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MIDO - L3

Differential equations

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Contents

Chapter 1

Existence and uniqueness of solutions

1.1 A simple example and remarks on notation.

A differential equation is an equation where the unknown is a function instead of a number. One of the simplest differential equations is

$$x'(t) = ax(t) \tag{1.1}$$

where $a, x'(t), x(t)$ are real numbers and $x'(\cdot)$ denotes the derivative with respect to t of the function $x : J \to \mathbb{R}$, $J \subset \mathbb{R}$ an open interval. This is indeed a functional equation. That is, a solution of (1.1) is not a number but a function. More precisely, a solution of (1.1) is described by an open time interval $J \subset \mathbb{R}$ and a differentiable function $x : J \to \mathbb{R}$ such that (1.1) holds for all times t in J. The solutions of (1.1) can be computed explicitly: these are the functions $x : \mathbb{R} \to \mathbb{R}$ of the form

$$x(t) = \lambda e^{at} \tag{1.2}$$

where $\lambda \in \mathbb{R}$, as well as the restrictions of these functions to any open time interval $J \subset \mathbb{R}$.[1]

We immediately see that the choice of J, the time interval of definition, is not unique. If, as we will soon do, we restrict our attention to solutions defined on a maximal time interval (i.e. the largest time interval of definition, hence in this case the whole set $\mathbb{R}$), for any initial time t_0 and initial position x_0, there exists a unique solution such that $x(t_0) = x_0$. Indeed, imposing the initial condition $x(t_0) = x_0$, we uniquely determine the value of λ, since $\lambda e^{at_0} = x_0$ gives $\lambda = x_0 e^{-at_0}$. In this case we say that the initial value problem $x'(t) = ax(t)$ and $x(t_0) = x_0$ is *well posed*. One of the main questions of this chapter is to determine conditions ensuring that initial value problems are well posed for more general differential equations.

Before doing so, let us stress that notation does not matter. Depending on application fields and authors, the derivative of the function $x(\cdot)$ may be denoted by $\dot{x}$ or $\frac{dx}{dt}$, so that (1.1) will be written: $\dot{x}(t) = ax(t)$ or $\frac{dx}{dt}(t) = ax(t)$. For conciseness, we could also skip the indication of time and write (1.1) in functional form: $x' = ax$ (or $\dot{x} = ax$, or $\frac{dx}{dt} = ax$). Though written in a different way, this is the same equation, and the solutions are still given by (1.2). Similarly, in books for first or second year students, the function is often denoted by y and the variable by x, so that (1.1) becomes

$$y'(x) = ay(x).$$

and the solutions $y(x) = \lambda e^{ax}$. Again, this is just a matter of notation: solutions are still functions whose derivative is equal to a times the function. We prefer the notation (1.1) because

[1] Indeed, if $(J, x(\cdot))$ is a solution, then for all t in J, $x'(t) - ax(t) = 0$, hence $e^{-at}(x'(t) - ax(t)) = 0$, $\frac{d}{dt}[e^{-at}x(t)] = 0$. Thus, there exists a constant λ such that for all t in J, $e^{-at}x(t) = \lambda$, that is, $x(t) = \lambda e^{at}$. Conversely, any function of this form is a solution.

we want to think of $x(t)$ as describing the position at time t of a particle (body) moving through space according to the law of motion (1.1). Here, $x(t)$ is a real number, so our particle is moving on the real line, but later, we will consider particles moving in $\mathbb{R}^2$, $\mathbb{R}^3$, or even $\mathbb{R}^d$, where d is a positive integer. We will then use the notation $X(t)$ for the particle's position. That is, our convention is that $x(t) \in \mathbb{R}$ and $X(t) \in \mathbb{R}^d$. Similarly, except if indicated otherwise, a function denoted by f takes its values in $\mathbb{R}$ while F takes its values in $\mathbb{R}^d$.[2]

We focus here on the mathematical theory rather than on applications, however differential equations arise in many different fields: physics, chemistry, ecology, demography, epidemiology, but also economics and finance. Equation (1.1), for instance, may be seen as a basic population growth model: the Malthusian model.[3] The same equation may be seen as an interest rate model: at a yearly interest rate r such that $e^a = 1 + r$, a deposit of 100 euros will yield a sum of $100e^{at}$ euros t years later. This is an instance of the unifying power of mathematics: a single equation for several applications.

1.2 First order differential equations

Though we will occasionally consider more general equations, this course focuses on explicit first-order differential equations, that is, equations of the form

$$X'(t) = F(t, X(t)) \tag{1.3}$$

with $X(t) \in \mathbb{R}^d$ and $F : \Omega \to \mathbb{R}^d$, where Ω is an open subset of $\mathbb{R} \times \mathbb{R}^d$, d a positive integer. We see $\mathbb{R}$ as a space of times and $\mathbb{R}^d$ as a space of possible positions of a particle moving through $\mathbb{R}^d$. Thus $X(t)$ is the position at time t of our imaginary particle. Eq. (1.3) is of *first-order* because second or higher derivatives of $X(\cdot)$ do not appear. It is *explicit* because it is of the form (1.3) as opposed to the *implicit* form $G(t, X(t), X'(t)) = 0$. The equation is *autonomous* if F does not depend on t, and *nonautonomous* otherwise (see Chapter 2).

Example 1.2.1. *$x'(t) = x(t)$ is autonomous; $x'(t) = tx(t)$ is nonautonomous.*

Definition 1.2.2. *A solution of (1.3) is described by a nonempty open time-interval J and a differentiable function $X : J \to \mathbb{R}^d$ such that for all t in J, $(t, X(t)) \in \Omega$ and $X'(t) = F(t, X(t))$.*[4]

So strictly speaking, a solution is a couple (interval, function defined on this interval), but we will often abuse vocabulary and refer to the function as a solution. Though we only require solutions of (1.3) to be differentiable, i.e. their derivative is defined for all $t \in J$ (that's why J has to be open!), they are typically much more regular:

Proposition 1.2.3. *If F is $C^k(\Omega)$, then all solutions of (1.3) are at least C^{k+1} on their domain of definition.*

[2] What we want to study are systems whose state at time t may be described by a finite number of parameters $x_1(t)$,..., $x_d(t)$, or equivalently by a vector $X(t) = (x_1(t), ..., x_d(t))$, evolving according to some differential equation. Such problems arise naturally in many different fields. In physics, a possible system would be the position and velocity of a planet (here $d = 2$); in chemistry, the concentrations of some chemical substances; in ecology, the population densities of d species constituting an ecosystem; in epidemiology, the population densities of three subgroups: those affected by a disease, those who never got the disease, and those who recovered from the disease and can no longer get it (here $d = 3$); in finance, the values of various assets; in macroeconomics, a number of macroeconomic variables such as the unemployment rate, the accumulated capital, a measure of the population know-how, etc., which together define the state of a simplified economy. Thus our imaginary particle with position $X(t)$ at time t actually represents the state of some system, and the space in which it evolves is the set of all possible states for this system. But we will usually not specify what the system is, and just think of a particle moving through space.

[3] (1.2) may be seen as the continuous time version of a discrete time model assuming that at each generation, the population is multiplied by a constant $k > 0$. Taking a time-scale such that generation time is $T = 1$ and letting $a = \ln k$ and λ be the the initial population size, then leads to (1.1) with $x(0) = \lambda$. Note that if $k > 1$, or equivalently $a > 0$, then the population grows without bounds, which was frightening Malthus.

[4] The condition $(t, X(t)) \in \Omega$ ensures that $F(t, X(t))$ is well defined. This condition is void when $\Omega = \mathbb{R} \times \mathbb{R}^d$ and boils down to $t \in I$ when $\Omega = I \times \mathbb{R}^d$ for some time interval $I \subset \mathbb{R}$.

Proof. By contradiction, assume that F is $C^k(\Omega)$ but $X : J \to \mathbb{R}^d$ is not $C^{k+1}(J)$. Note that $X(\cdot)$ is differentiable since $X'(\cdot)$ is defined, hence at least $C^0(J)$. Let $n \leq k$ be the highest integer such that $X(\cdot)$ is C^n. Since $n \leq k$, F is also $C^n(J)$, indeed F is $C^k(\Omega)$, hence $t \to F(t, X(t))$ is $C^n(J)$ as a composite function. Thus, $X'(\cdot)$ is $C^n(J)$ and $X(\cdot)$ is $C^{n+1}(J)$, contradicting the definition of n. □

Exercise 1.2.4. *Prove Proposition 1.2.3 by induction.*

Letting $(t_0, X_0) \in \Omega$, a solution of the *initial value problem*, also called *Cauchy problem*,

$$\left\{ \begin{array}{rcl} X'(t) & = & F(t, X(t)) \\ X(t_0) & = & X_0 \end{array} \right. \tag{1.4}$$

is a solution $(J, X(\cdot))$ of (1.3) such that $t_0 \in J$ and $X(t_0) = X_0$.

For instance, a solution of

$$\left\{ \begin{array}{rcl} x'(t) & = & x(t) \\ x(0) & = & 1 \end{array} \right. \tag{1.5}$$

is the exponential function, that is, the function $x : \mathbb{R} \to \mathbb{R}$ such that for all t in $\mathbb{R}$, $x(t) = e^t$. There are other solutions, indeed, an infinite number of them. This is because for any open interval J containing 0, the restriction of the exponential function to J is still a solution. Clearly, this multiplicity of solutions is spurious, since they are all defined by the same formula. To get rid of it, we introduce the concept of maximal solution.

Definition 1.2.5. *A solution $(J, X(\cdot))$ of (1.3) is* maximal *if it cannot be extended to a larger time-interval. That is, if there isn't any solution $(\tilde{J}, \tilde{X}(\cdot))$ such that J is strictly contained in $\tilde{J}$ and $\tilde{X}(t) = X(t)$ for all t in J.*

With this definition, we can now say that the initial value problem (1.5) has a unique maximal solution (that is, exactly one): the exponential function, defined on the whole $\mathbb{R}$. We say that this problem is *well posed*. In the following section, we give well posedness sufficient conditions for an initial value problem.

1.3 Existence and uniqueness of solutions

In this section, let d be a positive integer, Ω an open subset of $\mathbb{R} \times \mathbb{R}^d$, $F : \Omega \to \mathbb{R}^d$ and $(t_0, X_0) \in \Omega$.

