Propagation in kinetic reaction-transport equations

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Kinetic reaction-transport equations

Kinetic density $f(t,x,v)$: time t, position x and velocity v.

Space density

$$\rho := \int_V f(v)dv,$$

Set of velocities

$$V = [-v_{\max}, v_{\max}],$$

$$\boxed{\text{with } v_{max} \leq +\infty}.$$

3 ingredients :

- Run at velocity v = transport operator: $\partial_t + v\partial_x$,
- Tumbling events = velocity jump process: $\frac{1}{\tau}(M(v)\rho - f)$,
- Growth of the population = monostable nonlinearity: $r\rho(M(v) - f)$.

Altogether,

$$\partial_t f + v\partial_x f = \frac{1}{\tau}(M(v)\rho - f) + r\rho(M(v) - f)$$

Here, M is a given distribution which satisfies

$$\int_V M(v)dv = 1, \qquad \int_V vM(v)dv = 0, \qquad \int_V v^2 M(v)dv = \theta.$$

What we want to do :

→ Study qualitatively and quantitatively propagation phenomena in kinetic reaction-transport equations.

→ Are there special effects due to considering populations at the "mesoscopic" scale ?

We study the propagation from two points of view :

1. First approach : Study of (non-)existence of travelling wave solutions,
2. Second approach : Geometric optics point of view.

In Search of Kinetic Travelling Waves

$$\partial_t f + v\partial_x f = \frac{1}{\tau}\left(M(v)\rho - f\right) + r\rho\left(M(v) - f\right)$$

The equilibrium states: 0 and $M(v)$.

Definition

A travelling wave solution is of the form

$$f(t,x,v) = \mu\left(\xi = x - ct, v\right),$$

Speed : $c \in \mathbb{R}^+$, Profile : $\mu \in \mathcal{C}^2\left(\mathbb{R} \times V, \mathbb{R}^+\right)$.

Far field conditions : $\mu\left(-\infty,\cdot\right) = M, \qquad \mu\left(+\infty,\cdot\right) = 0.$

Main equation :

$$(v - c)\partial_\xi \mu = \frac{1}{\tau}\left(M(v)\nu - \mu\right) + r\nu\left(M(v) - \mu\right), \qquad \xi \in \mathbb{R},\ v \in V.$$

where ν is the macroscopic density associated to μ, that is $\nu\left(\xi\right) = \int_V \mu\left(\xi, v\right) dv$.

Why should we expect travelling waves ?

Macroscopic limit : We look at the situation when reorientations are much more frequent than reaction:

$$\mathbf{r} \mapsto \left(\mathbf{r}\varepsilon^{\mathbf{2}}\right).$$

M is unbiaised $\to$ Parabolic scaling $(t,x) \mapsto \left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)$:

$$\varepsilon^2 \partial_t f + \varepsilon v \partial_x f = \frac{1}{\tau}\left(M(v)\rho - f\right) + r\varepsilon^2 \rho\left(M(v) - f\right).$$

Then formally,

$$\lim_{\varepsilon \to 0} f^{\varepsilon}(t,x,v) = M(v)\rho(t,x),$$

The macroscopic limit is (at least formally) the **Fisher-KPP equation**

$$\partial_t \rho = \tau\theta\partial_{xx}\rho + r\rho(1-\rho)$$

Travelling waves for the Fisher-KPP equation (1937)

Combining **reaction** and **diffusion** creates propagation :

Theorem (Kolmogorov, Petrovsky, Piskunov, 1937)

- *There exists a minimal speed* $c^* := 2\sqrt{r\theta}$ *such that for all speed* $c \geq c^*$, *there exists a travelling wave solution (* $\rho(t,x) := \bar{\rho}(x - ct)$ *) of speed* c.
- *If the initial data has compact support then the front propagates with the minimal speed* c^*.

(Fisher, KPP, Kanel, Fife and McLeod, Aronson and Weinberger ...)

