Etude de phénomènes de propagation dans des modèles cinétiques et structurés

Emeric Bouin

UMPA - ENS de Lyon

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1. Models of kinetic type in biology
2. Travelling waves in kinetic equations (first example).
3. Hamilton-Jacobi limit of a structured model (second example)
4. Extensions and Perspectives.

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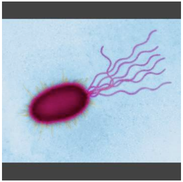
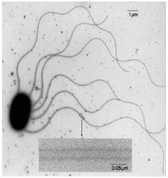
First example: Collective motion of bacteria

The bacteria E. Coli moves thanks to flagella :

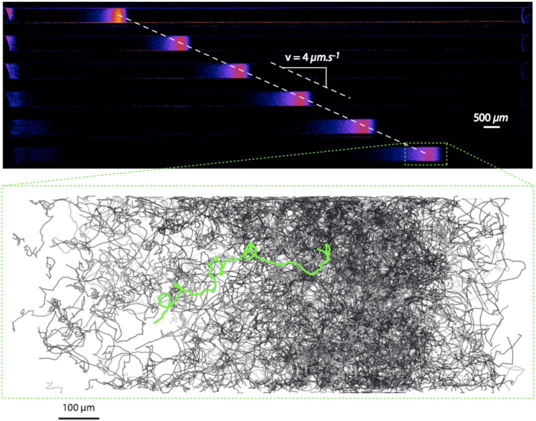




From Howard Berg's lab

and with a so-called *run and tumble* process :
straight swimming for $1s$
and
change of direction for $0.1s$.

Collective migration: Bacterial travelling pulses



The kinetic point of view is the most relevant for this situation.

J. Saragosti *et al*, Directional persistence of chemotactic bacteria in a travelling concentration wave, PNAS (2011).

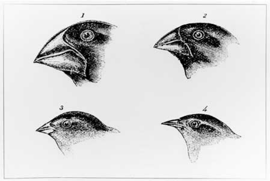
Second example: Modelling of Darwinian evolution

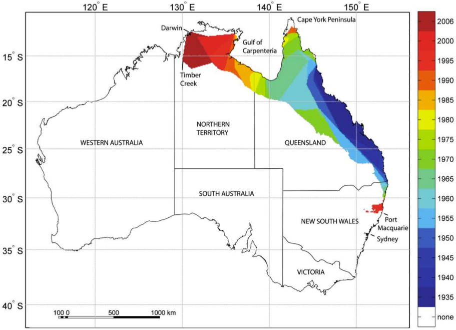
We study the

Darwinian evolution
of populations

which are structured by:

1. phenotypical traits,
2. position in space.





We need space-trait models!

Cane toads invasion

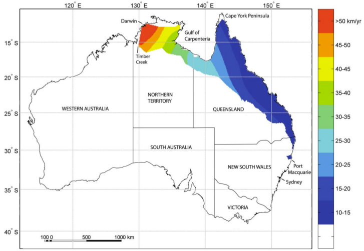


Figure : From Urban et al 2006

Evolution in fly wings

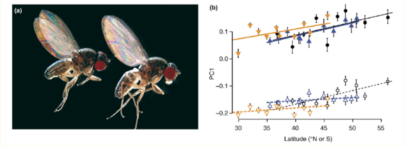


Figure : From Vellend et al 2007

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Kinetic reaction transport equations

- Density of bacteria $f(t,x,v)$ at time t, position x and speed v. Space density $\rho := \int_V f(v)dv$.
- The velocity set V : symmetric, bounded or unbounded ; $v_{max} \leq +\infty$.

The model (Schwetlick 2000 - Cuesta, Hittmeir, Schmeiser 2010):

$$\underbrace{\partial_t f + v\partial_x f}_{\text{Free run}} = \underbrace{(M(v)\rho - f)}_{\text{Tumbling}} + \underbrace{r\rho\,(M(v) - f)}_{\text{Growth with saturation}} \tag{1}$$

where the distribution M on the space V satisfies

$$\int_V M(v)dv = 1, \qquad \int_V vM(v)dv = 0, \qquad \int_V v^2 M(v)dv = D. \tag{2}$$

This is a kinetic analogous to the **Fisher-KPP equation**

$$\partial_t \rho = \partial_{xx}\rho + \rho(1-\rho)$$

Existence of travelling waves for bounded speeds

Parabolic limit result : (parabolic scaling) + ($r \to r\varepsilon^2$) :

Theorem (Cuesta, Hittmeir, Schmeiser)

Let the wave speed satisfy $s \geq 2\sqrt{rD}$. For ε small enough, there exists a travelling wave solution of speed s.

Existence result in the kinetic regime:

Theorem (B., Calvez, Nadin)

Assume that $v_{max} < +\infty$. There exists traveling front solutions for all $c \geq c^*$.