When the function F is not continuous, a solution to (1.4) could not exist:

Exercise 1.3.1. *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1$ if $x \leq 0$ and $f(x) = -1$ if $x > 0$. Show that the autonomous differential equation $x'(t) = f(x(t))$ doesn't have any solution such that $x(0) = 0$.*

When the function F is continuous, it can be shown that (1.4) has at least one solution, but this solution is not necessarily unique.

Theorem 1.3.2 (Cauchy-Peano-Arzela Global)**.** *Suppose F is continuous on Ω. Then the initial value problem (1.4) has at least a maximal solution $(J, X(\cdot))$, where J is an open interval.*

The proof of this theorem is beyond the scope of this course.

Example 1.3.3. *Consider the initial value problem $x'(t) = 3|x(t)|^{2/3}$ and $x(0) = 0$ (here, $d = 1$, $\Omega = \mathbb{R}^2$, and $f(t, x) = 3|x|^{2/3}$; in particular, f is continuous). The functions $x(\cdot)$ and $y(\cdot)$ defined on $\mathbb{R}$ by $x(t) = t^3$ and $y(t) = 0$, respectively, are both maximal solutions. Thus, this problem does not have a unique maximal solution.*

Exercise 1.3.4. *Show that the above initial value problem has an infinite number of maximal solutions!*

Multiplicity of maximal solutions is a huge problem for applications. This means that the knowledge of the law of motion and of the initial state of the system is not enough to predict its evolution. Fortunately, if the function F is regular enough, for instance of class C^1, then the initial value problem (1.4) has a unique maximal solution. This result is known as the Cauchy-Lipschitz theorem in France, and as the Picard or Picard-Lindelöf theorem in other parts of the world. Funnily, both Cauchy and Picard were French, so this is not a case of French chauvinism!

Theorem 1.3.5 (Cauchy-Lipschitz / Picard-Lindelöf Global)**.** *Consider the initial value problem (1.4).*

1) If F is Lipschitz or continuously differentiable on Ω, then the initial value problem (1.4) has a unique maximal solution. That is, a maximal solution exists and it is unique.

2) Solutions of (1.4) are restrictions of the unique maximal solution to smaller time-intervals: if $(J_{max}, X_{max}(\cdot))$ is the unique maximal solution, then $(J, X(\cdot))$ is a solution if and only if J is an open interval containing t_0 and included in J_{max}, and $X(t) = X_{max}(t)$ for all t in J.

Remark 1.3.6. *The second point means that we don't lose generality in focusing on maximal solutions. So from now on, except if indicated otherwise,* solution will always mean maximal solution.

Remark 1.3.7. *A stronger version of Theorem 1.3.5 is stated and proved in Section 1.5.*

A fundamental consequence of Theorem 1.3.5 is that two (maximal) solutions cannot intersect. That is, two solutions are either disjoint or equal.

Corollary 1.3.8. *Assume F is $C^1(\Omega)$ or Lipschitz. Let $X : J \to \mathbb{R}^d$ and $Y : J' \to \mathbb{R}^d$ be maximal solutions of (1.3). If there exists $t_0 \in J \cap J'$ such that $X(t_0) = Y(t_0)$, then $J = J'$ and $X(\cdot) = Y(\cdot)$.*

Proof. Let t_0 be a real number such that $X(t_0) = Y(t_0)$. Let $Z_0 = X(t_0) = Y(t_0)$. Then $(J, X(\cdot))$ and $(J', Y(\cdot))$ are two maximal solutions of the initial value problem $Z'(t) = F(t, Z(t))$ and $Z(t_0) = Z_0$. Due to the Picard-Lindelöf theorem, this problem has a unique maximal solution, hence $(J, X(\cdot)) = (J', Y(\cdot))$. □

In dimension 1, the theorem is even more powerful. This is because, on a given interval where both are defined, if two continuous functions do not intersect, then one remains above the other.

Corollary 1.3.9 (1D Comparison Principle)**.** *Let Ω be an open subset of $\mathbb{R}^2$, and let $f : \Omega \to \mathbb{R}$ be C^1 or Lipschitz. Let $x : J \to \mathbb{R}$ and $y : J' \to \mathbb{R}$ be two maximal solutions of $x'(t) = f(t, x(t))$. If there exists $t_0 \in J \cap J'$ such that $x(t_0) < y(t_0)$, then for all $t \in J \cap J'$, $x(t) < y(t)$.*

Proof. Assume by contradiction that there there exists $t_0 \in J \cap J'$ such that $x(t_0) < y(t_0)$ and t_1 in $J \cap J'$ such that $x(t_1) \geq y(t_1)$. Since $x(\cdot)$ and $y(\cdot)$ are differentiable, hence continuous, and since $J \cap J'$ is an open interval, it follows from the intermediate value theorem that there exists $t \in J \cap J'$ such that $x(t) = y(t)$. Due to Corollary 1.3.8, we thus have $x(\cdot) = y(\cdot)$ on $J = J'$. This contradicts the assumption $x(t_0) < y(t_0)$. □

Exercise 1.3.10. *Under the same assumptions of Corollary 1.3.9. Show that if there exists $t_0 \in J \cap J'$ such that $x(t_0) \leq y(t_0)$, then for all $t \in J \cap J'$, $x(t) \leq y(t)$.*

1.4 How can solutions cease to exist?

The previous section studied existence and uniqueness of solutions. This section studies a related question: how can solutions cease to exist?

Consider the special case of Eq. (1.3) when the function F is defined on a set Ω of the form $\Omega = I \times \mathbb{R}^d$, where I is a nonempty open interval of $\mathbb{R}$. We then say that a solution $(J, X(\cdot))$ of (1.3) is *global* if $J = I$. In the example below, $I = \mathbb{R}$.

Example 1.4.1. *a) The function $x : \mathbb{R} \to \mathbb{R}$ given by $x(t) = 0$ is a global solution of*

$$x'(t) = x^2(t) \tag{1.6}$$

b) The function $y :]-\infty; 1[\to \mathbb{R}$ given by $y(t) = 1/(1-t)$ is also a solution, but it is not global since it is not defined on the whole $\mathbb{R}$.

It is immediate that a global solution is maximal (prove it!). However, the converse is not true: a maximal solution need not be global. To see this, note that in the above example, the function $y(\cdot)$ is not global, but it is nonetheless maximal. Indeed, $y(\cdot)$ cannot be extended backward in time as it is already defined until $-\infty$, and it cannot be extended forward in time because it converges to $+\infty$ as $t \to 1$: a solution extending $y(\cdot)$ would have to be differentiable hence continuous by definition of solutions of (1.3), but $y(\cdot)$ has no continuous extension.

It is instructive to compare what happens at both extremities of the interval $]-\infty, 1[$. At $-\infty$, the solution $y(\cdot)$ cannot be extended because, so to speak, the time t reaches the boundary of the space in which it is allowed to evolve. At $t = 1$, the solution cannot be extended because, again speaking informally, the position $X(t)$ of the particle reaches the boundary ($+\infty$) of the space in which it is allowed to evolve. In both cases, $(t, X(t))$ reaches - or more precisely, converges to - the boundary of the space Ω in which it is allowed to evolve, implying that the solution cannot be extended. This turns out to be the only way in which a maximal solution may cease to exist. To make this precise, we need a definition of "convergence to the boundary". In the two definitions below, the closure $\bar{J}$ of the interval J is taken in $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.

Definition 1.4.2. *Let $Y(\cdot)$ be a function from an interval J of $\mathbb{R}$ to $\mathbb{R}^d$. Let $t^* \in \bar{J}$ and $Y^* \in \mathbb{R}^d$. Y^* is an* accumulation point *of $Y(\cdot)$ as $t \to t^*$ if there exists a sequence (t_n) of elements of J such that $t_n \to t^*$ and $Y(t_n) \to Y^*$ as $n \to +\infty$. This is equivalent to: for any neighborhoods V of t^* and W of Y^*, there exists t in $V \cap J$ such that $Y(t) \in W$.*

Definition 1.4.3. *Let Ω be an open subset of $\mathbb{R}^d$ and J an interval of $\mathbb{R}$. Let $Y : J \to \Omega$ and $t^* \in \bar{J}$. We say that $Y(t)$* converges to the boundary of Ω *as $t \to t^*$ if $Y(\cdot)$ has no accumulation point in Ω as $t \to t^*$. We then write $Y(t) \to Bd(\Omega)$ as $t \to t^*$.*

Remark 1.4.4. *To say that $Y(t)$ converges to the boundary of Ω as $t \to t^*$ is equivalent to say that either the set of accumulation points of $Y(\cdot)$ as $t \to t^*$ is empty or all the accumulation points of $Y(\cdot)$ as $t \to t^*$ belong to the boundary of Ω.*

Remark 1.4.5. *To say that $Y(t)$ converges to the boundary of Ω as $t \to t^*$ is equivalent to say that for any compact subset $K \subset \Omega$, it exists a neighborhood V of t^* such that $Y(t) \notin K$ for all $t \in V \cap J$.*

Example 1.4.6. *Let I be an open interval, t^* be in the closure of I and $X : I \to \mathbb{R}^d$ be an arbitrary function. If $\Omega = I \times \mathbb{R}^d$ and $Y(t) = (t, X(t))$, then $Y(t) \to Bd(\Omega)$ as $t \to t^*$ if and only if ($t^* \in \{\sup I, \inf I\}$ or $||X(t)|| \to +\infty$ as $t \to t^*$, or both). Here $||\cdot||$ is the euclidean norm in $\mathbb{R}^d$.*

We can now state our theorem. Note that we require here that F is C^1 or Lipschitz in order to use Cauchy-Lipschitz / Picard-Lindelöf Theorem 1.3.5 to show the existence of a solution. However, since uniqueness is not required, continuity is enough, due to Cauchy-Peano-Arzela Theorem 1.3.2.