The minimal speed c^* :

The front is created by small populations at the edge that reproduce almost exponentially. Seeking exponential decay in **the linearized equation** :

$$\rho(t,x) := e^{-\lambda(x-c(\lambda)t)} \implies c(\lambda) = \theta\lambda + \frac{r}{\lambda} \geq 2\sqrt{r\theta} := c^* .$$

References. R.A. Fisher, *The advance of advantageous genes*, (1937),
D.G. Aronson *et al. Nonlinear diffusion in population genetics* . . . , 1975.
A.N. Kolmogorov *et al. Etude de l'équation de la diffusion* . . . , (1937).

Existence of travelling waves for the kinetic model

1. Perturbative approach in the parabolic limit $(t, x, r) \mapsto \left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, r\varepsilon^2\right)$

Theorem (Cuesta, Hittmeir, Schmeiser)

Assume that V is compact. Let the wave speed satisfy $c \geq 2\sqrt{r\theta}$. For ε small enough, there exists a travelling wave solution of speed c.

2. Existence result in the kinetic regime:

Theorem (B., Calvez, Nadin)

Assume that V is compact. Suppose that M is continuous and positive.

1. *There exists a speed $c^* \in (0, v_{\max})$ such that there exists a travelling wave f solution of speed c for all $c \in [c^*, v_{\max})$.*
2. *The travelling wave is nonincreasing with respect to the space variable.*
3. *There exists no travelling wave of speed $c \in [0, c^*)$.*

Elements of proof

1. Find the minimal speed c^* : Given a spatial decay $\lambda \in \mathbb{R}^+$, we seek solutions of the linearized problem of type

$$f(t,x,v) = e^{-\lambda(x-c(\lambda)t)} Q_\lambda(v).$$

Associated speed : $c(\lambda) \in \mathbb{R}^+$, Expected profile at the edge : $Q_\lambda(v)$.

Proposition

We have $c^* = \min_{\lambda>0} c(\lambda)$, where $c(\lambda)$ is a solution of

$$\int_V \underbrace{\frac{(1+\tau r)M(v)}{1+\tau\lambda(c(\lambda)-v)}}_{=Q_\lambda(v)} \, dv = 1 \,.$$

2. Key tool : Comparison principle.

We can define, thanks to the dispersion relation, for $c \in (c^*, v_{max})$, an explicit couple of sub- and super- solutions.

Further properties

1. Spreading at finite speed (*a la* Aronson-Weinberger)
   1. For all $c > c^*$,

   $$(\forall v \in V) \quad \lim_{t \to +\infty} \left(\sup_{x \geq ct} f(t,x,v) \right) = 0\,,$$

   2. For all $c < c^*$,

   $$(\forall v \in V) \quad \lim_{t \to +\infty} \left(\sup_{x \leq ct} |M(v) - f(t,x,v)| \right) = 0\,,$$

2. Dynamical stability of the waves : *Rather explicit* weight $\phi(\xi, v)$ such that a travelling wave profile is *weakly linearly stable* in $L^2\left(e^{-2\phi(\xi,v)} d\xi dv\right)$.

Obstruction

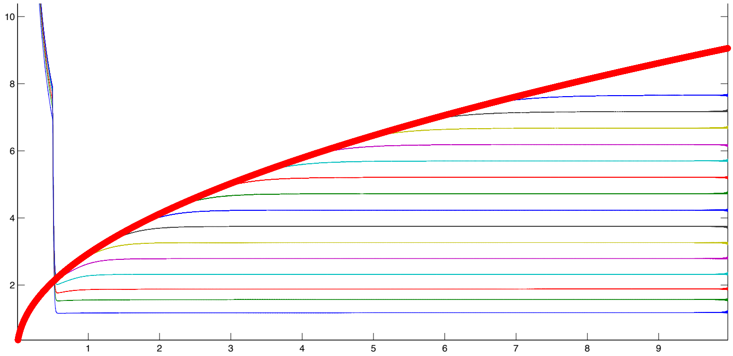
The dispersion relation for $\lambda \in \mathbb{R}^+$

$$\int_V \frac{(1+\tau r)M(v)}{1+\tau\lambda(c(\lambda)-v)}\,dv = 1\,.$$

has **no solution** when V is unbounded ($v_{max} = +\infty$).

