \exp\left(-\frac{(x\,[L])^2}{2\sigma}\right), \qquad x[L] = x \mod L, \;\; x \in \mathbb{R}.$$

In opposition with the Gaussian kernel because it has some Fourier coefficients with a negative real part. In this case the Fourier condition (B.10) is not fulfilled. Therefore according to the linear analysis in [118], and Proposition B.11, the constant state is unstable for problem (B.7)–(B.12) and we expect to observe pattern formation.

We have run both the Monte-Carlo and finite difference approximations with this periodic kernel. The numerical results are in accordance with those obtained in different contexts in [41, 190, 118, 117]. They can be found in Fig. B.3 and Fig. B.4 for Monte-Carlo simulations and Fig. B.5 for finite differences.

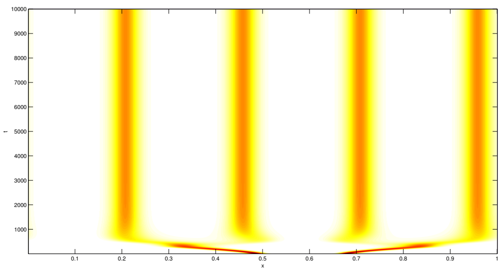


FIGURE B.5 – Numerical population density dynamics obtained by deterministic simulations for model (B.7)–(B.12) with periodic boundary conditions. We have used the following parameter values : $\lambda = 0.001$, $\sigma = 0.04$, $\Delta t = 0.0001$, $\Delta x = 0.001$, $L = 1$, $N = 1000$.

B.5 Conclusion

Mathematical models explaining how speciation occurs in biological population have been developed since the 1980's. They involve a population dynamics under local competition and with mutations. A self-contained formalism can be established. It allows to represent the speciation phenomena as the convergence of the solution to a sum of Dirac masses, either in the large time limit or the small mutation rate limit. However, competition models not always yield speciation and a population with a continuous set of traits can occur. It is difficult to predict between these two alternatives.

Numerical methods are therefore useful tools to observe the model prediction. We presented two numerical methods : finite differences and the individual based approach. These methods give compatible numerical results either in the case when a uniform trait distribution is produced by the model and when patterns are obtained.

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Abstract

This thesis is devoted to the study of propagation phenomena in PDE models arising from biology. We study kinetic equations coming from the modeling of the movement of colonies of bacteria, but also reaction-diffusion equations which are of great interest in ecology to reproduce several features of dynamics and evolution of populations.

The first part studies propagation phenomena for kinetic equations. We study existence and stability of travelling wave solutions for models where the dispersal part is given by an hyperbolic operator rather than by a diffusion. A set of admissible velocities comes into the game and we obtain various types of results depending on this set. In the case of a bounded set of velocities, we construct travelling fronts that propagate according to a speed given by a dispersion relation. When the velocity set is unbounded, we prove an accelerating propagation phenomena, for which we give the spreading rate. Then, we adapt to kinetic equations the Hamilton-Jacobi approach to front propagation. We show how to derive an effective Hamiltonian from the original kinetic equation, and prove some convergence results.

The second part is devoted to studying models for populations structured by space and phenotypical trait. These models are important to understand interactions between invasion and evolution. We first construct travelling waves that we study qualitatively to show the influence of the genetical variability on the speed and the distribution of phenotypes at the edge of the front. We also perform the Hamilton-Jacobi approach for these non-local reaction-diffusion equations.

Two appendices complete this work, one deals with the study of kinetic dispersal in unbounded domains, the other one being numerical aspects of competition models.

Keywords: Kinetic equations, reaction-diffusion equations, Hamilton-Jacobi equations, Front propagation, modelling.

Résumé

Cette thèse est consacrée à l'étude de phénomènes de propagation dans des modèles d'EDP venant de la biologie. On étudie des équations cinétiques inspirées par le déplacement de colonies de bactéries ainsi que des équations de réaction-diffusion importantes en écologie afin de reproduire plusieurs phénomènes de dynamique et d'évolution des populations.

La première partie étudie des phénomènes de propagation pour des équations cinétiques. Nous étudions l'existence et la stabilité d'ondes progressives pour des modèles ou la dispersion est donnée par un opérateur hyperbolique et non par une diffusion. Cela fait entrer en jeu un ensemble de vitesses admissibles, et selon cet ensemble, divers résultats sont obtenus. Dans le cas d'un ensemble de vitesses borné, nous construisons des fronts qui se propagent à une vitesse déterminée par une relation de dispersion. Dans le cas d'un ensemble de vitesses non borné, on prouve un phénomène de propagation accélérée dont on précise la loi d'échelle. On adapte ensuite à des équations cinétiques une méthode basée sur les équations de Hamilton-Jacobi pour décrire des phénomènes de propagation. On montre alors comment déterminer un Hamiltonien effectif à partir de l'équation cinétique initiale, et prouvons des théorèmes de convergence.

La seconde partie concerne l'étude de modèles de populations structurées en espace et en phénotype. Ces modèles sont importants pour comprendre l'interaction entre invasion et évolution. On y construit d'abord des ondes progressives que l'on étudie qualitativement pour montrer l'impact de la variabilité phénotypique sur la vitesse et la distribution des phénotypes à l'avant du front. On met aussi en place le formalisme Hamilton-Jacobi pour l'étude de la propagation dans ces équations de réaction-diffusion non locales.

Deux annexes complètent le travail, l'une étant un travail en cours sur la dispersion cinétique en domaine non-borné, l'autre étant plus numérique et illustre l'introduction.

Mots-clés: Equations cinétiques, équations de reaction-diffusion, équations de Hamilton-Jacobi, phénomènes de propagation, modélisation.