Remark: $c^* \leq 2\sqrt{rD}$.

Finding the speed : Dispersion relation

Proposition

We have $c^* = \min_{\lambda>0} c(\lambda)$, where $c(\lambda)$ is a solution of

$$(1+r)\int_V \frac{M(v)}{1+\lambda(c(\lambda)-v)}\,dv = 1\,. \tag{3}$$

No solution when V is unbounded ($v_{max} = +\infty$)

Approximation of $v_{max} = +\infty$:

Here $M(v) = C\left(V_{max}\right)\exp\left(-\frac{v^2}{2}\right)$.

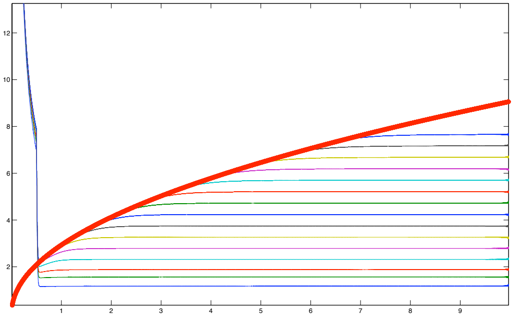

Figure : Speed as a function of time.

$$\mathbf{c(t) \sim \sqrt{t} \quad \Longrightarrow \quad x(t) \sim t^{\frac{3}{2}}}$$

Front acceleration when V is unbounded

Theorem (B., Calvez, Nadin)

Let $M(v) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{v^2}{2\sigma^2}\right)$*. Under suitable hypothesis on the initial data,*

1. ***Infinite speed of propagation:*** *For all* $c > 0$,

$$\lim_{t\to+\infty} \sup_{x\leq ct} |M(v) - f(t,x,v)| = 0 \,.$$

2. ***Propagation bounded by*** $t^{\frac{3}{2}}$***:*** *For all* $\varepsilon > 0$*, one has*

$$\lim_{t\to+\infty} \sup_{|x|\geq(1+\varepsilon)\sigma\sqrt{2r}(t+a)^{3/2}} \rho_f(t,x) \to 0 \,.$$

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A structured population model

$t \in \mathbb{R}^+$: time, $x \in \mathbb{R}^n$: space variable, $\theta \in \Theta$: phenotypical trait.

Space Diffusion, Mutations, Reproduction.

The model writes :

$$\begin{cases} \partial_t f - D\Delta_x f - \alpha\Delta_\theta f = r\, f\left(a(x,\theta) - \rho\right), \\ \rho(t,x) = \int_\Theta f(t,x,\theta')\, d\theta' . \end{cases}$$

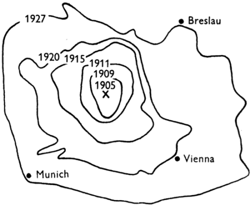
with Neumann boundary conditions in $\theta \in \Theta := [\theta_{min} > 0, \theta_{max} < +\infty]$.

Hamilton-Jacobi approach with θ as a "kinetic" variable

We expect finite speed propagation.

Long time - Large space asymptotics :

$$(t, x, \theta) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \theta\right).$$



Hopf-Cole transformation :

$$f^{\varepsilon}(t, x, \theta) := \exp\left(-\frac{u^{\varepsilon}(t, x, \theta)}{\varepsilon}\right).$$

The propagation is described by the nullset of $u^0 := \lim_{\varepsilon \to 0} u^{\varepsilon}$.

Convergence result for u_ε

Theorem (B., Mirrahimi)

When $\varepsilon \to 0$, u^ε converges locally uniformly to the unique **viscosity solution** *of*

$$\begin{cases} \min\left(\partial_t u + D|\nabla_x u|^2 + H, u\right) = 0, \\ u(0,\cdot) = u_0(\cdot), \end{cases}$$

where the effective Hamiltonian H is obtained through a **spectral problem** *in the θ variable.*

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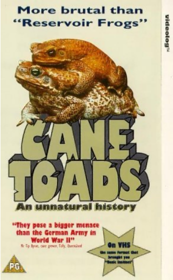
- Model for the cane toads invasion: Selection of dispersal

$$\begin{cases} \partial_t f - {\color{red}\theta} \Delta_x f - {\color{blue}\alpha} \Delta_\theta f = r\, f\,(1-\rho)\,, \\ \rho(t,x) = \int_\Theta f(t,x,\theta')\, d\theta'\,. \end{cases}$$



- Hamilton - Jacobi approach for kinetic equations:

$$f^\varepsilon(t,x,v) := M(v) e^{-\frac{\varphi^\varepsilon(t,x,v)}{\varepsilon}}\,.$$

- Front acceleration via Hamilton - Jacobi:

  Fancy scalings - Formulation of the limit problem can be tricky.

- Travelling fronts for multiscale models (*e.g.* shocks for BGK equations ..).

Merci de votre attention !