Approximation of $v_{max} = +\infty$: speed as a function of time

→ Gaussian equilibrium : $M(v) = C\,(v_{\max})\exp\left(-\frac{v^2}{2}\right)\mathbf{1}_{|v|\leq v_{max}}$



Conjecture :

$$\mathbf{c(t) \approx \sqrt{t} \quad \Longrightarrow \quad x(t) \approx t^{\frac{3}{2}}}$$

Acceleration phenomena

→ Fisher-KPP with initial decay slower than exponential :
F. Hamel, L. Roques, *Fast propagation for KPP equations with slowly decaying initial conditions*, (2010).

→ Accelerated propagation in fractionnal diffusion equations :

- X. Cabré, J.-M. Roquejoffre, *Propagation de fronts dans les équations de Fisher–KPP avec diffusion fractionnaire*, (2009).
- X. Cabré, J.-M. Roquejoffre, *The influence of fractional diffusion in Fisher-KPP equations,* (2013).
- A.-C. Coulon, J.-M Roquejoffre, *Transition between linear and exponential propagation in Fisher-KPP type reaction-diffusion equations*, (2012).

→ Acceleration in integro-differential equations with slowly decaying kernel :
J. Garnier, *Accelerating solutions in integro-differential equations*, (2011).

Infinite speed of propagation

Assume that : $\forall v \in \mathbb{R}, \qquad M(v) > 0.$

Theorem (B., Calvez, Nadin)

Assume that there exists $\gamma \in (0,1)$ *such that*

$$\forall x \leq 0, \qquad f(0,x,v) \geq \gamma M(v).$$

Then, one has, for all $c > 0$,

$$\lim_{t \to +\infty} \sup_{x \leq ct} |M(v) - f(t,x,v)| = 0\,.$$

Sketch of proof.

$\lim_{v_{\max} \to +\infty} c^*(v_{\max}) = +\infty$ and a sub-solution using the truncated problem. □

Rate of acceleration when M is a Gaussian on $V = \mathbb{R}$

Using a sub- and super- solutions technique, we prove

Theorem (B., Calvez, Nadin)

Let $M(v) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right)$. Under suitable hypothesis on the initial data, there exist two explicit constants c_1 and c_2 such that

$$\lim_{t \to +\infty} \left(\inf_{x \le c_1 t^{3/2}} \rho(t,x) \right) \ge \frac{1}{2}, \qquad \lim_{t \to +\infty} \left(\sup_{x \ge c_2 t^{3/2}} \rho(t,x) \right) = 0 .$$

Proposition

$$c_1 := \left(\frac{r}{r + \frac{3}{2}} \right)^{\frac{3}{2}} \le \underbrace{\frac{\left(\frac{2}{3} r\right)^{\frac{3}{2}}}{1+r}}_{\text{Expected constant}^a} \le \sqrt{2r} =: c_2 .$$

[a] See later.

Construction of the sub-solution

→ Idea : The free transport operator sends very few particles with very high velocity at the edge of the front. They are redistributed, and their density grows exponentially fast.

Reference. J. Garnier, *Accelerating solutions in integro-differential equations*, (2011).

