Large deviations for velocity-jump processes and Hamilton-Jacobi equations

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Collège de France - 25 novembre 2016

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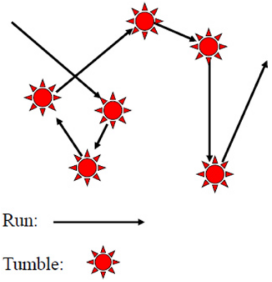
Velocity-jump processes



Persistent motion, with two phases, alternately:

$\rightarrow$ straight run,

$\rightarrow$ random change of velocity.

(This is the case for the most common bacteria, *E. coli*)

Biological reference. H.C. Berg, *E. coli in Motion*, (2004).

The (non-)linear kinetic BGK equation

The Chapman-Kolmogorov equation of this process is a kinetic equation.

Scattering + Reorientation + Growth with saturation (KPP type nonlinearity):

$$\underbrace{\partial_t f + v \cdot \nabla_x f}_{\text{Scattering}} = \underbrace{(M(v)\rho - f)}_{\text{Reorientation}} + \underbrace{r\rho\,(M(v) - f)}_{\text{Growth with saturation}}$$

Kinetic density $f(t,x,v)$: time $t \in \mathbb{R}^+$, position $x \in \mathbb{R}^n$ and velocity $v \in V$.

$$\rho := \int_V f(v)\,\mathrm{dv}, \qquad V = B(0, v_{\max}),\ v_{max} \leq +\infty.$$

Here, M is a given distribution which satisfies

$$\int_V M(v)\,\mathrm{dv} = 1, \qquad \int_V v\,M(v)\,\mathrm{dv} = 0, \qquad \int_V |v|^2\,M(v)\,\mathrm{dv} = \theta.$$

The diffusive limit when $r = 0$

M has **zero mean** & **finite variance** $\to$ parabolic scaling $(t, x, v) \mapsto \left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, v\right)$.

$$\varepsilon^2 \partial_t f + \varepsilon v \partial_x f = M(v)\rho - f.$$

Then,

$$\lim_{\varepsilon \to 0} f^\varepsilon(t, x, v) = M(v)\rho(t, x),$$

and the macroscopic density satisfies the **heat equation**,

$$\partial_t \rho = \theta \partial_{xx} \rho.$$

Can we study large deviations for the velocity-jump process ?

References

G. Blankenship and G. C. Papanicolaou, *Stability and control of stochastic systems with wide-band noise disturbance*, SIAM J. Appl. Math. 34 (1978), 437-476.

E. Larsen and J. B. Keller, *Asymptotic solutions of neutron transport problems*, J. Math. Phys. 15 (1974), 75-81.

Large deviations for the heat equation

Hyperbolic scaling: $(t,x) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$,

$$\varepsilon \partial_t \rho^\varepsilon = \varepsilon^2 \theta \Delta \rho^\varepsilon \,.$$

Perform the Hopf-Cole transformation $\rho^\varepsilon = \exp\left(-\varepsilon^{-1} u^\varepsilon\right)$ to find

$$\partial_t u^\varepsilon + \theta |\nabla_x u^\varepsilon|^2 = \varepsilon \theta \Delta u^\varepsilon .$$

When $\varepsilon \to 0$, the sequence u^ε converges towards the **viscosity solution** of the following **Hamilton-Jacobi equation**

$$\partial_t u + \theta |\nabla_x u|^2 = 0 \,.$$

We seek a similar result for the kinetic equation.

References W.H. Fleming, *Exit Probabilities and Optimal Stochastic Control*, (1978).

Diffusive limit when $r > 0$

Macroscopic limit: $(t, x) \mapsto \left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)$ and $\mathbf{r} \mapsto (\mathbf{r}\varepsilon^2)$:

The **Fisher-KPP equation**: $\partial_t \rho = \theta \partial_{xx} \rho + r\rho(1-\rho).$

Theorem (Kolmogorov, Petrovsky, Piskunov, 1937)

- *There exists a minimal speed $c^* := 2\sqrt{r\theta}$ such that for all speed $c \geq c^*$, there exists a travelling wave solution ($\rho(t,x) := \bar{\rho}(x - ct)$) of speed c.*
- *If the initial data has compact support then the front propagates with the minimal speed c^*.*