Theorem 1.4.7 (Characterization of maximal solutions)**.** *Let d be a positive integer, and Ω an open subset of $\mathbb{R} \times \mathbb{R}^d$. Let $(J, X(\cdot))$ be a solution (not necessarily maximal) of (1.3), with $F : \Omega \to \mathbb{R}^d$ assumed to be C^1 or Lipschitz. Then $(J, X(\cdot))$ is a maximal solution if and only if $(t, X(t))$ converges to the boundary of Ω, when $t \to \sup J$ and when $t \to \inf J$.*

Remark 1.4.8. *Note that in Definition 1.2.2 of solution of (1.3), we could also have taken J as a general interval of $\mathbb{R}$ (not necessarily open). In this case Theorem 1.4.7 would have implied that J has to be open if $(J, X(\cdot))$ is a maximal solution. Indeed, suppose $J =]\alpha, \beta]$ (the case $J = [\alpha, \beta[$ is similar). Then, continuity of X on $[\beta - \varepsilon, \beta] \subset]\alpha, \beta]$, implies $X(t) \to X(\beta) \in \mathbb{R}^d$ as $t \to \beta$ (hence $(\beta, X(\beta))$ is an accumulation point for $(t, X(t))$ as $t \to \sup J = \beta$) and since $\beta \in J$, $(\beta, X(\beta)) \in \Omega$ and not on its boundary. A contradiction.*

With the additional assumption $\Omega = I \times \mathbb{R}^d$, the theorem takes the form:

Theorem 1.4.9 (Explosion alternative/ Blow-up in finite time)**.** *Let $(J, X(\cdot))$ be a solution of (1.3). Then $(J, X(\cdot))$ is a maximal solution if and only if the following two conditions are both satisfied:*

1) $\sup J = \sup I$ *or (*$\sup J < \sup I$ *and* $||X(t)|| \to +\infty$ *as* $t \to \sup J$*).*

2) $\inf J = \inf I$ *or (*$\inf J > \inf I$ *and* $||X(t)|| \to +\infty$ *as* $t \to \inf J$*).*

Proof. This follows immediately from Theorem 1.4.7 and Example 1.4.6. Note that in the case $I = \mathbb{R}$, when the maximal solution ceases to be defined either t or $||X(t)||$, or both, go to infinity, hence the name "explosion alternative" for this result. In other words: finite time blow-up is equivalent to global nonexistence. □

Before proving Theorem 1.4.7, a few remarks are in order.

Remark 1.4.10. *Let $x(\cdot)$ be continuous and take its values in $\mathbb{R}$. Let $t^* \in \bar{\mathbb{R}}$. By continuity of $x(\cdot)$, $|x(t)| \to +\infty$ as $t \to t^*$ if and only if ($x(t) \to +\infty$ or $x(t) \to -\infty$). So when $d = 1$, we may replace in Theorem 1.4.9 the conclusion $|x(t)| \to \infty$ by ($x(t) \to +\infty$ or $x(t) \to -\infty$).*

By contrast, if $X(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$ then it can be that $||X(t)|| \to +\infty$ but neither $|x_1(t)|$, nor $|x_2(t)|$ goes to infinity. This is because $X(t)$ can be spiral. For instance, let $X(t) = (e^t \cos t, e^t \sin t)$. As $t \to +\infty$, $||X(t)|| \to +\infty$ but $e^t \cos t$ and $e^t \sin t$ keep taking the value 0 so their absolute value does not go to $+\infty$. This phenomenon may also arise when $\sup J < +\infty$, e.g., for $\tilde{X}(t) = X(\tan t)$, with $X(t)$ defined as above.

Remark 1.4.11. *Conditions 1) and 2) in Theorem 1.4.9 can be also stated as follows:*

1bis) If $\sup J < \sup I$*, then* $||X(t)|| \to +\infty$ *as* $t \to \sup J$*.*

2bis) If $\inf J > \inf I$*, then* $||X(t)|| \to +\infty$ *as* $t \to \inf J$*.*

Important: *When, for instance, $\sup J = \sup I$, it may be that we also have $||X(t)|| \to +\infty$ as $t \to \sup J$, but it is not necessary. For instance, this is the case for the solution $t \to e^t$ of $x'(t) = x(t)$, but not for the solution $t \to 0$.*

Theorem 1.4.9 implies that if the norm of the solution is bounded by a continuous function defined on the whole interval I, then the solution is global:

Corollary 1.4.12 (Non explosion criterion)**.** *Let $\Omega = I \times \mathbb{R}^d$. Let $(J, X(\cdot))$ be a maximal solution of (1.3). Let $g : I \to \mathbb{R}$ be continuous. Let $t_0 \in J$.*

1) If $||X(t)|| \leq g(t)$ on $[t_0, \sup J[$, then $\sup J = \sup I$.

2) If $||X(t)|| \leq g(t)$ on $]\inf J, t_0]$, then $\inf J = \inf I$.

3) If $||X(t)|| \leq g(t)$ on J, then $X(\cdot)$ is a global solution.

Proof. Let us prove 1). Assume by contradiction that $||X(t)|| \leq g(t)$ on $[t_0, \sup J[$ and $\sup J < \sup I$. We have that g is defined at $\sup J$. Since the interval $[t_0, \sup J]$ is compact and $g(\cdot)$ is continuous, there exists a constant K such that $g(t) \leq K$ on $[t_0, \sup J]$. Therefore $||X(t)|| \leq g(t) \leq K$ on $[t_0, \sup J[$, hence $||X(t)||$ cannot go to $+\infty$ as $t \to \sup J$. Since neither $\sup J = \sup I$ nor $||X(t)|| \to +\infty$ as $t \to \sup J$, this contradicts Theorem 1.4.9.

The proof of 2) is similar, and 3) follows from 1) and 2). □

Exercise 1.4.13. *Let J be a nonempty open interval and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a function of class C^1. Let $x : \mathbb{R} \to \mathbb{R}$, $y : \mathbb{R} \to \mathbb{R}$ and $z : J \to \mathbb{R}$ be solutions of $x'(t) = f(t, x(t))$. Assume that there exists $\tau \in J$ such that $x(\tau) < z(\tau) < y(\tau)$. Show that for all $t \in J$, $x(t) < z(t) < y(t)$. Conclude that if $x(\cdot)$ and $y(\cdot)$ are global then so is $z(\cdot)$.*

Proof of Theorem 1.4.7 (characterization of maximal solutions)

In this proof, "solution" refers to a solution which is not necessarily maximal. First note that if $X(t)$ can be extended to a solution of (1.3) after $\sup J$ then by continuity of this extension (remember Proposition 1.2.3), it must be that $(t, X(t))$ has a limit (hence an accumulation point) in Ω as $t \to \sup J$, and similarly in $\inf J$. It follows that if $(t, X(t)) \to Bd(\Omega)$ both in $\sup J$ and in $\inf J$ then $X(\cdot)$ is maximal, in other words, these are sufficient conditions of maximality. The difficulty is to show that these conditions are also necessary.

The idea of the proof is as follows. Assume that $(t, X(t))$ does not converge to the boundary of Ω as $t \to \sup J$. Then $(t, X(t))$ has an accumulation point (t^*, X^*) in Ω as $t \to \sup J$, and necessarily $\sup J = t^* < +\infty$. We have to show that this accumulation point is also a limit. Since Ω is open, we can find a compact neighborhood of (t^*, X^*) included in Ω. Since F is continuous, it follows that F is bounded on this neighborhood. Since $X'(t) = F(t, X(t))$, this implies that $X'(\cdot)$ is bounded when $(t, X(t))$ is in this neighborhood. For this reason, $X(\cdot)$ cannot evolve too wildly, and it can be shown that it converges to X^* as $t \to t^*$. We can then extend $X(\cdot)$ to a solution defined after $\sup J = t^*$ by sticking to $X(\cdot)$ a solution of the initial value problem: $X'(t) = F(t, X(t))$ and $X(t^*) = X^*$. This contradicts the maximimality of $X(\cdot)$. Note that we only need existence of a solution of $X'(t) = F(t, X(t))$ such that $X(t^*) = X^*$, not uniqueness. For this reason, *the theorem is still true if we only assume F to be continuous.*

Here is the full proof. We just prove that if $X(\cdot)$ is maximal, then $(t, X(t))$ converges to the boundary of Ω as $t \to \sup J$. The proof of the same result for $\inf J$ is similar.

Let $J =]\alpha, \beta[$, with α and β in $\bar{\mathbb{R}}$. Assume that $(t, X(t))$ does not converge to the boundary of Ω as $t \to \beta$, hence $Y(\cdot) := (\cdot, X(\cdot))$ has an accumulation point in Ω as $t \to \beta$. By the definition of accumulation point, there exists (t^*, X^*) in Ω and an increasing sequence (t_n) converging to β such that $(t_n, X(t_n))$ converges to (t^*, X^*) as $n \to +\infty$. This implies that $t^* = \beta$, hence $\beta < +\infty$, and that $X(t_n) \to X^*$ as $n \to +\infty$. We have to show that X^* is indeed a limit as $t \to \beta$ (and not only for the sequence t_n!):

Lemma 1.4.14. *$X(t) \to X^*$ as $t \to \beta$.*

Suppose this is true. Let $\tilde{X}(\cdot)$ be a solution of $X'(t) = F(t, X(t))$ such that $\tilde{X}(\beta) = X^*$, defined on $]\beta - \varepsilon, \beta + \varepsilon[$ for some $\varepsilon > 0$ (such a solution exists due to Picard-Lindelöf theorem).

Define $Z :]\alpha, \beta + \varepsilon[\to \mathbb{R}^d$ by $Z(t) = X(t)$ if $t \in]\alpha, \beta[$ and $Z(t) = \tilde{X}(t)$ if $t \in [\beta, \beta + \varepsilon[$. The function $Z(\cdot)$ is continuous, thanks to Lemma 1.4.14, and it is differentiable with derivative $Z'(t) = F(t, Z(t))$ on $]\alpha, \beta[$ and on $]\beta, \beta + \varepsilon[$. Moreover, $Z'(t) = F(t, Z(t)) \to F(\beta, Z(\beta))$ as $t \to \beta^{\pm}$ (with $t \neq \beta$). By a standard result from calculus, this implies that $Z(\cdot)$ is differentiable in β with derivative $Z'(\beta) = F(\beta, Z(\beta))$. Thus, $Z(\cdot)$ is a solution of (1.3). Since $Z(\cdot)$ is a strict extension of $X(\cdot)$, this contradicts the maximality of $X(\cdot)$.

