École Normale Supérieure de Lyon **Année 2013–2014**

Département de Mathématiques

Option B pour l'agrégation

TD/TP 10 : Equations aux dérivées partielles: Equation de la chaleur

Un peu de théorie pour commencer...

Exercice 1 (Equation d'Airy). What is for you the difference between the heat equation and the following Airy equation

$$\partial_t u - \partial_{xxx} u = 0.$$

Exercice 2. We consider the following

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x) & \text{in } (0,1) \times (0,T), \\ u(x,0) = u_0(x), & \text{on } (0,1), \end{cases}$$

with $f \in L^2(\Omega)$. Study the convergence of the solution when $t \to \infty$.

Exercice 3 (Turing patterns). In this exercice, we will study what may explain some patterns on animals, particularly on fish.

Figure 1: Examples of animals with spots and stripes. Left: Discus (http://animal-world.com/encyclo/fresh/cichlid/discus.php). Middle: Sergeant Major (http://www.lagons-plages.com/poissons-de-lagons.php) Right: Ocelot (http://notrenature.cowblog.fr/426-ocelot-3055620.html).

1. After what we discussed during the lectures, why is diffusion a stabilizing process?
2. We consider the following ODE system:

$$\begin{cases} \frac{du}{dt} = au + bv, \\ \frac{dv}{dt} = cu + dv, \end{cases}$$

where a, b, c, d are real constant coefficients. What are the standard assumptions to ensure that $(0, 0)$ is a stable attractive equilibrium point?

We now consider the reaction diffusion linear system on Ω a bounded subset of $\mathbb{R}^d$,

$$\begin{cases} \frac{\partial u}{\partial t} - \sigma_u \Delta u = au + bv, \\ \frac{\partial v}{\partial t} - \sigma_v \Delta v = cu + dv, \end{cases} \tag{1}$$

with either Dirichlet or Neumann boundary conditions and where the diffusivities are positive real constants. This should help for stability. Surprisingly, we have the following:

Theorem 1. *Consider the system* (1) *where we fix the domain* Ω*, the matrix* A *and* $\sigma_v > 0$*. We assume that the stability condition of the ODE system is fulfilled with* $a > 0, d < 0$*. Then, for* σ_u *small enough, the steady state* $(u, v) = (0, 0)$ *is linearly unstable. Moreover, only a finite number of eigenmodes are unstable.*

1. Project the reaction-diffusion system (1) on a suitable basis of eigenmodes.
2. For each eigenmode, look for a solution with exponential growth in time.
3. Prove that the linear system so-obtained can have an instability when σ_u is sufficiently small.

The usual interpretation of this result is as follows. Because $a > 0$ and $d < 0$, the quantity u is called an activator and v an inhibitor. We have

Turing instability: ***short*** *range activator,* ***long*** *range inhibitor,*

to be compared with

Traveling waves: ***long*** *range activator,* ***short*** *range inhibitor.*

Exercice 4. Prove that the backward heat equation Cauchy problem is ill-posed. For this purpose, provide a solution that explodes for ainsi given time t_0. The initial condition can be chosen.

Exercice 5 (Blow-up for heat equations). Let ϕ be a positive smooth function on $[0, \pi]$, such that $\phi(0) = \phi(\pi) = 0$. Let $\varepsilon > 0$. We assume that there exists a solution $u(t, x) \in \mathcal{C}^\infty([0, \pi[\times [0, T(\phi)[)$ of the following nonlinear parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = |u|^\varepsilon u, \qquad (t, x) \in [0, \pi] \times [0, T(\phi)[, \\ u(t, 0) = u(t, \pi) = 0, \qquad t \in [0, T(\phi)[, \\ u(0, x) = \phi(x), \qquad x \in [0, \pi]. \end{cases} \tag{2}$$

1. Prove that $u(t, x) \geq 0$ for all $(t, x) \in [0, \pi] \times [0, T(\phi)[$.
2. Find a nonnegative smooth function ψ on $[0, \pi]$ and $\lambda \in \mathbb{R}^{+*}$, satisfying

$$\begin{cases} \psi''(x) + \lambda \psi(x) = 0, \qquad x \in [0, \pi], \\ \psi(0) = \psi(\pi) = 0, \qquad \int_0^\pi \psi(x) dx = 1. \end{cases}$$

3. We assume that
$$\forall x \in \mathbb{R}^+, \qquad \left(\int_0^\pi \phi(x)\psi(x)dx\right)^\varepsilon > \lambda.$$
Prove that $f(t) := \displaystyle\int_0^\pi u(t,x)\psi(x)dx$ satisfies
$$\forall t \in [0, T(\phi)[, \qquad f'(t) \geq f(t)\left(f(t)^\varepsilon - \lambda\right),$$
and conclude that $T(\phi) < +\infty$.