The sub-solution has to solve :

$$\partial_t \underline{g} + v\partial_x \underline{g} + \underline{g} \quad \leq \quad \left(1 + \tfrac{r}{2}\right) M(v)\rho_{\underline{g}} \,.$$

1. **Transport :**

$$g_2 = \frac{1}{2} M(v) e^{-\frac{x}{v}} \,, \quad \text{if} \;\; v > \frac{x}{t} \,,$$

   and zero elsewhere, solves

$$\partial_t g_2 + v\partial_x g_2 + g_2 = 0.$$

   Partial mass contained in the branch $v > \frac{x}{t}$:

$$\mu_2(t,x) = \frac{1}{2} \int_{\frac{x}{t}}^{\infty} M(v) e^{-\frac{x}{v}} \, dv \,.$$

2. **Redistribution & Growth :** In the area $0 < v < \frac{x}{t}$, the partial mass is denoted by $\mu_1(t,x)$. It solves,

$$\partial_t \mu_1 + \mu_1 = \left(1 + \frac{r}{2}\right) \left(\min\left(\mu_1, \frac{1}{2}\right) \int_0^{\frac{x}{t}} M(v)\, dv + \mu_2 \right) \,.$$

The sub-solution has to solve :

$$\partial_t \underline{g} + v\partial_x \underline{g} + \underline{g} \quad \leq \quad 0 \qquad\qquad .$$

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$$\partial_t \underline{g} \qquad + \underline{g} \leq \left(1 + \frac{r}{2}\right) M(v) \rho_{\underline{g}} \,.$$

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Spreading : Estimation of μ_2 and then μ_1

Lemma

The following estimate holds true,

$$\mu_2(t,x) \geq \frac{1}{r_2(x)} \exp\left(-\frac{3}{2}x^{2/3}\right), \quad \textit{if} \quad x < t^{3/2}.$$

We define the zone

$$\mathcal{Y}_t = \left\{x \,:\, x \leq (\alpha t)^{3/2}\right\}.$$

Estimation of μ_1 for $x \in \mathcal{Y}_t$:

$$\mu_1(t,x) \gtrsim \frac{1}{\sqrt{t}} \exp\left(-\frac{3}{2}\left((\alpha t)^{3/2}\right)^{2/3}\right) e^{r(1-\alpha)t}.$$

For suitable α, for large times, the front has already passed through $\mathcal{Y}_t$.

Remark

"Just" need to estimate $\int_{\frac{x}{t}}^{\infty} M(v) e^{-\frac{x}{v}}\, dv$!

Conclusions

- Bounded velocities :
  - Minimal speed of propagation,
  - Profiles given by a spectral problem,
  - Linear spreading.

As for the Fisher-KPP equation.

- Unbounded velocities :
  - Accelerated propagation,
  - Almost exact rate in the Gaussian case ($\sim t^{\frac{3}{2}}$),

Unexpected result *since the diffusive limit is the Fisher-KPP equation.*

Hamilton-Jacobi approach

A method based on Hamilton-Jacobi equations has been used to study

1. Front propagation in models structured only by the space variable:

   Authors : Barles, Evans, Souganidis ... (89-94)

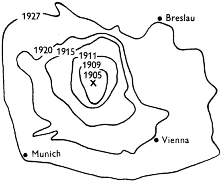
2. Dynamics of most favorable traits in populations structured only by a structural variable:

   Authors : Barles, Champagnat, Diekmann, Jabin, Lorz, Mirrahimi, Mischler, Perthame ...

Aim : Use this method to describe propagation phenomena in kinetic equations (populations structured by both space variable and velocity).

Geometric point of view - Fisher-KPP case

Hyperbolic scaling: $(t,x) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$:



$$(KPP_\varepsilon) \qquad \varepsilon\partial_t\rho^\varepsilon = \varepsilon^2\theta\Delta\rho^\varepsilon + r\rho^\varepsilon(1-\rho^\varepsilon)\,.$$

Hopf-Cole : $\rho^\varepsilon = \exp\left(-\varepsilon^{-1}\varphi^\varepsilon\right)$.