Sharp front limit ($\approx$ large deviations) for Fisher-KPP

Hopf-Cole transformation $\rho^\varepsilon = \exp\left(-\frac{u^\varepsilon}{\varepsilon}\right)$:

$$\partial_t u^\varepsilon + \theta|\nabla_x u^\varepsilon|^2 + r = \varepsilon\theta\Delta u^\varepsilon + r\rho^\varepsilon.$$

When $\varepsilon \to 0$, the sequence u^ε converges towards the **viscosity solution** of the following **constrained Hamilton-Jacobi equation**

$$\min\left(\partial_t u + \theta|\nabla_x u|^2 + r\,,\ u\right) = 0.$$

If $u(0,x) = \begin{cases} 0 & \text{if } x = 0 \\ +\infty & \text{else} \end{cases}$, then $u(t,x) = \max\left(\frac{x^2}{4\theta t} - rt, 0\right)$.

The **nullset** of u gives the information about the propagation:

$$u(t,x) = 0 \qquad \Longrightarrow \qquad |x| \leq 2\sqrt{r\theta}\ t = c^* t.$$

We seek a similar result for the kinetic equation.

References. M.I. Freidlin (1986), L.C. Evans and P.E. Souganidis (1989).

Aim of this talk

What we want to do :

→ Study qualitatively and quantitatively large deviations ($r = 0$)/propagation phenomena ($r > 0$) in kinetic reaction-transport equations of the type

$$\partial_t f + v \cdot \nabla_x f = M\rho - f + r\rho(M - f) \,.$$

→ Does it make any difference with the macroscopic limit ?

In this talk, we stick to the 1d case for both space and velocity :

$$x \in \mathbb{R}, \qquad v \in V = [-v_{\max}, v_{\max}].$$

Large deviations in the kinetic framework

Hyperbolic scaling : $(t, x, v) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$

$$\varepsilon \left(\partial_t f^\varepsilon + v \partial_x f^\varepsilon\right) = M\rho^\varepsilon - f^\varepsilon + r\rho^\varepsilon \left(M - f^\varepsilon\right).$$

Hopf-Cole transform :

$$f^\varepsilon(t, x, v) = e^{-\frac{u^\varepsilon(t,x,v)}{\varepsilon}}.$$

New equation for u^ε :

$$\partial_t u^\varepsilon + v\partial_x u^\varepsilon - 1 = -(1+r)\int_V M(v) e^{\frac{u^\varepsilon(v) - u^\varepsilon(v')}{\varepsilon}} dv' + r\rho^\varepsilon,$$

$\rightarrow$ **Question :** can we pass to the limit ? Does it make a difference with the macroscopic case ?

Passing to the limit

Theorem (B. & Calvez (2012),B. (2014))

Let $V = [-v_{\max}, v_{\max}]$ **bounded**. *Assume* $M > 0$ *and* $u^\varepsilon(0,x,v) = u_0(x)$. *Then* $(u^\varepsilon)_\varepsilon$ *converges locally uniformly towards* u, *where* u **does not depend on** v. *Moreover* u *is the unique viscosity solution of* **the constrained Hamilton-Jacobi equation**

$$\begin{cases} \min\left\{\partial_t u + (1+r)\mathcal{H}\left(\frac{\nabla_x u}{1+r}\right) + r, u\right\} = 0, & \forall (t,x) \in \mathbb{R}_+^* \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

The Hamiltonian solves the relation

$$\int_V \frac{M(v')}{1 + \mathcal{H}(p) - v'p} dv' = 1.$$

We combine **kinetic equations** and **viscosity solutions**!

See also a Lagrangian viewpoint in the finite velocity case and $r = 0$ (Faggionato *et al* (2008)). If M vanishes, see Caillerie (2016).

Sketch of proof

$$\partial_t u^\varepsilon + v \cdot \nabla_x u^\varepsilon - 1 = -(1+r) \int_V M(v) e^{\frac{u^\varepsilon(v) - u^\varepsilon(v')}{\varepsilon}} dv' + r\rho^\varepsilon,$$

1. Uniform Lipschitz estimates give the locally uniform convergence of u^ε (up to extraction).
2. The boundedness of

$$\int_V M(v) e^{\frac{u^\varepsilon(v) - u^\varepsilon(v')}{\varepsilon}} dv'$$

implies in the limit

$$(\forall v, v' \in V) \qquad u(v) - u(v') \leq 0,$$

and this implies the independence of v in the limit $\varepsilon \to 0$.