Exercice 6 (Gradient blow-up). Let ϕ be a positive smooth function on $\bar{\Omega} \subset \mathbb{R}^n$, such that $\phi(x) = 0$ on $\partial\Omega$. We assume that there exists a solution $u(t,x) \in \mathcal{C}^\infty(\Omega \times [0, T(\phi)[)$ of the following nonlinear parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = |\nabla u|^{2+\varepsilon}, & (x,t) \in \Omega \times [0, T(\phi)[, \\ u(t,x) = 0, & (x,t) \in \partial\Omega \times [0, T(\phi)[, \\ u(0,x) = \phi(x), & x \in \Omega. \end{cases} \tag{3}$$

1. Prove that the solution remains positive and bounded for all positive times.
2. Prove the following gradient estimate:
$$\forall\, T < T(\phi), \qquad \sup_{[0,T]} \|\nabla u\|_\infty = \sup_{\mathcal{P}_T} |\nabla u|,$$
where $\mathcal{P}_T$ denotes the parabolic boundary.

Let $1 \leq q < +\infty$. We want to show that there exists $C := C(\varepsilon, q, \Omega)$, such that if $\|u_0\|_{L^q(\Omega)} \geq C$, then $T(\phi) < +\infty$.

1. We define $q_0 := \frac{2(1+\varepsilon)}{\varepsilon}$. Why is it sufficient to prove the assertion for $q_0 \leq q < +\infty$?
2. Let thus define $k = q - 1 \left(\geq \frac{2+\varepsilon}{\varepsilon}\right)$. Compute $\frac{1}{k+1}\frac{d}{dt}\left(\int_\Omega u^{k+1}dx\right)$.
3. Prove that
$$\int_\Omega |\nabla u|^{2+\varepsilon} u^k dx \geq C_1 \int_\Omega u^{k+2+\varepsilon} dx, \qquad \int_\Omega |\nabla u|^2 u^{k-1} dx \leq C_2 \left(\int_\Omega |\nabla u|^{2+\varepsilon} u^k dx\right)^{\frac{k+1}{k+2+\varepsilon}}.$$
4. Deduce that
$$\frac{d}{dt}\left(\int_\Omega u^q dx\right) \geq C_3 \left(\int_\Omega u^q dx\right)^{\frac{q+1+\varepsilon}{q}} - C_4$$
and conclude.

Simulations numériques

Exercice 7 (Basic heat equation). Prepare a script that solves the classical heat equation:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 & \text{in } (0,1) \times (0,T), \\ u(x,0) = u_0(x), & \text{on } (0,1), \\ u(0,t) = u(1,t) = 0 & \text{on } t > 0. \end{cases}$$

The script will contain the *explicit* Euler, *implicit* Euler, and Crank-Nicolson. Please give the stability conditions (when expected) and the precisions of the schemes.

Exercice 8 (Polarization equation). Let $M > 0$ be the mass of markers. The one-dimensional polarisation equation writes:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = u(t,0)\partial_x u, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ \partial_x u(t,0) = 0, \qquad t > 0, \\ u(0,x) = C_M \exp\left(-x^2\right), \end{cases}$$

where C_M is a renormalization constant such that $\int_0^\infty u^0(x)dx = M$. Solve numerically the equation with a Neumann boundary condition in $x = 0$. Play with the value of the mass M, what do you observe?

Exercice 9 (Evolution of temperature in the ground). Propose a model that describes the evolution of temperature in the ground given that the temperature of the atmosphere changes during the year. Make a numerical simulation.

Exercice 10 (Fisher-KPP equation). Prepare a script that solves the following Fisher-KPP equation:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = u(1-u) & \text{in } (0,1) \times (0,T), \\ u(x,0) = \mathbf{1}_{x<0} \end{cases}$$

What qualitative behavior do you observe? Quantify the speed of propagation.

Exercice 11 (Turing patterns). Prepare a script that solves system (1) (in one space dimension, or two, if you are awesome!).

Exercice 12 (Blow-up). Prepare a script that solves system (2).