# A kinetic eikonal equation

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We consider the following kinetic equation with BGK relaxation operator

$$\partial_t f + v \cdot \nabla_x f = M(v)\rho - f\,, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V\,, \tag{1}$$

where $f(t,x,v)$ denotes the density of particles moving with speed $v \in V$ at time $t$ and position $x$. The function $\rho(t,x)$ denotes the macroscopic density of particules:

$$\rho(t,x) = \int_V f(t,x,v)\,dv\,, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n\,.$$

Here $V$ denotes a bounded symmetric subset of $\mathbb{R}^n$. The Maxwellian $M$ is symmetric and satisfies:

$$\int_V M(v)dv = 1\,, \qquad \int_V vM(v)dv = 0\,, \qquad \int_V v^2 M(v)dv = \theta^2\,.$$

## The hydrodynamic limit of the BGK equation

We focus on the large scale hyperbolic limit $(t,x) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$, $\varepsilon \to 0$. The kinetic equation (1) reads as follows in the new scaling:

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon}\left(M(v)\rho^\varepsilon - f^\varepsilon\right), \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V\,. \tag{2}$$

We introduce of the following Hopf-Cole transformation:

$$f^\varepsilon(t,x,v) = M(v)e^{-\frac{\varphi^\varepsilon(t,x,v)}{\varepsilon}}\,.$$

where we expect the phase $\varphi^\varepsilon$ to become independent of $v$ as $\varepsilon \to 0$.

## The Hamilton-Jacobi equation

**Theorem 1.** *Let $V \subset \mathbb{R}^n$ be* ***bounded*** *and symmetric, and $M \in L^1(V)$ be nonnegative and symmetric. Then $\varphi^\varepsilon$ converges (locally) uniformly towards $\varphi^0$, where $\varphi^0$ does not depend on $v$. Moreover $\varphi^0$* ***is the viscosity solution of the following Hamilton-Jacobi equation****:*

$$\int_V \frac{M(v)}{1 - \partial_t \varphi^0(t,x) - v \cdot \nabla_x \varphi^0(t,x)}\,dv = 1\,, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n\,. \tag{3}$$

*The denominator of the integrand is positive for all $v \in V$.*

The last observation in Theorem 1 is not compatible with an unbounded velocity set.

### Explicit computations of the effective Hamiltonian.

Using the implicit function theorem, equation (3) rewrites as $\partial_t \varphi^0 + H\left(\nabla_x \varphi^0\right) = 0$, where $H$ is *the effective Hamiltonian*. The effective Hamiltonian is convex.

- We can compute the effective Hamiltonian $H$ in one dimension for a constant Maxwellian $M \equiv \frac{1}{2}$ on $V = (-1,1)$. We obtain $H(p) = \frac{p - \tanh(p)}{\tanh(p)}$.
- When $M(v) = \frac{1}{2}(\delta_1 + \delta_{-1})$, though it is not a $L^1$ function, we are able to obtain the relativistic hamiltonian $H(p) = \frac{\sqrt{1+4p^2}-1}{2}$.

### Kinetic Eikonal *vs* Classical Eikonal

We obtain a Hamilton-Jacobi equation which differs from the classical eikonal equation, see the Figure below. This is unexpected since the formal limit of equation (2) at order $O(\varepsilon)$ is the heat equation with small diffusivity:

$$\partial_t \rho^\varepsilon = \varepsilon\theta^2 \Delta_x \rho^\varepsilon\,, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n\,.$$

It is well-known that the phase $\phi^\varepsilon = -\varepsilon \log \rho^\varepsilon$ satisfies in the limit $\varepsilon \to 0$ the classical eikonal equation in the sense of viscosity solutions [4, 3]:

$$\partial_t \phi^0 + \theta^2 |\nabla_x \phi^0|^2 = 0\,. \tag{4}$$

We only have asymptotic equivalence between the two approaches for small $|p|$ by Taylor expansion: $H(p) \sim \theta^2 |p|^2$.