Equation for φ^ε :

$$\partial_t\varphi^\varepsilon + \theta|\nabla_x\varphi^\varepsilon|^2 + r = \varepsilon\theta\Delta\varphi^\varepsilon + r\,\rho^\varepsilon.$$

When $\varepsilon \to 0$, the sequence φ^ε converges towards the **viscosity solution** of the following **constrained Hamilton-Jacobi equation**

$$\min\left(\partial_t\varphi^0 + \theta|\nabla_x\varphi^0|^2 + r\,,\ \varphi^0\right) = 0\,.$$

If

$$\varphi^0(0,x) = \begin{cases} 0 & \text{if } x = 0, \\ +\infty & \text{else,} \end{cases}$$

then

$$\varphi^0(t,x) = \max\left(\frac{x^2}{4\theta t} - rt, 0\right),$$

The **nullset** of φ^0 gives the information about the propagation:

$$\varphi^0(t,x) = 0 \qquad \Longrightarrow \qquad |x| \leq 2\sqrt{r\theta}\, t.$$

Locally on

- $\text{Int}\left(\varphi^0 = 0\right)$, $\qquad \lim_{\varepsilon\to 0} \rho^\varepsilon = 1.$
- $\text{Int}\left(\varphi^0 > 0\right)$, $\qquad \lim_{\varepsilon\to 0} \rho^\varepsilon = 0.$

References. M.I. Freidlin, *Geometric optics approach . . .* , (1986)
L.C. Evans and P.E. Souganidis, *A PDE approach to geometric . . .* , (1989)

In the kinetic framework with bounded velocities.

Hyperbolic scaling : $(t, x, v) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$

$$\varepsilon \left(\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon\right) = \frac{1}{\tau}\left(M(v)\rho^\varepsilon - f^\varepsilon\right) + r\rho^\varepsilon \left(M(v) - f^\varepsilon\right),$$

By analogy with Fisher-KPP, our **kinetic WKB ansatz** writes

$$f^\varepsilon(t, x, v) = M(v) e^{-\frac{\varphi^\varepsilon(t,x,v)}{\varepsilon}}.$$

New equation for φ^ε :

$$\tau \left(\partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon\right) - 1 = -(1 + \tau r) \int_V M(v) e^{\frac{\varphi^\varepsilon(v) - \varphi^\varepsilon(v')}{\varepsilon}} dv' + \tau r \rho^\varepsilon,$$

$\rightarrow$ Can we pass to the limit ? Does it make a difference with the macroscopic case ?

Passing to the limit.

Theorem (B. & Calvez)

Let $V = [-v_{\max}, v_{\max}]$. Suppose that the initial data is well-prepared,

$$\forall (x,v) \in \mathbb{R} \times V, \qquad \varphi^\varepsilon(0,x,v) = \varphi_0(x),$$

Then $(\varphi^\varepsilon)_\varepsilon$ converges locally uniformly towards φ^0, where φ^0 **does not depend on v**. *Moreover φ^0 is the unique viscosity solution of the constrained Hamilton-Jacobi equation*

$$\begin{cases} \min\left\{\partial_t \varphi^0 + \mathcal{H}\left(\nabla_x \varphi^0\right), \varphi^0\right\} = 0, & \forall (t,x) \in \mathbb{R}_+^* \times \mathbb{R}, \\ \varphi^0(0,x) = \varphi_0(x), & x \in \mathbb{R}. \end{cases}$$

We combine **kinetic equations** and **viscosity solutions**!

Sketch of proof

$$\tau\left(\partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon\right) - 1 = -(1+\tau r)\int_V M(v) e^{\frac{\varphi^\varepsilon(v) - \varphi^\varepsilon(v')}{\varepsilon}} dv' + \tau r \rho^\varepsilon,$$

1. Uniform Lipschitz estimates give the locally uniform convergence of φ^ε (up to extraction).
2. The boundedness of

$$\int_V M(v) e^{\frac{\varphi^\varepsilon(v) - \varphi^\varepsilon(v')}{\varepsilon}} dv'$$

forces in the limit

$$(\forall v, v' \in V) \qquad \varphi^0(v) - \varphi^0(v') \leq 0,$$

an this implies the independence of v in the limit $\varepsilon \to 0$.

About the Hamiltonian.

