A kinetic Fisher-KPP equation : traveling waves and front acceleration

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A kinetic model for the Fisher-KPP equation

We consider the following kinetic model

$$\varepsilon^2 \partial_t f + \varepsilon v \cdot \nabla_x f = \underbrace{(M(v)\rho - f)}_{\text{Scattering}} + \varepsilon^2 \underbrace{r\rho\,(M(v) - f)}_{\text{Monostable growth}}, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V\,, \quad (1)$$

where $f(t,x,v)$ denotes the density of particles moving with speed $v \in V$ at time t and position x. The function $\rho(t,x)$ denotes the macroscopic density of particules:

$$\rho(t,x) = \int_V f(t,x,v)\,dv\,, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n\,.$$

Here V denotes a symmetric subset of $\mathbb{R}^n$. The Maxwellian M is symmetric and satisfies:

$$\int_V M(v)dv = 1\,, \qquad \int_V vM(v)dv = 0\,, \qquad \int_V v^2 M(v)dv = D\,.$$

The formal macroscopic limit $\varepsilon \to 0$ is the classical Fisher-KPP equation:

$$\partial_t \rho - D\partial_{xx}\rho = r\rho\,(1-\rho)\,. \quad (2)$$

Traveling waves when V is bounded, w.l.o.g $V = [-1;1]$.

Definition 1. *We say that a function $f(t,x,v)$ is a traveling front solution of speed $c \in \mathbb{R}^+$ of equation (1) if it can be written $f(t,x,v) = \mu\,(\xi = x - ct, v)$, where the profile $\mu \in \mathcal{C}^2\,(\mathbb{R} \times V)$ is nonnegative, satisfies $\mu\,(-\infty, \cdot) = M$, $\mu\,(+\infty, \cdot) = 0$, and μ solves*

$$\varepsilon(v - c\varepsilon)\partial_\xi \mu = (M(v)\nu - \mu) + r\varepsilon^2 \nu\,(M(v) - \mu)\,, \qquad \xi \in \mathbb{R},\ v \in V. \quad (3)$$

where ν is the macroscopic density associated to μ, that is $\nu\,(\xi) = \int_V \mu\,(\xi, v)\,dv$.

Theorem 1. *Assume that $\varepsilon > 0$ and that $\mathrm{Supp}(M) = [-1,1]$. There exists a **minimal** speed $c^*(\varepsilon) \in (0, \frac{1}{\varepsilon})$ such that **there exists a traveling wave solution of** (1) **of speed** c **for** $c \in \left[c^*(\varepsilon), \frac{1}{\varepsilon}\right]$. Moreover, this traveling wave is **nonincreasing** with respect to x.*

This is **not** a perturbative result from the Fisher-KPP equation.

Finding the speed : Dispersion relation

As for the Fisher-KPP equation, the rate of exponential decay in space and the speed are given by the linearized problem at the edge of the front. Yields the following ***spectral problem*** : For all λ, find $c(\lambda)$ such that there exists a Maxwellian Q_λ such that

$$\forall v \in V, \quad (1 + \varepsilon\lambda\,(c(\lambda)\varepsilon - v))\,Q_\lambda(v) = (1 + r\varepsilon^2)\int_V M(v)Q_\lambda(v)dv. \quad (4)$$

Proposition 1. *For all $\varepsilon > 0$, the minimal speed $c^*(\varepsilon)$ is given by $c^*(\varepsilon) = \min_{\lambda>0} c(\lambda)$ where $c(\lambda)$ is for all λ a solution of the following **dispersion relation**:*

$$(1 + r\varepsilon^2)\int_V \frac{M(v)}{1 + \varepsilon\lambda(c(\lambda)\varepsilon - v)}\,dv = 1\,. \quad (5)$$

This relation is **not** compatible with and unbounded velocity set.

