A Hamilton-Jacobi approach for a model of population structured by space and trait

Emeric Bouin*
Joint work with Sepideh Mirrahimi**

* École Normale Supérieure de Lyon, UMR CNRS 5669 'UMPA' and INRIA project 'NUMED', 46, allée d'Italie, F-69364 Lyon Cedex 07, France.
** CNRS, Institut de Mathématiques de Toulouse UMR 5219, 31062 Toulouse, France.

A structured population model

We consider the following non-local reaction-diffusion model :

$$\begin{cases} \partial_t n(t,x,\theta) = \underbrace{D\partial^2_{xx} n(t,x,\theta)}_{①} + \underbrace{\alpha\partial^2_{\theta\theta} n(t,x,\theta)}_{②} + \underbrace{rn(t,x,\theta)\left(a(x,\theta)-\rho(t,x)\right)}_{③}, \\ \qquad (t,x,\theta) \in (0,\infty)\times\mathbb{R}^d\times\Theta, \\ \dfrac{\partial n}{\partial \mathbf{n}} = 0 \qquad \text{on } (0,\infty)\times\mathbb{R}^d\times\partial\Theta, \\ n(0,x,\theta) = n^0(x,\theta), \qquad (x,\theta)\in\mathbb{R}^d\times\Theta. \end{cases}$$

① Spatial diffusion with constant diffusivity D.

② Mutations modeled by a Laplacian operator in the trait variable.

③ Heterogeneous reproduction in space and trait : The optimal trait for reproduction can vary with space. Competition independent of genetics : A non-locality enters into the game.

We want to study propagation using a Hamilton-Jacobi formalism.

Rescaling and Hopf-Cole transformation

We are interested in the following scaling (long time, small mutations) :

$$t \mapsto \frac{t}{\varepsilon}, \qquad D \mapsto \varepsilon^2 D.$$

We make use of the following *Hopf-Cole transformation* :

$$u_\varepsilon := -\varepsilon \ln n_\varepsilon, \qquad \text{equivalently,} \qquad n_\varepsilon = \exp\left(-\frac{u_\varepsilon}{\varepsilon}\right).$$

We want to pass to the limit $\varepsilon \to 0$ in

$$\begin{cases} \partial_t u_\varepsilon = \varepsilon D \Delta_{xx} u_\varepsilon + \frac{\alpha}{\varepsilon}\Delta_{\theta\theta} u_\varepsilon - D|\nabla_x u_\varepsilon|^2 - \frac{\alpha}{\varepsilon^2}|\nabla_\theta u_\varepsilon|^2 - r(a(x,\theta)-\rho_\varepsilon), \\ \qquad (t,x,\theta)\in(0,\infty)\times\mathbb{R}^d\times\Theta, \\ \dfrac{\partial u_\varepsilon}{\partial \mathbf{n}} = 0 \qquad \text{on } (0,\infty)\times\mathbb{R}^d\times\partial\Theta, \\ u_\varepsilon(0,x,\theta) = u^0_\varepsilon(x,\theta) \qquad (x,\theta)\in\mathbb{R}^d\times\Theta. \end{cases}$$

The convergence result

Theorem 1 (B. & Mirrahimi). *With some additional structural hypothesis,*

(i) The family $(u_\varepsilon)_\varepsilon$ converges locally uniformly towards $u : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}$, the unique viscosity solution of

$$\begin{cases} \min\left(\partial_t u + D|\nabla_x u|^2 + \mathrm{H}, u\right) = 0, \qquad \textit{in } (0,\infty)\times\mathbb{R}^d, \\ u(0,\cdot) = u_0(\cdot) \qquad \textit{in } \mathbb{R}^d. \end{cases} \tag{1}$$

(ii) Uniformly on compact subsets of Int $\{u<0\}\times\Theta$, $\quad \lim_{\varepsilon\to 0} n^\varepsilon = 0$,

(iii) For every compact subset of Int $(\{u(t,x)=0\}\cap\{\mathrm{H}(x)>0\})$, there exists $\overline{C} > 1$ such that,

$$\liminf_{\varepsilon\to 0} \rho_\varepsilon(t,x) \geq \frac{\mathrm{H}(x)}{r\overline{C}}, \qquad \textit{uniformly on } K. \tag{2}$$

Formal expansions and heuristics

Write formally

$$u_\varepsilon(t,x,\theta) = u_0(t,x,\theta) + \varepsilon u_1(t,x,\theta) + \mathcal{O}(\varepsilon^2).$$

Order -2 :

$$\boxed{|\nabla_\theta u_0(t,x,\theta)|^2 = 0.}$$

Then order 0 :

$$\alpha\left(\Delta_\theta u_1 - |\nabla_\theta u_1|^2\right) - ra(x,\theta) = \left[\partial_t u_0 + D|\nabla_x u_0|^2 - r\rho_0\right](t,x). \tag{3}$$

We recognize here a spectral problem in the trait variable θ that defines u_1 and $\mathrm{H}(x)$:

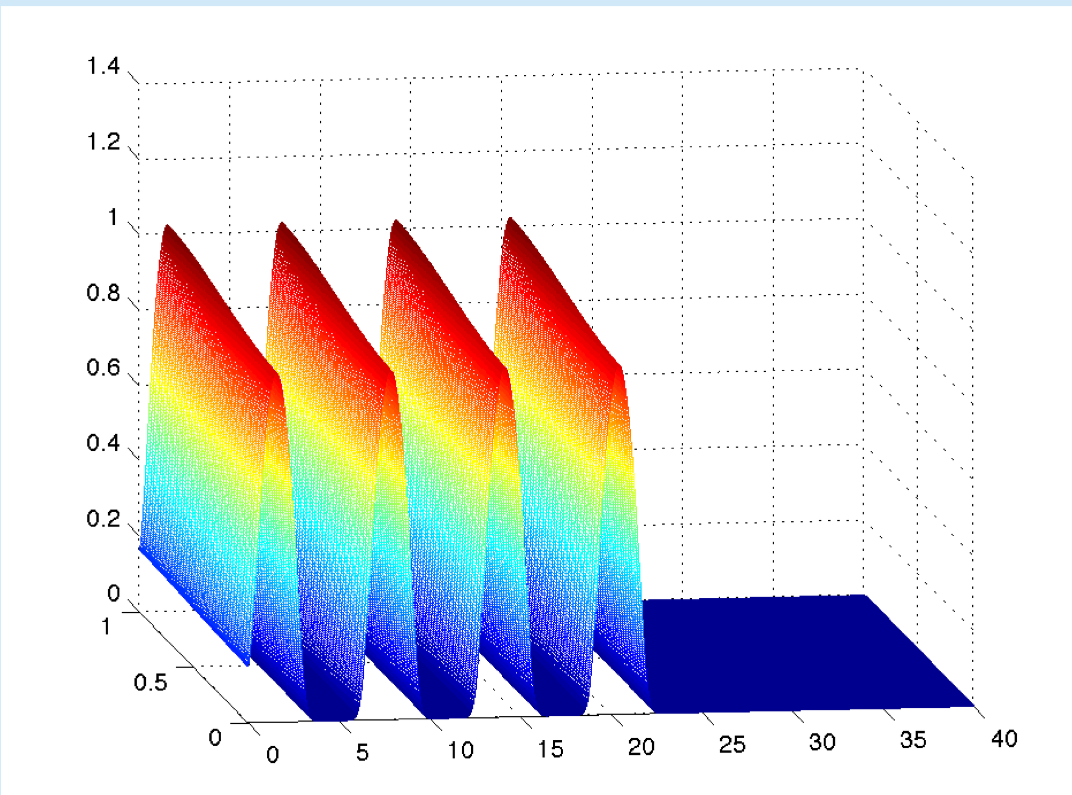
$$\boxed{\forall x\in\mathbb{R}^d, \qquad \alpha\Delta\left(e^{-u_1}\right) + ra(x,\cdot)e^{-u_1} = \mathrm{H}(x)e^{-u_1}.}$$

We now formally compute ρ^0 :

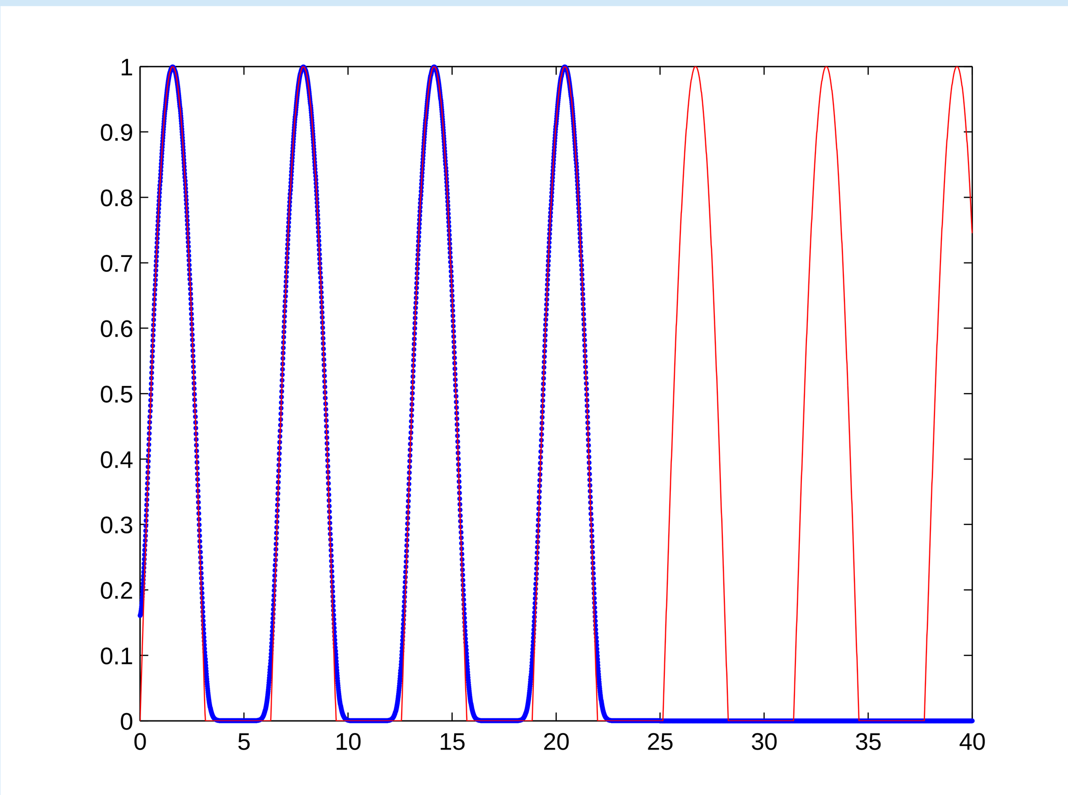
$$\begin{cases} \rho_0(t,x) = 0 \quad \Longrightarrow \quad \partial_t u_0(t,x) + D|\nabla_x u_0|^2(t,x) + \mathrm{H}(x) = 0, \\ \rho_0(t,x) > 0 \quad \Longrightarrow \quad u_0(t,x) = 0 \quad \text{and} \quad r\rho_0(t,x) = \mathrm{H}(x), \end{cases}$$