About the Hamiltonian

Now, write formally (we then use the **perturbed test function method**)

$$u^\varepsilon(t,x,v) = u(t,x) - \varepsilon \ln\left(Q(t,x,v)\right) + \mathcal{O}(\varepsilon),$$

One gets, at least formally,

$$\left(1 - \partial_t \varphi^0 - v \cdot \nabla_x \varphi^0\right) Q(v) = (1+r) \int_V M(v')Q(v')dv'$$

Being given $\nabla_x u$, find $\partial_t u$ as an eigenvalue of a cell problem in the velocity space.

Implicit equation in this case:

$$\int_V \frac{M(v')}{1 + \mathcal{H}(\nabla_x u) - v'\nabla_x u} dv' = 1.$$

Similar to homogenization theory : x **slow variable**, v **fast variable**.

The procedure can be written with a more general setting [B. 2014]:

$$M\rho - f \qquad \longrightarrow \qquad P(f).$$

where P has a maximum principle.

- The equation writes

$$\min\left\{\partial_t u + \mathcal{H}\left(\nabla_x u\right) + r, u\right\} = 0,$$

- The Hamiltonian is obtained after solving a **spectral problem** in the velocity variable via a Krein-Rutman type argument :
  "For all $p \in \mathbb{R}^n$, there exists a unique $\mathcal{H}(p)$ such that there exists a positive normalized eigenvector $Q_p \in L^1(V)$ such that

$$\forall v \in V, \qquad \mathcal{L}(Q_p)(v) + (v \cdot p)\, Q_p(v) = \mathcal{H}(p) Q_p(v)."$$

$\mathcal{H}$ satisfies $|\mathcal{H}'(p)| \leq v_{\max}$: It keeps in mind the finite speed of propagation at the kinetic level.
Reminder : Performing the diffusion limit first gives $\theta|p|^2 + r$.

References. Non L^1 case: Caillerie (2016).

Existence of travelling wave solutions

1. Perturbative approach in the parabolic limit $(t, x, r) \mapsto \left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, r\varepsilon^2\right)$

Theorem (Cuesta, Hittmeir, Schmeiser 2012)

Assume that V is compact. Take $c \geq 2\sqrt{r\theta}$. For ε small enough, there exists a travelling wave solution of speed c.

2. Existence result in the kinetic regime (for all ε, optimal (no TW below c^*)):

Theorem (B., Calvez, Nadin 2013)

Assume that V is compact. Suppose that M is continuous and positive. Define

$$c^* = \inf_{p>0} \frac{\mathcal{H}(p)}{p}$$

There exists a travelling wave f solution of speed c for all $c \in [c^*, v_{\max})$.

Further properties

Spreading at finite speed (*a la* Aronson-Weinberger)

1. For all $c > c^*$,

$$(\forall v \in V) \quad \lim_{t \to +\infty} \left(\sup_{x \geq ct} f(t,x,v) \right) = 0 \,,$$

2. For all $c < c^*$,

$$(\forall v \in V) \quad \lim_{t \to +\infty} \left(\sup_{x \leq ct} |M(v) - f(t,x,v)| \right) = 0 \,,$$

But why V has to be compact ?!

The dispersion relation that one has to solve to compute the Hamiltonian $\mathcal{H}$ is

$$\int_V \frac{M(v')}{1+\mathcal{H}(p)-v'p}dv' = 1.$$

This equation has **no solution meeting the constraint of non-negativity** when V is unbounded ($v_{max} = +\infty$)

Remark

Up to here, the strategy would be exactly the same in any velocity dimension. However, now, the integrability condition reads differently depending on the dimension and different effects might appear (see Caillerie (2016)).