We can provide estimates for the critical speed $c^*(\varepsilon)$. In particular,

Proposition 2. *Assume that Supp$(M) = [-1,1]$, then $c^*(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 2\sqrt{rD} := c_{KPP}$.*

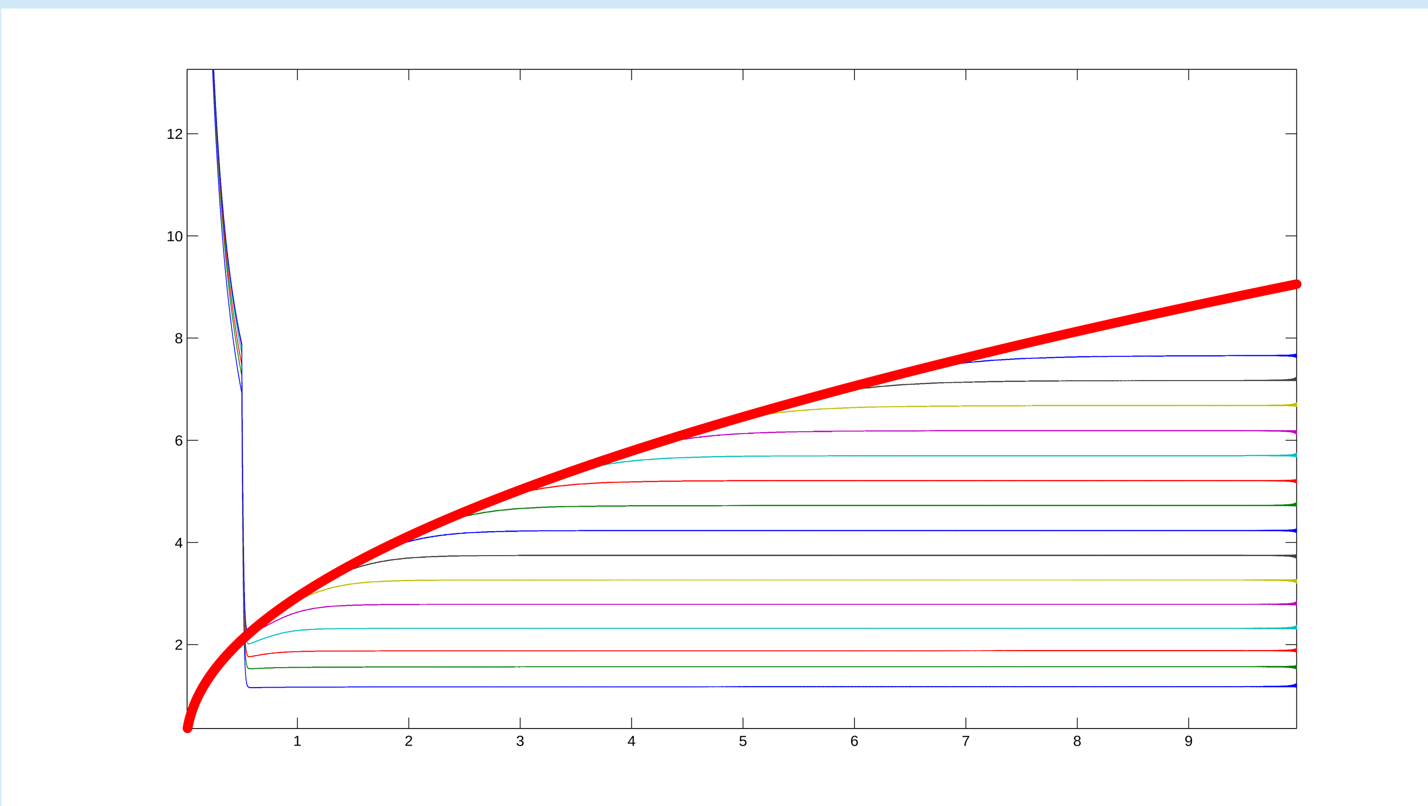
The fronts are **stable** in suitable Lebesgue spaces.

Unbounded V : Front acceleration (w.l.o.g. $\varepsilon = 1$).