It remains to prove Lemma 1.4.14. Since Ω is open, there exists $\varepsilon > 0$ such that the closed ball B of center (β, X^*) and radius ε is included in Ω. Since B is compact, the continuous function F is bounded on B by some constant K. It follows that if $t \in J$ and $(t, X(t)) \in B$,

then
$$||X'(t)|| = ||F(t, X(t))|| \leq K \tag{1.7}$$
Now recall that $t_n \to \beta$ and $X(t_n) \to X^*$. Therefore, we can find an integer N such that $t_N \geq \beta - \varepsilon$, $K(\beta - t_N) < \varepsilon/2$ and $||X(t_N) - X^*|| < \varepsilon/2$.

We claim that:

Claim 1.4.15. *For all t in $[t_N, \beta[$, $||X(t) - X^*|| < \varepsilon$.*

Indeed, otherwise, let τ be the smallest time greater than t_N such that $||X(t) - X^*|| \geq \varepsilon$ (this smallest time exists by continuity of $X(\cdot)$ and of $||X(\cdot) - X^*||$). By definition, $||X(\tau) - X^*|| \geq \varepsilon$ and $||X(t) - X^*|| < \varepsilon$ for all $t \in [t_N, \tau[$. The latter implies that for all $t \in [t_N, \tau[$, $(t, X(t)) \in B$, hence $||X'(t)|| \leq K$. Therefore

$$||X(\tau) - X(t_N)|| \leq \int_{t_N}^{\tau} ||X'(t)||dt \leq K(\tau - t_N) < K(\beta - t_N) < \varepsilon/2.$$

Thus,
$$||X(\tau) - X^*|| \leq ||X(\tau) - X(t_N)|| + ||X(t_N) - X^*|| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$
contradicting the definition of τ.

We now conclude: due to (1.7), Claim 1.4.15 and the fact that $t_N \geq \beta - \varepsilon$, $(t, X(t)) \in B$ hence $||X'(t)|| \leq K$ for all t in $[t_N, \beta[$. Therefore

$$\int_{t_N}^{\beta} ||X'(t)||dt \leq K(\beta - t_N) < +\infty.$$

Therefore, the integral $\int_{t_N}^{\beta} X'(t)dt$ is convergent, which implies that $X(\cdot)$ has a limit when $t \to \beta$. Finally, since X^* is an accumulation point, the only possible limit is X^*. This concludes the proof of Lemma 1.4.14, hence of Theorem 1.4.7.

1.5 Proof of Picard-Lindelöf theorem

This Section might be useful for later courses on dynamical systems.

Theorem 1.3.5 is a simplified version of the standard Picard-Lindelöf theorem. Before stating and proving the standard theorem, we need to introduce the notion of locally Lipschitz functions.

Definition 1.5.1. *A function F defined on a subset Ω of $\mathbb{R}^n$ is locally Lipschitz if every X in Ω has a neighborhood V such that F is Lispschitz on $V \cap \Omega$,*

Remark 1.5.2. *A function F defined on a subset Ω of $\mathbb{R}^n$ is locally Lipschitz if and only if for all compact sets $C \subset \Omega$, the restriction of F to C is Lipschitz.*

Of course, any Lipschitz function is locally Lipschitz, but the converse is not true. This is because in the definition, the Lipschitz constant of the restriction of F to C may depend on C, and may go to infinity as C becomes larger. For instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is locally Lipschitz but not Lipschitz. Indeed, since $f'(x) \to +\infty$ as $x \to +\infty$, it follows that f is not Lipschitz on $\mathbb{R}$. However, on every compact subset C of $\mathbb{R}$, $f'(x)$ is bounded, which implies that the restriction of f to C is Lipschitz. Generalizing this reasoning, it is easy to see that any C^1 function is locally Lipschitz. Thus, locally Lipschitz functions generalize both Lipschitz and C^1 functions. This generalization is strict in the sense that there are locally Lipschitz functions that are neither Lipschitz nor C^1. For instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ if $x > 0$ and $f(x) = x^2$ if $x \leq 0$. Moreover, being locally Lipschitz is stronger than being merely continuous. For instance, the function $f : \mathbb{R}_+ \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is continuous but not locally Lipschitz.

Exercise 1.5.3. *Being differentiable and being locally Lipschitz are two conditions that are stronger than continuity but weaker than being C^1. Show that these conditions cannot be compared: give an example of a function which is locally Lipschitz but not differentiable, and an example of a function which is differentiable but not locally Lipschitz (hint: try $f(x) = x^n \sin^m(\frac{1}{x})$ for $x \neq 0$, and $f(0) = 0$, for a good choice of n and m).*

The assumption of the standard Cauchy-Lipschitz theorem is not exactly that the function F in (1.3) is Lipschitz or locally Lipschitz, but that it is (jointly) continuous, and locally Lipschitz with respect to the space variable X.[5] Being locally Lipschitz in X means that for all compact sets $C \subset \Omega$, there exists a constant K_C such that, for all $(t, X, Y) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$:

$$\text{if } (t, X) \text{ and } (t, Y) \text{ are in } C, \text{ then } ||F(t, X) - F(t, Y)|| \leq K_C ||X - Y||.$$

We can now state the standard Picard-Lindelöf theorem:

Theorem 1.5.4 (Cauchy-Lipschitz / Picard-Lindelöf Global, Strong form)**.** *Let d be a positive integer, and Ω an open subset of $\mathbb{R} \times \mathbb{R}^d$. Let $F : \Omega \to \mathbb{R}^d$. If $F(t, X)$ is jointly continuous in (t, X) and locally Lipschitz with respect to X, then the initial value problem (1.4) is well posed, that is, it has one and only one maximal solution. Moreover, all solutions of (1.4) are restrictions of the maximal solution.*

The proof of the theorem proceeds in two steps. We first prove existence and uniqueness of a local solution, that is, defined on some smaller closed interval $[t_0 - a, t_0 + a]$. We then show how to extend this solution to a maximal time interval.

Lemma 1.5.5 (Cauchy-Lipschitz / Picard-Lindelöf Local, Strong form)**.** *Under the same assumptions of Theorem 1.5.4, if $a > 0$ is small enough, then the initial value problem (1.4) has a solution defined on $[t_0 - a, t_0 + a]$ and this solution is unique.*

Remark 1.5.6. *The use of the term solution in the above theorem is improper according to Definition 1.2.2. What Lemma 1.5.5 means, is that there exists a function $X : [t_0 - a, t_0 + a] \to \mathbb{R}^d$ such that for all t in $[t_0 - a, t_0 + a]$, $(t, X(t)) \in \Omega$ and $X'(t) = F(t, X(t))$. Of course, in order that differentiability makes sense for $X(\cdot)$ up to the boundary of $[t_0 - a, t_0 + a]$, it is better to see the solution as defined on a bigger nonempty open time-interval J containing $[t_0 - a, t_0 + a]$. This will not be a problem since we are not saying that the solution given by Lemma 1.5.5 is maximal. See also Remark 1.4.8 which ensures that maximal solutions are defined on open intervals.*

Proof. To simplify the exposition, assume that $t_0 = 0$. We begin with a fundamental remark: let $Y : [-a, a] \to \mathbb{R}^d$. Then $Y(\cdot)$ is a solution of (1.4) if and only if $Y(\cdot)$ is continuous and satisfies the integral equation

$$\forall t \in [-a, a], \ Y(t) = X_0 + \int_0^t F(s, Y(s))\, ds. \tag{1.8}$$

Indeed, if $Y(\cdot)$ is a solution of (1.4) we can find (1.8) just integrating (1.4) over 0 and t. On the other hand, if Y, given by (1.8), is continuous on $[-a, a]$, then $F(\cdot, Y(\cdot))$ is continuous on $[-a, a]$. Hence $Y(\cdot)$ is continuously differentiable on $[-a, a]$ and $Y'(t) = F(t, Y(t))$ for all $t \in [-a, a]$. Moreover $Y(0) = X_0$. Hence $Y(\cdot)$ is a solution of (1.4).

Thus, it suffices to prove that the integral equation (1.8) has a unique (continuous) solution. To do so, the idea is to define a function Φ which associates to $Y(\cdot)$ the right-hand side of (1.8) and to show that this function has a unique fixed point, this will prove that (1.8) has a unique solution. The assumptions we will make during the proof ensure that this auxiliary function is

[5] If F is continuous in t and locally Lipschitz in X, then F is jointly continuous, so we may replace the assumption'jointly continuous in (t, X)" by "continuous in t".

well defined and contracting, so that the existence and uniqueness of a fixed point follows from Banach fixed point Theorem [6] (also called contraction mapping Theorem).

Let $\bar{B}_r$ denote the closed ball with center $(0, X_0)$ and radius r. Since Ω is open:

Claim 1: there exists $r > 0$ such that $\bar{B}_r \subset \Omega$.

Since F is locally Lipschitz with respect to X:

Claim 2: there exists $K_r > 0$ such that F is K_r-Lipschitz with respect to X on $\bar{B}_r$.

Since F is jointly continuous and $\bar{B}_r$ compact:

Claim 3: F is bounded on $\bar{B}_r$ by some constant M.

Finally, we may choose $a > 0$ such that:

Condition 4: $aK_r < 1$, $aM < r$ and $a < r$.

Let E_a denote the set of continuous functions $X(\cdot) : [-a, a] \to \mathbb{R}^d$ such that $X(0) = X_0$ and

$$||X(\cdot) - X_0||_\infty \leq r \tag{1.9}$$

Note that using the supremum norm, defined as $||X(\cdot)||_\infty = \sup_{t\in[a,a]} ||X(t)||$, this is a complete metric space (e.g., as a closed subset of the set of functions from $[-a, a]$ to $\mathbb{R}^d$). Indeed $C^0([a, a])$ is a complete metric space with respect to the supremum norm (note that $C^0([a, a])$ is even more than metric, it is a complete normed space, i.e. a Banach space), and E_a is a closed subset of $C^0([a, a])$.

