# A kinetic Fisher-KPP equation : traveling waves and front acceleration.

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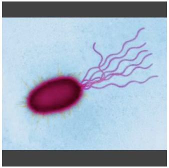
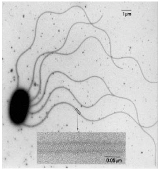
UMPA - ENS de Lyon

La Londe les Maures - 2012

# How do bacteria move ...

The bacteria E. Coli moves thanks to flagella :



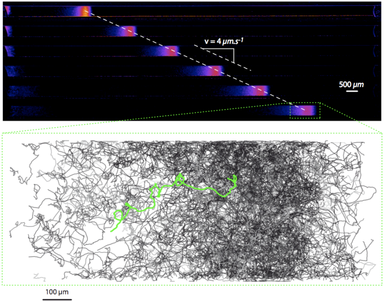


From Howard Berg's lab

and with a so-called *run and tumble* process :
straight swimming for $1s$
and
change of direction for $0.1s$.

# Collective migration

Bacterial traveling pulses :




**The kinetic point of view is the most relevant** for this situation.

Saragosti J, Calvez V, Bournaveas N, Buguin A, Silberzan P, et al. (2010) Mathematical Description of Bacterial Traveling Pulses. PLoS Comput Biol 6(8)

# Kinetic equations

- Probability density of bacteria $f(t,x,v)$ at time $t$, position $x$ and speed $v$. Total density $\rho := \int_V f(v)dv$.
- The velocity set $V$ : symmetric, bounded or unbounded.

The model ( e.g. Schmeiser and al, 2010 ) :

$$\underbrace{\varepsilon\partial_t f + v\partial_x f}_{\text{Free run}} = \underbrace{\frac{1}{\varepsilon}\left(M(v)\rho - f\right)}_{\textit{Tumbling}} + \underbrace{\varepsilon r\rho\left(M(v) - f\right)}_{\text{Growth with saturation}} \qquad (1)$$

where the Maxwellian $M$ on the space $V$ satisfies

$$\int_V M(v)dv = 1, \qquad \int_V vM(v)dv = 0, \qquad \int_V v^2 M(v)dv = D. \qquad (2)$$

# Aim of the talk : traveling waves

**Formally**, in the limit $\varepsilon \to 0$, $\rho$ satisfies the classical Fisher-KPP equation (1937) :

$$\partial_t \rho - D\partial_{xx}\rho = r\rho(1-\rho). \tag{3}$$

It is now well known that (3) admits traveling waves solutions with minimal speed $c_{KPP} = 2\sqrt{rD}$.

**Aim of the talk** : Discuss **existence** and **stability** of traveling fronts for (2), for all values of $\varepsilon$ ( i.e. in the **full kinetic** regime ).

# Plan.

1. Introduction.
2. Traveling waves when $V$ is bounded
3. Finally, what happens when $V$ is unbounded ?
4. Conclusion.

Without loss of generality, we assume that $V = [-1;1]$.

### Definition

We say that a function $f(t,x,v)$ is a traveling front solution of speed $c \in \mathbb{R}^+$ of equation (1) if it can be written $f(t,x,v) = \mu(\xi = x - ct, v)$, where *the profile* $\mu \in \mathcal{C}^2(\mathbb{R} \times V)$ is nonnegative, satisfies $\mu(-\infty, \cdot) = M$, $\mu(+\infty, \cdot) = 0$, and $\mu$ solves

$$\varepsilon(v - c\varepsilon)\partial_\xi \mu = (M(v)\nu - \mu) + r\varepsilon^2 \nu (M(v) - \mu), \qquad \xi \in \mathbb{R}, \ v \in V. \tag{4}$$

where $\nu$ is the macroscopic density associated to $\mu$, that is $\nu(\xi) = \int_V \mu(\xi, v)\, dv$.

# Existence result

**Theorem**

*Assume that $\varepsilon > 0$ and that $\mathrm{Supp}(M) = [-1, 1]$. There exists a **minimal** speed $c^*(\varepsilon) \in (0, \frac{1}{\varepsilon})$ such that **there exists a traveling wave solution of** $(1)$ **of speed** $c$ **for** $c \in \left[c^*(\varepsilon), \frac{1}{\varepsilon}\right]$. Moreover, this traveling wave is **nonincreasing** with respect to $x$.*

This is ***not*** a perturbative result from the Fisher-KPP equation.

# Finding the speed : Dispersion relation

- As for the Fisher-KPP equation, the rate of exponential decay in space and the speed are given by the linearized problem at the edge of the front. In this case, we have $D\lambda^2 - c\lambda + r = 0$.

- Similarly, looking for a solution of the linearized problem of type $e^{-\lambda\xi}Q(v)$ yields the following ***spectral problem*** : For all $\lambda$, find $c(\lambda)$ such that there exists a Maxwellian $Q_\lambda$ such that

$$\forall v \in V, \quad \left(1 + \varepsilon\lambda\left(c(\lambda)\varepsilon - v\right)\right) Q_\lambda(v) = (1 + r\varepsilon^2) \int_V M(v) Q_\lambda(v) dv. \quad (5)$$

### Proposition

For all $\varepsilon > 0$, the minimal speed $c^*(\varepsilon)$ is given by $c^*(\varepsilon) = \min_{\lambda > 0} c(\lambda)$ where $c(\lambda)$ is for all $\lambda$ a solution of the following **dispersion relation** :

$$(1 + r\varepsilon^2) \int_V \frac{M(v)}{1 + \varepsilon\lambda(c(\lambda)\varepsilon - v)}\, dv = 1\,. \tag{6}$$

This relation is ***not*** compatible with and unbounded velocity set.

# Remarks on the results

1. The existence result is proved using a sub- and super-solutions technique (see e.g. Berestycki and Hamel).
2. We can provide estimates for the critical speed $c^*(\varepsilon)$. In particular,

**Proposition**

Assume that $\mathrm{Supp}(M) = [-1, 1]$, then $c^*(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 2\sqrt{rD} := c_{\mathrm{KPP}}$.

# Numerical simulations

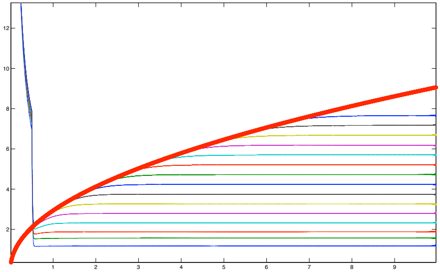



Figure: Evolution of the speed of the front for different values of the maximal speed. The Maxwellian here is a Gaussian : $M(v) = C\left(V_{max}\right)\exp\left(-\frac{v^2}{2}\right)$. "Bell" initial condition.

# Concluding remarks.

- When $V$ is bounded we construct traveling wave profiles. The speed is obtained through a dispersion relation.
- On the contrary, when $V$ is unbounded, accelaration pattern is observed.

# Thank you for your attention !

# References


- E. Bouin, V. Calvez, G. Nadin, *A kinetic KPP equation : traveling waves and front acceleration*, in preparation,
- C. Cuesta, S. Hittmeir, C. Schmeiser, *Traveling waves of a kinetic transport model for the KPP-Fisher equation*, preprint, (2010).
- D.G. Aronson and H.F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. Math. 30, pp. 33-76, (1978).