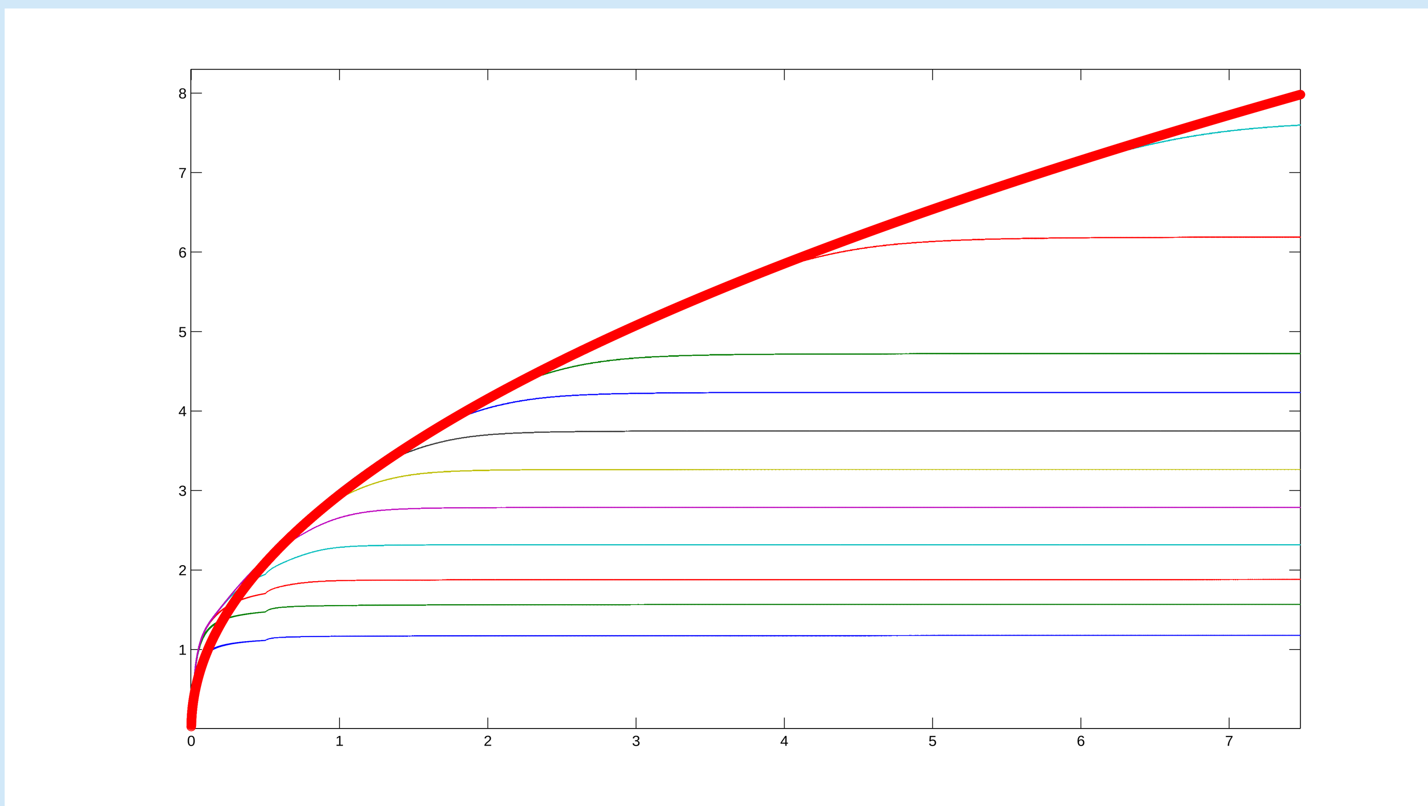
In the Figures below, we plot the evolution of the value of the speed of the front, for different values of V_{max}. We deduce that c^* increases with V_{max}.

Moreover, we observe that the envelop of the curves (which represents the instantaneous speed of the front with an unbounded V) behaves like $t^{\frac{1}{2}}$.

It yields that **the density ρ propagates likes $x \propto t^{\frac{3}{2}}$** .



"Bell" initial condition.



Heaviside initial condition.

Evolution of the speed of the front for different values of the maximal speed. The Maxwellian here is a Gaussian : $M(v) = C\,(V_{max})\exp\left(-\frac{v^2}{2}\right)$.

References

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