### Elements of the proof

**Step 1. Existence and uniform bounds.**

**Proposition 1.** *Let $V \subset \mathbb{R}^n$ be a bounded subset. Assume $M \in L^1(V)$ and $\varphi_0 \in W^{1,\infty}(\mathbb{R}^n)$. The kinetic equation has a unique solution $\varphi^\varepsilon \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^n \times V)$ uniformly in $\varepsilon$ (locally in time).*

**Step 2. Viscosity solution procedure.**

1. **Locally uniform convergence.** We can extract from the familly $(\varphi^\varepsilon)_\varepsilon$ a locally uniformly converging subsequence. We denote by $\varphi^0$ the limit. We prove that $\varphi^0$ does not depend on $v$.
2. **Viscosity solution and correcting term.** Let $\psi^0 \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{R}^n)$ be a test function such that $\varphi^0 - \psi^0$ has a local maximum at $(t^0, x^0)$. We define a corrective term $\eta$ not depending on $\varepsilon$, as in the Evans perturbed test function method [2]: $\psi^\varepsilon = \psi^0 + \varepsilon\eta$.
3. **Maximum principle.** We prove the subsolution property thanks to an adapted maximum principle.

## References


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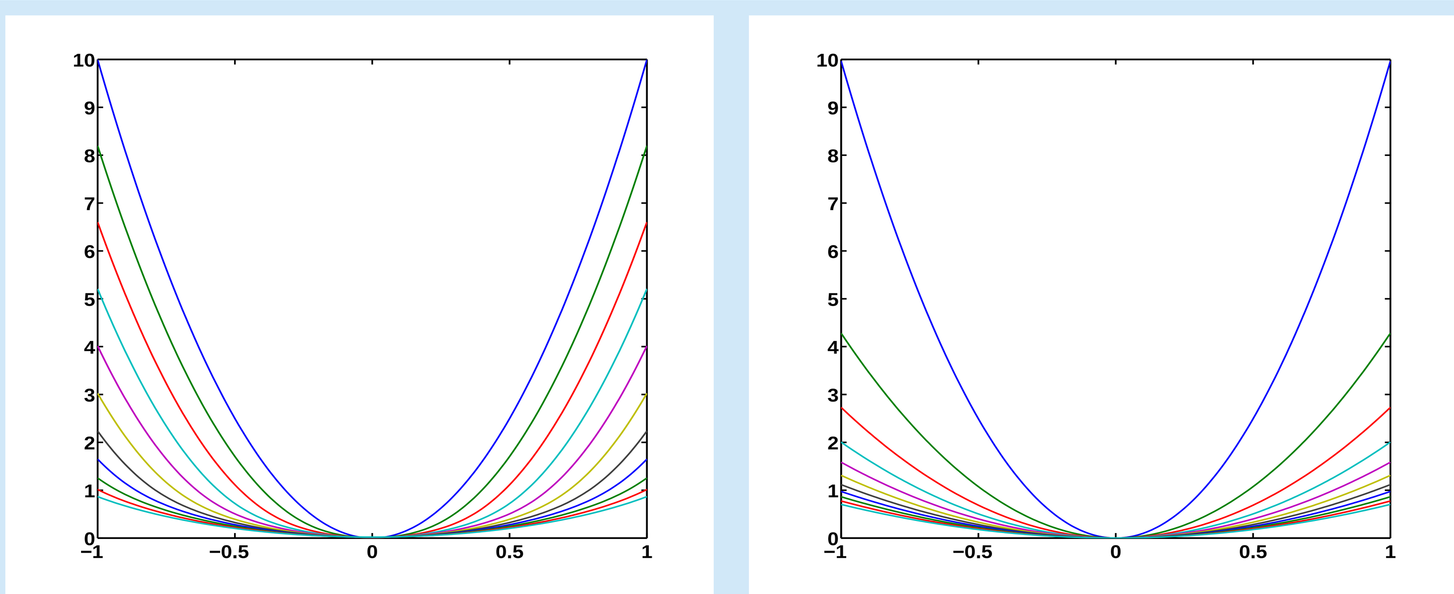


Kinetic eikonal     Classical eikonal

Comparison between the kinetic eikonal equation (3) and the classical eikonal equation (4), with a constant Maxwellian on $V = (-1,1)$.