Step 1: The auxiliary mapping. If $X(\cdot) \in E_a$, define $\tilde{X} := \Phi(X)$ by

$$\forall t \in [-a, a], \quad \tilde{X}(t) = \Phi(X)(t) := X_0 + \int_0^t F(s, X(s))\, ds$$

Clearly, $\tilde{X}(\cdot)$ is continuous and such that $\tilde{X}(0) = X_0$. Moreover, for all t in $[-a, a]$,

$$||\tilde{X}(t) - X_0|| = || \int_0^t F(s, X(s))\, ds\, || \leq |t|M \leq aM < r \tag{1.10}$$

where the inequalities follow from Claim 3 and Condition 4. Thus $\tilde{X}(\cdot) \in E_a$ and we may see $\Phi : E_a \to E_a$ as a mapping from E_a to itself.

Step 2: This mapping is contracting. Indeed, if $X(\cdot)$ and $Y(\cdot)$ are in E_a, then due to Claim 2 and to $a < r$,

$$\forall s \in [-a, a],\ ||F(s, X(s)) - F(s, Y(s))|| \leq K_r||X(s) - Y(s)|| \leq K_r||X(\cdot) - Y(\cdot)||_\infty.$$

Thus letting $\tilde{X} = \Phi(X)$ and $\tilde{Y} = \Phi(Y)$, for all $t \in [-a, a]$

$$||\tilde{X}(t)-\tilde{Y}(t)|| = || \int_0^t [F(s, X(s))-F(s, Y(s))]\, ds\, || \leq |t|K_r||X(\cdot)-Y(\cdot)||_\infty \leq aK_r||X(\cdot)-Y(\cdot)||_\infty$$

so $||\Phi(X) - \Phi(Y)||_\infty \leq \lambda||X(\cdot) - Y(\cdot)||_\infty$ with $\lambda = aK_r < 1$ thanks to Condition 4.

Therefore, Banach fixed point Theorem ensures that the mapping Φ has a unique fixed point, that is, there exists exactly one function in E_a that solves (1.8). Thus we have the existence of a solution, and its uniqueness within E_a.

In order to prove that (1.8) has only one solution, we still need to show that all solutions are in E_a (there could be solutions outside this set, for example solutions for which $||X(\tau)-X_0|| > r$

[6] **Banach fixed point Theorem** Let (X, d) be a non-empty complete metric space with a contraction mapping $T : X \to X$, i.e. there exists $\delta \in [0, 1)$ such that

$$d(T(x), T(y)) \leq \delta d(x, y)$$

for all x, y in X. Then T admits a unique fixed-point x^* in X (i.e. $T(x^*) = x^*$). Furthermore, x^* can be found as follows: start with an arbitrary element x_0 in X and define a sequence $(x_n)_{n\in\mathbb{N}}$ by $x_n = T(x_{n-1})$, then $x_n \to x^*$.

for a $\tau \in [-a, a]$). To do so, consider a continuous solution $Y : [-a, a] \to \mathbb{R}^d$ of (1.8). Necessarily, $Y(0) = X_0$, so if $Y(\cdot) \notin E_a$, this is because (1.9) is not satisfied. This implies that there exists a time $\tau \in [-a, a]$ such that $||Y(\tau) - X_0|| > r$. To fix the ideas, assume $\tau > 0$. By continuity of $Y(\cdot)$, there exists a first (i.e. minimal) time $t_1 > 0$ such that $||Y(t_1) - X_0|| > r$. Since $||Y(t) - X_0|| \le r$ for all $t \in [0, t_1[$, by the same computation as in (1.10), $||Y(t_1) - X_0|| \le t_1 M \le aM < r$, a contradiction. Therefore, all solutions are in E_a, this concludes the proof of Lemma 1.5.5. □

Remark 1.5.7. *Under the same assumptions as in the above lemma, we can construct a sequence of functions, called Picard's sequence, $X^n : [-a, a] \to \mathbb{R}$, fixing $X^0 \equiv X_0$ and for all $n \ge 0$,*

$$\forall t \in [-a, a],\ X^{n+1}(t) = X_0 + \int_0^t F(s, X^n(s))\, ds. \tag{1.11}$$

The reader can show (see the exercise below) that $\forall n \in \mathbb{N}$, X^n is a sequence of functions in E_a, where E_a is defined as in the proof of the above Lemma. Hence by the Banach fixed point Theorem, X^n uniformly converges to the unique solution X. Thus, this sequence can be used to compute the solution numerically.

Exercise 1.5.8 (Proof of Lemma 1.5.5 using Picard's method)**.** *Under the same assumptions of Theorem 1.5.4, let $(X^n)_{n\in\mathbb{N}}$ be the Picard's sequence of functions defined above. Let E_a, r, K_r, M be defined as in the proof of Lemma 1.5.5.*

1) *Prove by induction that $\forall n \in \mathbb{N}$, $X^n \in E_a$, i.e. that for all $n \in \mathbb{N}$, X^n is continuous on $[-a, a]$,*
$$X^n(0) = X_0, \quad \text{and} \quad \forall t \in [-a, a] ||X^n(t) - X_0|| \le r.$$

2) *Prove by induction that $\forall n \ge 1$, $\forall t \in [-a, a]$,*
$$||X^n(t) - X^{n-1}(t)|| \le M \frac{(K_r)^{n-1}|t|^n}{n!}.$$

3) *Prove that the series of functions $\sum_{n=1}^{+\infty} (X^n - X^{n-1})$ normally converges on $[-a, a]$, hence the convergence is uniform on $[-a, a]$.*

4) *Deduce that (X_n) uniform converges on $[-a, a]$, to a function $X : [-a, a] \to \mathbb{R}$, that is continuous and such that $X(0) = X_0$, $||X(\cdot) - X_0||_\infty < r$.*

5) *Using Claim 2, show that the sequence of functions $F(\cdot, X^n(\cdot))$ converges uniformly to $F(\cdot, X(\cdot))$ on $[-a, a]$.*

6) *Conclude that*
$$X(t) = X_0 + \int_0^t F(s, X(s))ds$$
and hence $X(\cdot)$ is a solution of the initial value problem (1.4).

Lemma 1.5.9. *Let (J_Y, Y) and (J_Z, Z) be two solutions of the initial value problem (1.4). Then $Y(t) = Z(t)$ for all t in $J = J_Y \cap J_Z$.*

Proof. The only set where it makes sense to consider both solutions is $J = J_Y \cap J_Z$. Let $J^* \subset J$ be the set of points on which Y and Z agree. Note that $t_0 \in J^*$, hence J^* is not empty. Since Y and Z are continuous, J^* is closed in J (being the set of t such that $Y(t) - Z(t) = 0$). Let $t_1 \in J^*$, hence $Y(t_1) = Z(t_1)$. Then, fixing $X_1 = Y(t_1)$, Y and Z are two solutions of $X'(t) = F(t, X(t))$ with the same initial condition $X(t_1) = X_1$. Therefore, by Lemma 1.5.5 (uniqueness part) Y and Z agree on $]t_1 - a, t_1 + a[$, for a small enough. Hence J^* is open in J (for all $t_1 \in J^*$ we have found a neighborhood entirely contained in J^*). Since the only open and closed set in J are either J or $\emptyset$, and J^* is non empty, we have that J^* is equal to J. □

We can now prove Theorem 1.5.4. Consider the set S of all solutions (J, Y) of the initial value problem (1.4). This set is nonempty by Lemma 1.5.5, indeed this Lemma gives in particular a solution defined on the open interval $]t_0 - a, t_0 + a[\subset [t_0 - a, t_0 + a]$. Define the "maximal" time interval

$$J_{max} := \bigcup_{(J,Y)\in S} J. \tag{1.12}$$

That is, $J_{\max}$ is the union of all (open) intervals on which the initial value problem (1.4) has a solution. Note that $J_{\max}$ is open.[7] Define the function $Y_{\max} : J_{\max} \to \mathbb{R}^d$ by

$$\forall\, t \in J_{\max}, \quad Y_{\max}(t) = Z(t) \quad \text{for all solutions} \quad (J, Z) \text{ such that } t \in J.$$

The function $Y_{\max}$ is well defined: indeed, if $t \in J_{\max}$, then there exists a solution (J_Y, Y) such that $t \in J_Y$, so we may give a value to $Y_{\max}(t)$, and if (J_Z, Z) is another solution such that $t \in J_Z$, then due to Lemma 1.5.9, $Y(t) = Z(t)$, so there is no ambiguity on the value of $Y_{\max}(t)$.

The fact that around each $t \in J_{\max}$, $Y_{\max}$ agrees with a solution of the initial value problem (1.4) implies that Y_{max} is a solution of this problem. Finally, the definition of $J_{\max}$ and Lemma 1.5.9 imply that this solution is maximal and that all other solutions are restrictions of this maximal solution. This concludes the proof.

Adapting the proof of Lemma 1.5.5 we can prove

Theorem 1.5.10. *Let d be a positive integer, and I an open subset of $\mathbb{R}$. Let $F : I \times \mathbb{R}^d \to \mathbb{R}^d$. If $F(t, X)$ is jointly continuous in (t, X), locally Lipschitz with respect to X, and there exist $A, B \geq 0$ such that*

$$||F(t, X)|| \leq A||X|| + B \quad \forall (t, X) \in I \times \mathbb{R}^d,$$

then the unique solution of the initial value problem (1.4) is defined on the whole I.

Remark 1.5.11. *This theorem is a particular case of Corollary 4.2.1.*

[7] Since our definition of solutions requires that they be defined on an open interval, it follows from (1.12) that $J_{\max}$ is open, as an uncountable union of open sets. Alternatively, if we had not required J open in the definition of a solution, we could have proved that J_{max} is open as in Remark 1.4.8.

Chapter 2

Autonomous equations

This chapter shows that the theory of Chapter 1 allows us, without any computation, to understand the behavior of autonomous equations in dimension 1. We begin with some generalities.

2.1 Generalities

An autonomous equation is an equation of the form[1]

$$X'(t) = G(X(t)) \tag{2.1}$$

with $G : \Omega_d \to \mathbb{R}^d$, where Ω_d is an open subset of $\mathbb{R}^d$. It may be seen as a nonautonomous equation (1.3) with F independent of t by letting $\Omega = \mathbb{R} \times \Omega_d$ and $F(t, X) = G(X)$ for all $t \in \mathbb{R}$. Thus, any result on nonautonomous equations has an equivalent for autonomous equations. For instance, it follows from Picard-Lindelöf theorem that if G is Lipschitz or C^1, then for any $t_0 \in \mathbb{R}$ and $X_0 \in \Omega_d$, (2.1) has a unique maximal solution $(J, X(\cdot))$ such that $X(t_0) = X_0$. Moreover, by the Explosion alternative, $\sup J = +\infty$ or $||X(t)|| \to +\infty$ as $t \to \sup J$, or both, and similarly for $\inf J$.