$\rightarrow$ Looks like homogenization theory : x slow variable, v fast variable.

Now, write formally

$$\varphi^\varepsilon(t,x,v) = \varphi^0(x) - \varepsilon \ln\left(\frac{Q(t,x,v)}{M(v)}\right) + \mathcal{O}(\varepsilon),$$

Homogenization: being given $\nabla_x \varphi^0$, find $\partial_t \varphi^0$ as an eigenvalue of a cell problem in the velocity space:

$$\left(1 - \tau\left(\partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon\right)\right) Q(v) = (1+\tau r) M(v) \int_V Q(v')dv'$$

Explicit solution in this case:

$$\int_V \frac{(1+\tau r)M(v')}{1 - \tau(\underbrace{\partial_t \varphi^\varepsilon}_{-\mathcal{H}(\nabla_x \varphi^0)} + v' \cdot \nabla_x \varphi^\varepsilon)} dv' = 1.$$

The procedure can be written with a more general setting:

$$M(v)\rho - f \qquad \longrightarrow \qquad P(f) - \Sigma(v) f.$$

where P has a maximum principle.

- The equation writes

$$\min\left\{\partial_t \varphi^0 + \mathcal{H}\left(\nabla_x \varphi^0\right) + r, \varphi^0\right\} = 0,$$

- The Hamiltonian is obtained after solving a **spectral problem** in the velocity variable via a Krein-Rutman type argument :
"For all $p \in \mathbb{R}^n$, there exists a unique $\mathcal{H}(p)$ such that there exists a positive normalized eigenvector $Q_p \in L^1(V)$ such that

$$\forall v \in V, \qquad \mathcal{L}(Q_p)(v) + (v \cdot p)\, Q_p(v) = \mathcal{H}(p) Q_p(v)."$$

- $\mathcal{H}$ is globally **Lipschitz** with respect to p : It keeps in mind the finite speed of propagation at the kinetic level. Performing the diffusion limit first gives $\theta|p|^2$.

And if $V = \mathbb{R}$ and M is a Gaussian ?

The hyperbolic scaling is not relevant.

The only interesting scaling is

$$(t, x, v) = \left(\frac{t'}{\varepsilon}, \frac{x'}{\varepsilon^{\mathbf{3/2}}}, \frac{v'}{\varepsilon^{\mathbf{1/2}}} \right).$$

Theorem (B.,Calvez,Grenier,Nadin)

Assume $r = 0$. Then the limit system when $\varepsilon \to 0$ shall be :

$$\begin{cases} \max\left(\partial_t \varphi^0 + v \cdot \nabla_x \varphi^0 - 1, \varphi^0 - \min\limits_{w \in \mathbb{R}^n} \varphi^0 - \dfrac{|v|^2}{2} \right) = 0\,, \\ \partial_t \left(\min\limits_{w \in \mathbb{R}^n} \varphi^0 \right) \leq 0\,, \\ \partial_t \left(\min\limits_{w \in \mathbb{R}^n} \varphi^0 \right) = 0\,, \qquad \textit{if} \quad \textit{argmin}(\varphi^0)(t, x) = \{0\}\ . \\ \varphi^0(0, x, v) = \varphi_0(x, v)\,. \end{cases}$$

Sketch of proof

The equation to solve is (when $r = 0$):

$$\partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon = 1 - \frac{1}{\sqrt{2\pi\varepsilon}} \int_{\mathbb{R}^n} \exp\left(\frac{\varphi^\varepsilon(v) - \varphi^\varepsilon(v') - |v|^2/2}{\varepsilon} \right) dv',$$

Uniform bounds imply, as $\varepsilon \to 0$,

$$(\forall v, v') \quad \varphi^0(v) \le \varphi^0(v') + \frac{|v|^2}{2} \qquad \Longrightarrow \qquad \varphi^0(v) \le \min_{\mathbf{v}' \in \mathbb{R}} \varphi^{\mathbf{0}}(\mathbf{v}') + \frac{|v|^2}{2}$$

Non-local constraint!