The Cauchy problem with unbounded velocities

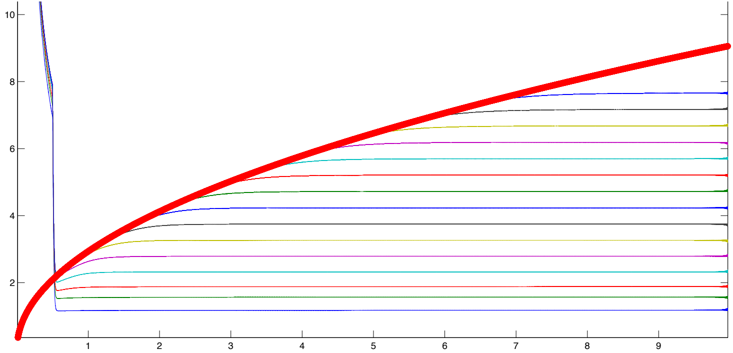
Consider now the case $V = \mathbb{R}$.

$$\begin{cases} \partial_t f + v\partial_x f = M(v)\rho - f + r\rho\left(M(v) - f\right), \\ f(t = 0, x, v) = M(v)\mathbf{1}_{x \le 0}. \end{cases}$$

Question : what is the speed of propagation ?

Approximation of $v_{max} = +\infty$: speed as a function of time

$$M(v) = C\left(v_{\max}\right)\exp\left(-\frac{v^2}{2}\right)\mathbf{1}_{|v|\leq v_{max}}$$



Conjecture :

$$\boxed{\mathbf{c(t)} \approx \sqrt{\mathbf{t}} \quad \Longrightarrow \quad \mathbf{x(t)} \approx \mathbf{t}^{\frac{3}{2}}}$$

Infinite speed of propagation

$$\boxed{\text{Assume that : } \quad \forall v \in \mathbb{R}, \qquad M(v) > 0.}$$

Theorem (B., Calvez, Nadin)

Assume that there exists $\gamma \in (0,1)$ such that

$$\forall x \leq 0, \qquad f(0,x,v) \geq \gamma M(v).$$

Then, one has, for all $c > 0$,

$$\lim_{t \to +\infty} \sup_{x \leq ct} |M(v) - f(t,x,v)| = 0\,.$$

Sketch of proof.

$\lim_{v_{\max} \to +\infty} c^*(v_{\max}) = +\infty$ and a sub-solution using the truncated problem. □

Rate of acceleration when M is a Gaussian on $V = \mathbb{R}$

Using a sub- and super- solutions technique, we prove

Theorem (B., Calvez, Nadin)

Let $M(v) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right)$. *Under suitable hypothesis on the initial data, there exist two explicit constants* c_1 *and* c_2 *such that*

$$\lim_{t \to +\infty} \left(\inf_{x \leq c_1 t^{3/2}} \rho(t,x) \right) \geq \frac{1}{2}, \qquad \lim_{t \to +\infty} \left(\sup_{x \geq c_2 t^{3/2}} \rho(t,x) \right) = 0 .$$

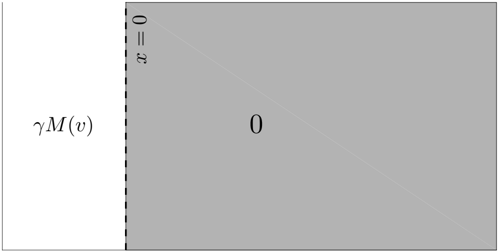
Proposition

$$c_1 := \left(\frac{r}{r + \frac{3}{2}} \right)^{\frac{3}{2}} \leq \underbrace{\frac{\left(\frac{2}{3} r\right)^{\frac{3}{2}}}{1 + r}}_{\text{Expected constant}^a} \leq \sqrt{2r} =: c_2 .$$

[a] See later.

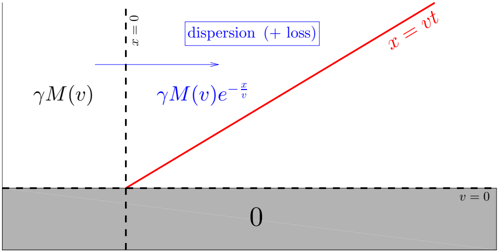
Construction of the sub-solution

1. Start with the initial data $\gamma M(v) \mathbf{1}_{x \leq 0}$.



Construction of the sub-solution

- Start with the initial data $\gamma M(v)\mathbf{1}_{x\leq 0}$.
- Transport very few particles with very high velocity at the edge of the front.