ce qui nous amène directement à la formulation variationnelle:

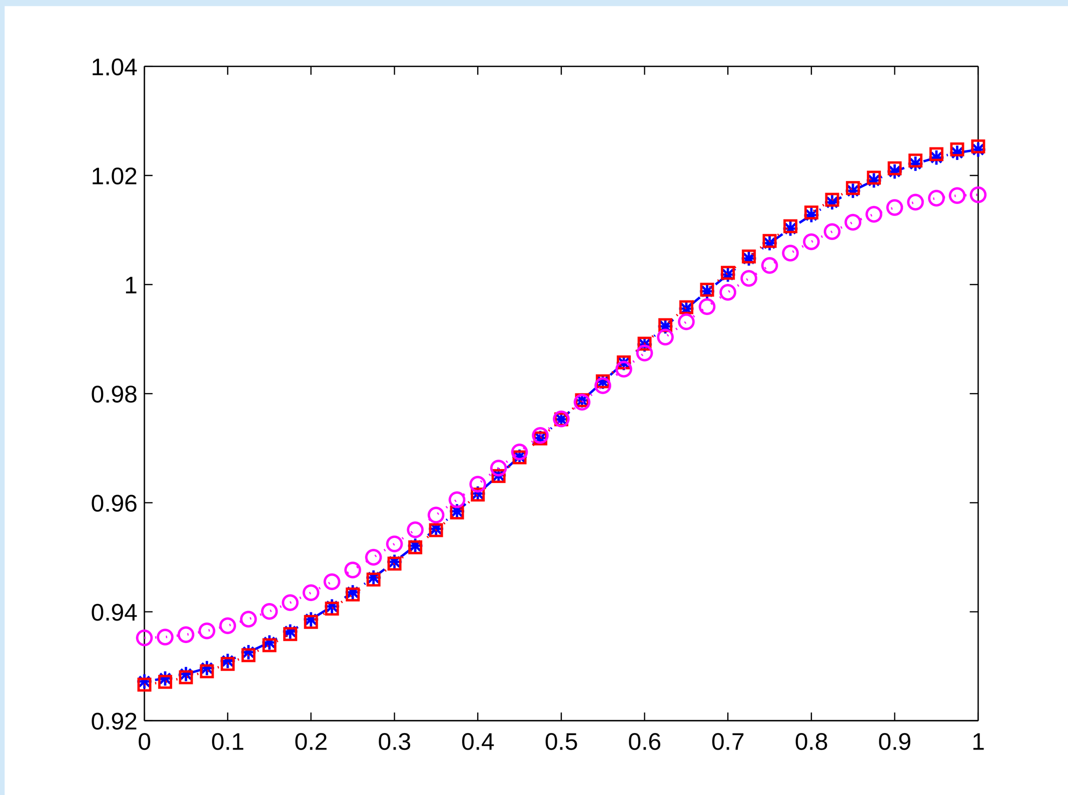
$$\boxed{\min\left(\partial_t u_0 + D|\nabla_x u_0|^2 + \mathrm{H}(x)\,,\, u_0\right) = 0.}$$



The profile



The macroscopic density



Trait distributions

Pulsating wave obtained for ε small with the fitness $a_2(x,\theta) = a_1(\theta) + \left(\sin(x) - \frac{1}{2}\right)$.

References

[1] M. Alfaro, J. Coville, G. Raoul, *Travelling waves in a nonlocal equation as a model for a population structured by a space variable and a phenotypic trait*, Communications in Partial Differential Equations, Vol 38, No 12 (2013).

[2] E. Bouin, S. Mirrahimi, *A Hamilton-Jacobi limit for a model of population structured by space and trait*, Accepted for publication in Communications in Mathematical Sciences (2014).