Autonomous equations are important for three reasons: first, they often arise in applications.[2] Second, they have special properties that make them easier to study. Third, any nonautonomous equation may be seen as an autonomous equation. Indeed, consider a nonautonomous equation

$$X'(t) = F(t, X(t)) \tag{2.2}$$

with $F : \Omega \to \mathbb{R}^d$, where $\Omega \subset \mathbb{R}^{d+1}$. Define $G : \Omega \to \mathbb{R}^{d+1}$ by $G(Y) = (1, F(Y))$ so that if $Y = (t, X)$, then

$$G(Y) = G(t, X) = (1, F(t, X)).$$

Associate, now, to $X : J \to \mathbb{R}^d$ the function $Y : J \to \mathbb{R}^{d+1}$ defined by $Y(t) = (t, X(t))$.

Claim 2.1.1. *$(J, X(\cdot))$ is a solution of (2.2) such that $X(t_0) = X_0$ if and only if $(J, Y(\cdot))$ is a solution of the autonomous equation $Y'(t) = G(Y(t))$ such that $Y(t_0) = (t_0, X_0)$.*

Proof. $X(t_0) = X_0 \Leftrightarrow Y(t_0) = (t_0, X_0)$, and $X'(t) = F(t, X(t)) \Leftrightarrow Y'(t) = (1, F(t, X(t)) \Leftrightarrow Y'(t) = G(Y(t))$. $\square$

The reader can check that if we start from a maximal solution of one equation, we end up with a maximal solution of the other. We conclude that solving $Y'(t) = G(Y(t))$ is equivalent to solving (2.2). In practice, this is not a good method to solve nonautonomous equations because

[1] More precisely, (2.1) is a first-order autonomous differential equation in implicit form.

[2] The term autonomous comes from the fact that, in physics, a system which is “autonomous” in the sense that it is isolated from the rest of the world should be described by autonomous equations.

the dimension increases: instead of having a variable $X(t)$ in $\mathbb{R}^d$, we end up with a variable $Y(t)$ in $\mathbb{R}^{d+1}$. But the fact that we can use this method, in theory means, that for many theoretical results, it suffices to prove the result for autonomous equation, and then to apply the above trick to show that the result still holds for nonautonomous equations. This is one of the reasons why we will mostly focus on autonomous equations.

Exercise 2.1.2. *Write the following equations in their equivalent autonomous form:*

a) $x'(t) = t + x(t)$

b) $X'(t) = (x'(t), y'(t)) = (t + x(t), t - y(t))$*, with* $X(t) = (x(t), y(t)) \in \mathbb{R}^2$.

c) $x'(t) = 3x^2(t)$ *(why is this a silly example?)*

Invariance by translation through time.

Imagine a particle whose position $X(t)$ at time t evolves according to a first order differential equation. If this equation is autonomous, that is, of the form $X'(t) = G(X(t))$, then $X'(t)$ depends only on $X(t)$, and not explicitly on t. That is, the direction and speed of the particle depend only on its current position.[3] It follows that solutions of autonomous equations are invariant by translation through time. That is, if $X : J \to \mathbb{R}^d$ is solution of (2.1), then for any $\Delta \in \mathbb{R}$, the function defined on $J_Y =]\inf J + \Delta, \sup J + \Delta[$ by $Y(t) = X(t - \Delta)$, (or equivalently $Y(t + \Delta) = X(t)$, is also solution defined in $]\inf J - \Delta, \sup J - \Delta[$). Indeed,

$$Y'(t) = X'(t - \Delta) = G(X(t - \Delta)) = G(Y(t))$$

Moreover, it is easy to see that if $X(\cdot)$ is maximal, then so is $Y(\cdot)$. It follows that if $X(\cdot)$ is the maximal solution taking the value X_0 at time τ_0, then $Y(\cdot)$ is the maximal solution taking the value X_0 at time $\tau_1 = \tau_0 + \Delta$. For future reference, we write this as a proposition:

Proposition 2.1.3. *Consider an autonomous equation* (2.1) *with* G *of class* C^1 *or Lipschitz. Let* $X_0 \in \mathbb{R}^d$, $\tau_0 \in \mathbb{R}$ *and* $\Delta \in \mathbb{R}$. *Let* $(J_X, X(\cdot))$ *and* $(J_Y, Y(\cdot))$ *be the maximal solutions taking the value* X_0 *at times* τ_0 *and* $\tau_0 + \Delta$*, respectively. Then* $J_Y = \{t + \Delta, t \in J_X\}$ *and:*

$$\forall t \in J_X,\; Y(t + \Delta) = X(t).$$ [4]

In particular (take $\tau_0 = 0$ and $\Delta = t_0$), for any t_0 in $\mathbb{R}$, we may recover from the solution $X(\cdot)$ taking the value X_0 at time 0 the solution taking the value X_0 at time t_0: this is the function $Y(\cdot)$ defined by $Y(t) = X(t - t_0)$. Thus, for autonomous equations, we may assume without loss of generality that the initial condition is given at time 0.

Exercise 2.1.4. *What is the relation between the graph of* $x : J \to \mathbb{R}$ *and of the function* $y(\cdot)$ *defined by* $y(t) = x(t - t_0)$*? Recall that the solution from* $x'(t) = x(t)$ *such that* $x(0) = 1$ *is the exponential function. Without computing it, deduce from this fact the graph of the solution of* $x'(t) = x(t)$ *such that* $x(3) = 1$*?*

Consequences for trajectories. The trajectory [5] (or orbit) generated by a solution $(J, X(\cdot))$ of (2.1) is the set $\{X(t), t \in J\} \subset \mathbb{R}^d$ of positions taken by $X(t)$ for t in J.

Proposition 2.1.5. *Consider an autonomous equation as in* (2.1) *and assume* G *of class* C^1 *or Lipschitz. Let* $X_0 \in \mathbb{R}^d$.

a) If two solutions take the value X_0*, then they follow the same trajectory.*

b) The trajectories generated by two solutions are either equal or disjoint.

[3] If the equation is nonautonomous, then the direction and speed of the particle also depend on time. Consider for instance the equation $x'(t) = x(t)\sin t$. If $x(t_0) = x_0 = 1$ and $t_0 = \pi/2$, then $x'(t_0) = 1$, and the particle goes forward; if $x(t_0) = 1$ and $t_0 = -\pi/2$, then $x'(t_0) = -1$, and the particle goes backward. So indeed, $x'(t_0)$ does not depend only on $x(t_0)$.

[4] The interpretation is that, if two particles with law of motion (2.1) start from the same position X_0, the second Δ units of time later than the first one, then they will visit the same points in space, the second particle always Δ units of time after the first one.

[5] The concept of trajectory is used also for nonautonomous equations

Proof. a) Let $(J_X, X(\cdot))$ and $(J_Y, Y(\cdot))$ be solutions of (2.1) such that, respectively, $X(t_0) = Y(t_1) = X_0$, for $t_0 \in J_X, t_1 \in J_Y$. Then $t_1 = t_0 + \Delta$, for a $\Delta \in \mathbb{R}$. By Proposition 2.1.3, $J_Y = \{t + \Delta, t \in J_X\}$ and $Y(t + \Delta) = X(t)$ for all t in J_X. Thus, letting T_X and T_Y be the trajectories generated by $X(\cdot)$ and $Y(\cdot)$, respectively:

$$T_Y = \{Y(t), t \in J_Y\} = \{Y(t + \Delta), t \in J_X\} = \{X(t), t \in J_X\} = T_X.$$

b) If the trajectories generated by two solutions are not disjoint, then there exists X_0 in $\mathbb{R}^d$ such that both solutions take the value X_0, possibly at different times. Then by a), these solutions generate the same trajectory. □

Remark 2.1.6. *Note that, for nonautonomous equations, trajectories can intersect. Assume F is $C^1(\Omega)$ or Lipschitz. Let $X : J \to \mathbb{R}^d$ and $Y : J' \to \mathbb{R}^d$ be maximal solutions of (1.3). Then, there could exist $J \ni t_1 \neq t_2 \in J'$ such that $X(t_1) = Y(t_2)$. (Find an example!) This is not in contradiction with Corollary 1.3.8, which is about solutions and not about trajectories!*

2.2 Equilibria

A solution $(J, X(\cdot))$ of (2.1) is *stationary* if $J = \mathbb{R}$ and $X(\cdot)$ is constant. A point $X^* \in \Omega_d$ is an *equilibrium* of (2.1) if $G(X^*) = 0$. The link between these notions is that:

Claim 2.2.1. *If X^* is an equilibrium, then $X(t) = X^*$ for all $t \in \mathbb{R}$ defines a stationary solution. Moreover, if G is C^1 or Lipschitz, then for any $t_0 \in \mathbb{R}$, the stationary solution is the unique maximal solution such that $X(t_0) = X^*$.*

Indeed, if X^* is an equilibrium, we then have that $X(t) = X^*$ for all $t \in \mathbb{R}$ satisfies $X'(t) = 0 = G(X^*) = G(X(t))$ for all $t \in \mathbb{R}$, and uniqueness follows from Picard-Lindelöf theorem. Assuming that G is C^1 or Lipschitz is important: otherwise, as in Example 1.3.3, solutions starting at an equilibrium could go away from it.

Remark 2.2.2. *Equilibria are only defined for autonomous equations. Stationary (= constant) solutions are also defined for nonautonomous equations. For instance, $x(t) = 0$ is a stationary solution of $x'(t) = tx(t)$.*

Equilibria are very important, because they are the only possible (finite) limits of solutions.