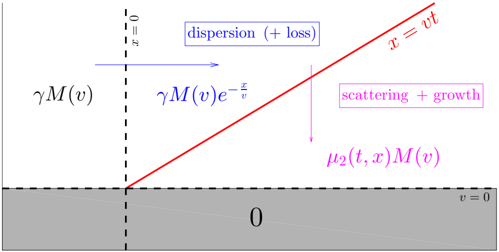


Partial mass contained in the branch $v > \frac{x}{t}$:

$$\mu_1(t,x) = \gamma\int_{\frac{x}{t}}^{\infty} M(v)e^{-\frac{x}{v}}\,dv \geq \frac{1}{r_2(x)}\exp\left(-\frac{3}{2}x^{2/3}\right), \quad \text{if} \quad x < t^{3/2}.$$

Construction of the sub-solution

- Start with the initial data $\gamma M(v) \mathbf{1}_{x \leq 0}$.
- Transport very few particles with very high velocity at the edge of the front.
- Redistribute and grow the previous density.



Estimation of μ_2 for $x \in \left\{ x \, : \, x \leq (\alpha t)^{3/2} \right\}$:

$$\mu_2(t,x) \gtrsim \frac{1}{\sqrt{t}} \exp\left(-\frac{3}{2} \left((\alpha t)^{3/2} \right)^{2/3} \right) e^{r(1-\alpha)t}.$$

Conclusions

- Bounded velocities :
  - Minimal speed of propagation,
  - Profiles given by a spectral problem,
  - Linear spreading.

Qualitatively similar to the Fisher-KPP equation,
but quantitative differences!

- Unbounded velocities :
  - Accelerated propagation,
  - Almost exact rate in the Gaussian case ($\sim t^{\frac{3}{2}}$),

Unexpected result *since the diffusive*
limit is the Fisher-KPP equation.

A large deviation framework with unbounded velocities

(M is still a Gaussian)

The hyperbolic scaling is not relevant.

The relevant scaling is

$$(t, x, v) = \left(\frac{t'}{\varepsilon}, \frac{x'}{\varepsilon^{\mathbf{3/2}}}, \frac{v'}{\varepsilon^{\mathbf{1/2}}} \right).$$

Consider first $r = 0$. The equation to solve is

$$\partial_t u^\varepsilon + v \cdot \nabla_x u^\varepsilon = 1 - \frac{1}{\sqrt{2\pi\varepsilon}} \int_{\mathbb{R}^n} \exp\left(\frac{u^\varepsilon(v) - u^\varepsilon(v') - |v|^2/2}{\varepsilon} \right) dv'.$$

Towards a limit system...

$$\partial_t u^\varepsilon + v \cdot \nabla_x u^\varepsilon = 1 - \frac{1}{\sqrt{2\pi\varepsilon}} \int_{\mathbb{R}^n} \exp\left(\frac{u^\varepsilon(v) - u^\varepsilon(v') - |v|^2/2}{\varepsilon} \right) dv'$$

Uniform bounds imply, as $\varepsilon \to 0$,

$$(\forall v, v') \quad u(v) \le u(v') + \frac{|v|^2}{2} \qquad \Longrightarrow \qquad u(v) \le \min_{\mathbf{v}' \in \mathbb{R}} \mathbf{u}(\mathbf{v}') + \frac{|v|^2}{2}$$

Non-local constraint!

$\to$ The minimum value is attained at $v = 0$.
$\to$ Free transport in the unconstrained area:

$$\partial_t u + v \cdot \nabla_x u - 1 = 0\,.$$

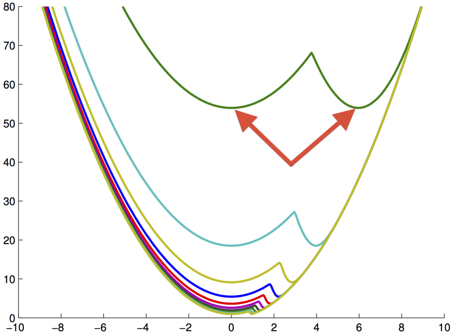
This gives the first part of the system:

$$\max\left(\partial_t u + v \cdot \nabla_x u - 1, u - \min_{w \in \mathbb{R}^n} u - \frac{|v|^2}{2} \right) = 0\,,$$

Problem: dynamics of the minimum value $\min_v u(t, x, v)$?

Numerical illustration

It can vary only if there is another minimum point, $v^* \neq 0$, in the unsaturated area !.



Time snapshots of the velocity profile (fixed x)

Information about $\min_v u(t,x,v)$?