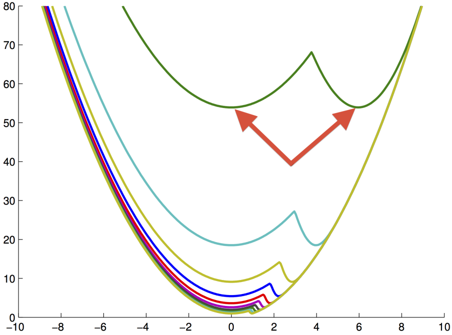
→ The minimum value is attained at $v = 0$.
→ Free transport in the unconstrained area:

$$\partial_t \varphi^0 + v \cdot \nabla_x \varphi^0 - 1 = 0\,.$$

Problem: dynamics of the minimum value $\min_v \varphi^0(t, x, v)$?

Numerical illustration

It can vary only if there is another minimum point, $v^* \neq 0$, in the unsaturated area !.



Time snapshots of the velocity profile (fixed x)

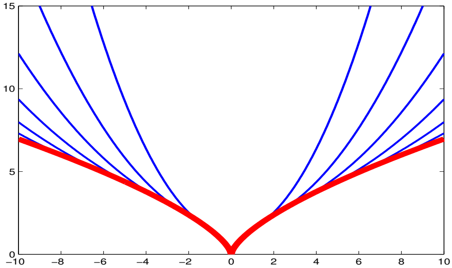
Fundamental solution of the limit system

Initial data:

$$\varphi^0(0,x,v) = \mathbf{0}_{x=0} + \mathbf{0}_{v=0}\,.$$

The minimum value satisfies:

$$\min_v \varphi^0(t,x,v) = \min_{0\leq s\leq t}\left(\frac{x^2}{2s}+s\right) = \begin{cases} \frac{3}{2}|x|^{2/3} & \text{if } |x|\leq t^{3/2} \\ \frac{|x|^2}{2t^2}+t & \text{if } |x|\geq t^{3/2} \end{cases}$$



Time snapshots of the minimum value $\min_v \varphi^0(t,x,v)$

Front acceleration when $r > 0$.

The equation is :

$$\partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon = 1 - (1+r) \int_{\mathbb{R}^n} \frac{1}{\sqrt{2\pi\varepsilon}} e^{\frac{\varphi^\varepsilon - \varphi'^\varepsilon - |v|^2/2}{\varepsilon}} dv' + \frac{r}{\sqrt{\varepsilon}} \int_{\mathbb{R}^n} e^{-\frac{\varphi'^\varepsilon}{\varepsilon}} dv'.$$

Constraint coming from the nonlinear problem : $\min_{\mathbf{v} \in \mathbb{R}} \varphi^{\mathbf{0}}(\mathbf{t}, \mathbf{x}, \mathbf{v}) \geq \mathbf{0}$.

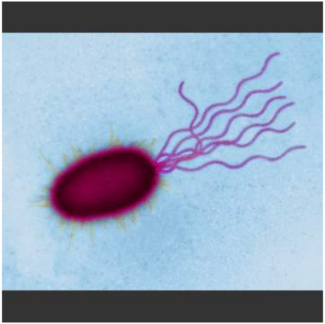
We can prove that we can proceed as for the Fisher-KPP equation case:

We truncate the fundamental solution of the linearized limit system such that is satisfies the constraint.

We obtain

$$\min_{v \in \mathbb{R}} \varphi^0 = 0 \quad \Longrightarrow \quad \frac{3}{2} \left((1+r)x\right)^{\frac{2}{3}} = rt \quad \Longrightarrow \quad x = \frac{\left(\frac{2}{3}\mathbf{r}\right)^{\frac{3}{2}}}{\mathbf{1}+\mathbf{r}} t^{\frac{3}{2}}.$$

Thank you for your attention ...



... and happy 40^{th} *birthday to you, Francis !*