Proposition 2.2.3 (Importance of equilibria)**.** *Let $(J, X(\cdot))$ be a solution of $X'(t) = G(X(t))$ with G continuous. Let $X^* \in \mathbb{R}^d$. If $X(t) \to X^*$ as $t \to \sup J$ (or as $t \to \inf J$), then $G(X^*) = 0$. That is, X^* is an equilibrium.*

Proof. Assume that $X(t) \to X^*$ as $t \to \sup J$. By the Explosion alternative, $\sup J = +\infty$. Moreover, since G is continuous, $G(X(t)) \to G(X^*)$ as $t \to \sup J = +\infty$. Since $X'(t) = G(X(t))$, this implies that $X'(t) \to G(X^*)$, hence, letting $Y = G(X^*)$, $X_i'(t) \to Y_i$ for all $i \in \{1, .., d\}$. If $Y \neq 0$, then there exists $i \in \{1, ..., d\}$ such that $Y_i \neq 0$. Assume $Y_i > 0$ (the case $Y_i < 0$ is similar). Since $X_i'(t) \to Y_i$, there exists T such that for all $t \geq T$, $X_i'(t) \geq Y_i/2$. For $t \geq T$,

$$X_i(t) = X_i(T) + \int_T^t X_i'(s)ds \geq X_i(T) + \int_T^t Y_i/2\, ds = X_i(T) + (t - T)Y_i/2.$$

Thus, $X_i(t) \to +\infty$ as $t \to +\infty$, contradicting our assumption $X(t) \to X^*$. Therefore $Y = 0$. □

Remark 2.2.4. *Proposition 2.2.3 shows that if a solution of an autonomous equation converges to a point in $\inf J$ or $\sup J$, then this point is an equilibrium. But solutions of autonomous equation do not necessarily converge: they may grow to infinity, cycle (when $d \geq 2$), or have a more complicated behavior.*

Example 2.2.5. *a) The exponential function is solution on* $\mathbb{R}$ *of the autonomous equation* $x'(t) = x(t)$*. As* $t \to -\infty$*, it converges to* 0 *which, as implied by Proposition 2.2.3, is an equilibrium. But as* $t \to +\infty$*, it grows to infinity.*

b) The function $X : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $X(t) = (\cos t, \sin t)$ *is solution of the autonomous equation* $X'(t) = F(X(t))$ *with* $F(x, y) = (-y, x)$*. It cycles endlessly, and in particular, does not converge to an equilibrium.*

There are different kinds of equilibria. In particular, an equilibrium is *attracting* if nearby solutions move towards it, and *repelling* if nearby solutions go away from it. Formally:

Definition 2.2.6. *Let* X^* *be an equilibrium of (2.1). Then the equilibrium is said to be attracting if it has a neighborhood* V *such that for any solution* $(J, X(\cdot))$ *for which* $X(t_0) \in V$ *for a* $t_0 \in J$*, then* $\sup J = +\infty$ *and* $X(t) \to X^*$ *as* $t \to +\infty$*. The equilibrium is said to be repelling if it has a neighborhood* V *such that for any solution* $(J, X(\cdot))$ *with* $X(t_0) \in V \backslash \{X^*\}$ *for some* $t_0 \in J$*, there exists* $T \in J$ *such that* $X(t) \notin V$ *for all* $t \in J \cap [T, +\infty[$*.*

Other notions of equilibrium stability will be studied in later chapters.

2.3 Autonomous equations in dimension 1

Recall that if the law of motion of a particle is described by an autonomous equation (2.1), then the direction and speed of this particle depends only on its position. It follows that we may define, and draw, the direction of movement of the particle when the particle is placed at X_0. Consider for instance a particle moving through the real line according to the equation $x'(t) = x(t)$. If $x(t) = x_0 = 0$, then its speed is zero: the particle is at rest. If $x(t) = x_0 > 0$, then $x'(t) > 0$: the particle goes forward. If $x(t) = x_0 < 0$, then $x'(t) < 0$: the particle goes backward. We can indicate this information on the *phase line*, that is, the real line seen as the set of possible positions of the particle, by dots for equilibria and arrows for directions of movement.

Example of such drawings will be given in the course. The next proposition shows that, for autonomous equations in dimension 1, they tell us almost everything on the qualitative behavior of solutions. In the following proposition, saying that $x : J \to \mathbb{R}$ increases from $x_1 \in \bar{\mathbb{R}}$ to $x_2 \in \bar{\mathbb{R}}$ means that $x(\cdot)$ is increasing, $x(t) \to x_1$ as $t \to \inf J$ and $x(t) \to x_2$ as $t \to \sup J$.

Proposition 2.3.1. *Consider an autonomous equation in dimension 1:* $x'(t) = g(x(t))$*, with* $g : \mathbb{R} \to \mathbb{R}$ *assumed* C^1 *or Lipschitz. Let* $(t_0, x_0) \in \mathbb{R}^2$*. Let* $(J, x(\cdot))$ *be the solution such that* $x(t_0) = x_0$*. Let* x_+ *be the smallest equilibrium greater than* x_0 *if there is one, and* $x_+ = +\infty$ *otherwise. Let* x_- *be the largest equilibrium smaller than* x_0 *if there is one, and* $x_- = -\infty$ *otherwise.*

1. If $g(x_0) = 0$, that is, if x_0 is an equilibrium, then $J = \mathbb{R}$ and $x(t) = x_0$ for all $t \in \mathbb{R}$.
2. If $g(x_0) > 0$, then $x(t)$ increases from x_- to x_+.
3. If $g(x_0) < 0$, then $x(t)$ decreases from x_+ to x_-.
4. If $x(t)$ has a finite limit in $\sup J$ (resp. $\inf J$), then $\sup J = +\infty$ (resp. $\inf J = -\infty$).

Proof. 1) follows from Claim 2.2.1. For 2), let us first show that $x(t) < x_+$ for all $t \in J$. If $x_+ = +\infty$, this is obvious. If $x_+ < +\infty$, then x_+ is an equilibrium, and the result follows from Claim 2.2.1 and the fact that two distinct solutions cannot cross. Similarly, $x(t) > x_-$ for all $t \in J$. By definition of x_+ and x_-, there are no equilibria in $]x_-, x_+[$, so $g(x)$, being continuous, has a constant sign on this interval. Therefore $g(x(t))$, hence $x'(t)$, has a constant sign. Since

$g(x_0) > 0$, this sign is positive. Therefore, $x(\cdot)$ is increasing and converges as $t \to \sup J$ to some finite or infinite limit x^*, such that $x_0 \leq x^* \leq x_+$ since $x(\cdot)$ is increasing and $x(t) < x_+$ for all t.

Case 1: If $x_+ < +\infty$, then $x(\cdot)$ is bounded from above, so x^* is finite. Therefore, x^* is an equilibrium, due to Proposition 2.2.3; due to the fact that $x_0 \leq x^* \leq x_+$ and that the only equilibrium in this interval is x_+, it follows that $x^* = x_+$.

Case 2: If $x_+ = +\infty$, then there are no equilibria greater than x_0. Therefore, x^* cannot be finite, because as a finite limite x^* should be an equilibrium. Therefore $x^* = +\infty = x_+$. Thus in both cases, $x(t) \to x_+$ as $t \to \sup J$.

The proof of $x(t) \to x_-$ as $t \to \inf J$ is similar, and so is the proof of 3). Finally, point 4) follows from the explosion alternative. □

Exercise 2.3.2. *Show that if x_- and x_+ are finite, that is, if there are equilibria smaller than x_0 and equilibria greater than x_0, then $x(\cdot)$ is bounded and global.*

Example 2.3.3. *Assume $g(x) = x$ (so $x'(t) = x(t)$). The only equilibrium is 0. If $x_0 = 0$, then by 1), $J = \mathbb{R}$ and $x(t) = x_0$ for all t. If $x_0 > 0$, then $g(x_0) > 0$, so by 2), $x(\cdot)$ is strictly increasing, converges to $+\infty$ as $t \to \sup J$ and to 0 as $t \to \inf J$, so that $\inf J = -\infty$ by 4). Theorem 1.5.10 can be applied to prove that $\sup J = +\infty$. If $x_0 < 0$, then $g(x_0) < 0$, so by 3), $x(\cdot)$ is strictly decreasing, converges to $-\infty$ as $t \to \sup J$, and to 0 as $t \to \inf J$, so that $\inf J = -\infty$ by 4). Theorem 1.5.10 can be applied to prove that $\sup J = +\infty$.*

Example 2.3.4. *Consider the logistic growth equation $x'(t) = rx(t)(1 - x(t))$.[6] Then $g(x) = rx(1-x)$. There are two equilibria: 0 and 1. The function g is negative below 0, positive between 0 and 1, and negative again above 1. If $x_0 = 0$ or $x_0 = 1$, then $x(\cdot)$ is stationary. If $x_0 < 0$, then $x(\cdot)$ decreases, converges to $-\infty$ as $t \to \sup J$, and to 0 as $t \to \inf J$ (so $\inf J = -\infty$). If $x_0 \in]0,1[$, then $x(\cdot)$ is bounded and global, and increases from 0 to 1. If $x_0 > 1$, then $x(\cdot)$ decreases from $+\infty$ to 1 (hence $\sup J = +\infty$).*

Proposition 2.3.1 shows that, up to velocity and blow-up considerations, the qualitative behavior of $x'(t) = g(x(t))$ only depends on the sign of g. The method to use this proposition is to study this sign, then draw the phase line accordingly, and "read" it. Note that the phase line does not give information on a single solution, but on all solutions. Different arrows (or dots) correspond to different solutions.

Exercise 2.3.5. *Let x^* be an equilibrium of $x'(t) = g(x(t))$. Show that if there exists $\varepsilon > 0$ such that g is positive on $]x^* - \varepsilon, x^*[$ and negative on $]x^*, x^* + \varepsilon[$, then x^* is attracting. Conclude that, whenever $g'(x_0)$ is well defined, if $g'(x^*) < 0$, then x^* is attracting. Show that if $g'(x^*) > 0$, then x^* is repelling.*

Exercise 2.3.6. *For $x'(t) = x^2(t)$, show that the equilibrium 0 is neither attracting nor repelling.*

Exercise 2.3.7. *Sketch the phase line for $x' = \sin x$ and show that all solutions are global.*

Exercise 2.3.8. *For $x' = x^2(x-1)(x+1)$, which equilibria are attracting? repelling?*

Exercise 2.3.9. *Draw the phase line of $x' = x(1-x) - p$ for $p > 1/4$, $p = 1/4$ and $p < 1/4$.*

[6] In ecology, this models a population whose growth depends on a limited resource. When the population is small, it grows with a growth rate close to r. As it gets larger, the resource becomes limiting and the growth-rate diminishes. When its density is higher than the carrying capacity K, that is, the maximal capacity that the environment may sustain, then its growth rate becomes negative. We chose units so that $K = 1$.