Heuristic : $\min_v u(t,x,v)$ **encodes the** *mass* **at** (t,x) **!**

Lemma

Let $I \subset \mathbb{R}^n$. One has, locally uniformly in $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$,

$$\lim_{\varepsilon \to 0} \left(-\varepsilon \ln \underbrace{\left(\int_I e^{-\frac{u^\varepsilon(t,x,v')}{\varepsilon}} dv' \right)}_{=\sqrt{\varepsilon}\rho^\varepsilon} \right) = \min_{w \in I} u(t,x,w).$$

We thus use **mass conservation** at the f^ε level to get the information :

$$\varepsilon\left(\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon\right) = M_\varepsilon(v)\rho^\varepsilon - f^\varepsilon,$$

Integrate in velocity over $B(0,\delta) \subset \mathbb{R}^n$:

$$- \varepsilon \partial_t \left(\int_{B(0,\delta)} f^\varepsilon dv \right) + \int_{B(0,\delta)} \left(v \cdot \nabla_x u^\varepsilon \right) f^\varepsilon dv = \int_{B(0,\delta)} f^\varepsilon dv - \left(\int_{B(0,\delta)} M_\varepsilon(v) dv \right) \rho^\varepsilon ,$$

$$\partial_t \left(-\varepsilon \ln \left(\int_{B(0,\delta)} e^{-\frac{u^\varepsilon(v)}{\varepsilon}} dv \right) \right) + \int_{B(0,\delta)} \left(v \cdot \nabla_x u^\varepsilon \right) \frac{f^\varepsilon dv}{\int_{B(0,\delta)} f^\varepsilon(v') dv'} = 1 - \left(\int_{B(0,\delta)} M_\varepsilon(v) dv \right) \frac{\rho^\varepsilon}{\int_{B(0,\delta)} f^\varepsilon dv} ,$$

From this we deduce the last part of the system :

$$\begin{cases} \partial_t \left(\min\limits_{w \in B(0,\delta)} u = \min\limits_{\mathbf{w} \in \mathbb{R}^\mathbf{n}} \mathbf{u} \right) \leq 0 \,, \\ \partial_t \left(\min\limits_{w \in \mathbb{R}^n} u \right) = 0 \,, \qquad \text{if} \quad \operatorname{argmin}(u)(t,x) = \{0\} \,. \end{cases}$$

Typical dynamics of solutions

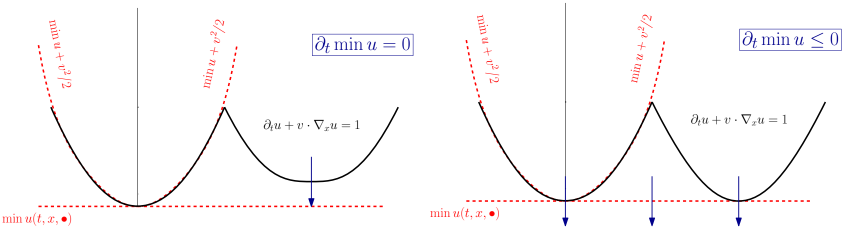


Figure: Typical dynamics of solutions

Proposition (B.,Calvez,Grenier,Nadin 2016)

Assume $r = 0$. Then the limit system when $\varepsilon \to 0$ shall be :

$$\begin{cases} \max\left(\partial_t u + v \cdot \nabla_x u - 1, u - \min\limits_{w \in \mathbb{R}^n} u - \dfrac{|v|^2}{2}\right) = 0\,, \\ \partial_t \left(\min\limits_{w \in \mathbb{R}^n} u\right) \leq 0\,, \\ \partial_t \left(\min\limits_{w \in \mathbb{R}^n} u\right) = 0\,, \qquad \text{if} \quad \operatorname{argmin}(u)(t,x) = \{0\}\ . \\ u(0,x,v) = u_0(x,v)\,. \end{cases}$$

In which sense ?!

Cauchy theory and convergence

We need to introduce a suitable notion of sub- and super- solutions !

Theorem (Comparison principle)

(Suitably defined,) let $\underline{u}$ (resp. $\overline{u}$) be a viscosity sub-solution (resp. super-solution) on $[0, T) \times \mathbb{R}^{2n}$. Assume that $\underline{u}$ and $\overline{u}$ are such that

$$\overline{u} - |v|^2/2\,, \quad \underline{u} - |v|^2/2 \in L^\infty\left([0, T) \times \mathbb{R}^{2n}\right)\,.$$

Then $\underline{u} \leq \overline{u}$ on $[0, T) \times \mathbb{R}^{2n}$.

We can extend this uniqueness result to solutions with at most quadratic growth (in space) at infinity.

Theorem (Convergence)

Take $u_0 \in \frac{|v|^2}{2} + L^\infty$. Then u^ε converges locally uniformly towards u, which is the unique viscosity solution of the limit system, as $\varepsilon \to 0$.

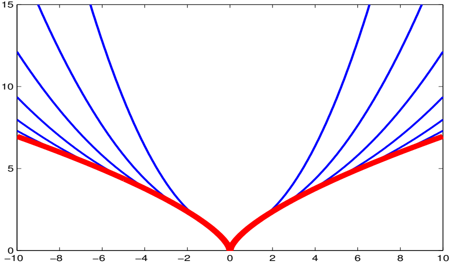
Particular solution of the limit system

Initial data:

$$u(0,x,v) = \mathbf{0}_{x=0} + \mathbf{0}_{v=0}\,.$$

The minimum value satisfies:

$$\min_v u(t,x,v) = \min_{0\le s\le t}\left(\frac{x^2}{2s} + s\right) = \begin{cases} \frac{3}{2}|x|^{2/3} & \text{if } |x| \le t^{3/2} \\ \frac{|x|^2}{2t^2} + t & \text{if } |x| \ge t^{3/2} \end{cases}$$



Time snapshots of the minimum value $\min_v u(t,x,v)$

Limit system when $r > 0$.

The equation is :

$$\partial_t u^\varepsilon + v \cdot \nabla_x u^\varepsilon = 1 - (1+r) \int_{\mathbb{R}^n} \frac{1}{\sqrt{2\pi\varepsilon}} e^{\frac{u^\varepsilon(v) - u^\varepsilon(v') - |v|^2/2}{\varepsilon}} dv' + \frac{r}{\sqrt{\varepsilon}} \int_{\mathbb{R}^n} e^{-\frac{u^\varepsilon(v')}{\varepsilon}} dv'.$$

Constraint coming from the nonlinear problem : $\mathbf{\min_{v \in \mathbb{R}} u(t,x,v) \geq 0}$.

Limit system:

1. If $\min_{w \in \mathbb{R}^n} u(t,x,w) = 0$ then, for all $v \in \mathbb{R}^n$, one has $u = \frac{|v|^2}{2}$.
2. If $\min_{w \in \mathbb{R}^n} u(t,x,w) > 0$,

$$\begin{cases} \max\left(\partial_t u + v \cdot \nabla_x u - 1, u - \min\limits_{w \in \mathbb{R}^n} u - \dfrac{|v|^2}{2}\right) = 0\,, \\ \partial_t \left(\min\limits_{w \in \mathbb{R}^n} u\right) \leq -r\,, \\ \partial_t \left(\min\limits_{w \in \mathbb{R}^n} u\right) = -r\,, \qquad \text{if } \operatorname{argmin}(u)(t,x) = \{0\}\,. \end{cases} \tag{1}$$

3. $u(0,\cdot,\cdot) = u_0(\cdot,\cdot)$.

Acceleration rate when $r > 0$.

We conjecture that we can proceed as for the Fisher-KPP equation case:

> We truncate the fundamental solution of the linearized limit system such that is satisfies the constraint.

Recall that the minimum value satisfies:

$$\min_v u(t,x,v) = \begin{cases} \frac{3}{2}|(1+r)x|^{2/3} - rt & \text{if } |x| \le (1+r)^{\frac{1}{2}} t^{3/2} \\ \frac{|x|^2}{2t^2} + t & \text{if } |x| \ge (1+r)^{\frac{1}{2}} t^{3/2} \end{cases}$$

We obtain

$$\min_{v\in\mathbb{R}} u = 0 \quad \Longrightarrow \quad \frac{3}{2}\left((1+r)x\right)^{\frac{2}{3}} = rt \quad \Longrightarrow \quad x = \frac{\left(\frac{2}{3}\mathbf{r}\right)^{\frac{3}{2}}}{\mathbf{1}+\mathbf{r}} t^{\frac{3}{2}}.$$

Thank you for your attention ... !