Chapter 3

Explicit solutions in dimension 1

In this chapter we will see some techniques to be used in order to find solutions of differential equations in dimension 1.

3.1 Guessing !

Guessing the solution is a perfectly rigorous way to solve a differential equation. Consider for instance the initial value problem

$$x'(t) = -x^2(t) \quad ; \quad x(0) = 1. \tag{3.1}$$

Since the function $x \to x^2$ is C^1, this problem has a unique maximal solution. Assume that for whatever reason you correctly guess that this solution is the function $u :]-1, +\infty[\to \mathbb{R}$ defined by $u(t) = 1/(t+1)$. Differentiating shows that, indeed, $u'(t) = -1/(t+1)^2 = -u^2(t)$. Checking that $0 \in]-1, +\infty[$ and $u(0) = 1/(0+1) = 1$ then proves that $u(\cdot)$ is a solution of (3.1). Is this solution maximal? Noting that $u(\cdot)$ is defined till $+\infty$ and that $u(t) \to +\infty$ as $t \to -1$ shows that this solution cannot be extended, hence is maximal. Thus, $u(\cdot)$ is the maximal solution of (3.1).

Note that the proof is perfectly rigorous. The only problem is that you first need to guess the solution, and this is not necessarily easy! This being said, if in a test you encounter a question such as "prove that the solution to this initial value problem is this function", that is, if the teacher provides the correct guess, do not venture in complicated methods: just check that the guess is a true solution!

3.2 Separation of variables

Separation of variables in practice. Consider an equation of the form

$$x'(t) = g(x(t))h(t) \tag{3.2}$$

where g and h are real valued functions. Equations that take this form are called "equation with separated variables". (Note that they are not autonomous in general.) They may be solved as follows: without worrying about what this means, write the equation as

$$\frac{dx}{dt} = h(t)g(x)$$

then "separate variables" to get:

$$\frac{dx}{g(x)} = h(t)dt$$

Now integrate

$$\int \frac{dx}{g(x)} = \int h(t)dt$$

to obtain $\Psi(x) = H(t) + K$ where Ψ is a primitive of $1/g$, H a primitive of h, and K a constant. Inverting Ψ and letting $x = x(t)$ leads to the guess that the general solution is

$$x(t) = \Psi^{-1}(H(t) + K) \tag{3.3}$$

which is defined on the maximal interval containing x_0 and such that this expression is well defined. If we want to guess the solution such that $x(t_0) = x_0$, then we just need to choose the constant K appropriately, that is, $K = \Psi(x_0) - H(t_0)$. We then check that our guesses are correct.

Example 3.2.1. *Assume that we want to solve the initial value problem $x'(t) = -x^2(t)$ and $x(0) = 1$ (in this example $g(x) = -x^2$ and $h(t) = 1$). We write*

$$\frac{dx}{dt} = -x^2,$$

then we integrate

$$-\int \frac{dx}{x^2} = \int dt,$$

obtaining

$$\frac{1}{x} = t + K \quad \Rightarrow \quad x(t) = \frac{1}{(t+K)}.$$

Since we want $x(0) = 1$, we choose K such that $1/(0 + K) = 1$, that is, $K = 1$. We thus guess that the solution is given by

$$x(t) = 1/(t + 1)$$

and it is defined on the maximal time interval containing $t_0 = 0$ and such that this expression is well defined, that is, $J =]-1, +\infty[$. We then check that the function $x :]-1, +\infty[\to \mathbb{R}$ defined by $x(t) = 1/(t + 1)$ is indeed solution, is maximal, and the unique maximal solution, as in Section 3.1.

Example 3.2.2. *Assume that we want to find all maximal solutions of the equation $x'(t) = tx(t)$. We write:*

$$\frac{dx}{dt} = tx \quad \Rightarrow \quad \int \frac{dx}{x} = \int tdt \quad \Rightarrow \quad \ln|x| = \frac{t^2}{2} + K \quad \Rightarrow \quad |x| = e^K e^{t^2/2}$$

hence $x(t) = Ce^{t^2/2}$ for some constant C. Since this expression is defined on the whole $\mathbb{R}$, we guess that solutions are the functions $x : \mathbb{R} \to \mathbb{R}$ defined by $x(t) = Ce^{t^2/2}$, with $C \in \mathbb{R}$. Checking that these are indeed solutions is a simple computation which is left to the reader. Moreover, since these solutions are global, they are maximal. It remains to check that there are no other maximal solutions. To do this, let $(t_0, x_0) \in \mathbb{R}^2$, and note that taking $C = x_0 \exp(-t_0^2/2)$ gives a maximal solution such that $x(t_0) = x_0$. Since any maximal solution must take some value x_0 at some time t_0, this shows that all maximal solutions are indeed of the above form.[1]

Why does it work? The method may seem weird, but the following proposition shows that when g and h are regular enough, and when we take care of the case $g(x_0) = 0$, the method always works.

[1]Indeed, let $(J, u(\cdot))$ be a maximal solution, let $t_0 \in J$ and let $x_0 = u(t_0)$. Then for $C_0 = x_0 \exp(-t_0^2/2)$, $(J, u(\cdot))$ and $(\mathbb{R}, t \to C_0 e^{t^2/2})$ are two maximal solutions with the same initial condition, hence they are equal, since the function $(t, x) \to tx$ is C^1. Therefore, any maximal solution is one of those we had found.

Proposition 3.2.3. *Consider the initial value problem $x'(t) = g(x(t))h(t)$ and $x(t_0) = x_0$, with $g : \mathbb{R} \to \mathbb{R}$ and $h : I \to \mathbb{R}$, where I is an open interval of $\mathbb{R}$. Assume g is of class C^1 or Lipschitz, and h continuous so that, by Theorem 1.5.4, this problem has a unique maximal solution $(J, \bar{x}(\cdot))$.*

1) if $g(x_0) = 0$, the solution is stationary: $J = I$ and $\bar{x}(t) = x_0$ for all t in I.

2) if $g(x_0) \neq 0$, then for all t in J, $g(\bar{x}(t)) \neq 0$ and

$$\int_{x_0}^{\bar{x}(t)} \frac{du}{g(u)} = \int_{t_0}^{t} h(s)ds \tag{3.4}$$

Proof. 1) is a variant of Claim 2.2.1. For 2), assume by contradiction that $g(\bar{x}(t_1)) = 0$ for some $t_1 \in J$, and let $x_1 = \bar{x}(t_1)$. Then $g(x_1) = 0$ and $(J, \bar{x}(\cdot))$ is a solution of $x'(t) = g(x(t))h(t)$ and $x(t_1) = x_1$. By Case 1) applied to this initial value problem, this implies that for all t in J, $\bar{x}(t) = x_1$, hence $g(x_0) = g(\bar{x}(t_0)) = g(x_1) = 0$. This contradicts our assumption $g(x_0) \neq 0$. Therefore $g(x(t))$ does not vanish, and we can divide our equation by $g(x(t))$. We obtain that for all t in J,

$$\frac{x'(t)}{g(x(t))} = h(t)$$

so that:

$$\int_{t_0}^{t} \frac{x'(s)}{g(x(s))} ds = \int_{t_0}^{t} h(s)ds$$

The change of variable $u = x(s)$ in the first integral and $x(t_0) = x_0$ then lead to (3.4). $\square$

Some remarks are in order. First, let $\Psi(x) = \int_{x_0}^{x} \frac{du}{g(u)}$. From the fact that g is continuous and that $g(x(t))$ does not vanish, we know that for all t in J, $g(u)$ has a constant sign between x_0 and $x(t)$. It follows that on the domain of interest, Ψ is strictly monotone, hence invertible. So (3.4) indeed leads to (3.3). More precisely, it leads to (3.3) with $K = 0$ because we chose specific primitives Ψ and H: more precisely, such that $\Psi(x_0) = H(t_0)$. If we do not take this precaution, then we still get (3.3) but then we have to fix K using the initial condition.

Second, we see from Proposition 3.2.3 that the case $g(x_0) = 0$ is special and should in principle be separated from the case $g(x_0) \neq 0$. To see what can go wrong, imagine that we want to find the solution of $x(t) = -x^2(t)$ such that $x(0) = 0$. If we proceed as in Example 3.2.1, then once we obtain $x(t) = 1/(t + K)$ we have a problem. Indeed, there is no value of K such that $x(0) = 0$. We may overcome the problem in two ways: by noting from the beginning that the solution is stationary, as follows from point 1) of Proposition 3.2.3, or by noting that to get $x(0) = 0$, we would need to have $K = \infty$. Giving to K an infinite value in $x(t) = 1/(t + K)$ leads to the correct guess that the solution is $x(t) = 0$.[2]

The origin of this difficulty is that when writing $dx/g(x) = h(t)dt$, we implicitly assume that $g(x)$ is nonzero. In Example 3.2.2, we are lucky, as we recover the stationary solution $x(t) = 0$ from the formula $x(t) = Ce^{t^2/2}$ by letting $C = 0$. While, in Example 3.2.1, we must be careful.

Finally, if g is not C^1 or Lipschitz but just continuous, we may still apply the method of separation of variables, but with the following modifications: first, if $g(x_0) = 0$, then there is a stationary solution equal to x_0, but there might also be other solutions. Second, if $g(x_0) \neq 0$, the formula (3.4) is only valid as long as $g(x(t))$ does not vanish. See Example 1.3.3 and Exercise 1.3.4.

Exercise 3.2.4. *Solve the initial value problem $x'(t) = tx(t)$ and $x(1) = 1$.*

[2]We will see later that solutions of differential equations depend continuously on the initial condition. This explains why we may recover the solutions corresponding to the case $g(x_0) = 0$ as limits of solutions corresponding to the case $g(x_0) \neq 0$.

Remark 3.2.5. *An autonomous equation $x'(t) = f(x(t))$ with $f : I \to \mathbb{R}$ of class C^1 is a particular case of equations with separated variables: the case $g(x) = f(x)$ and $h(t) = 1$. Thus, the method of separation of variables allows us to reduce the problem of computing solutions of these equations to a problem of computation of primitives (however this can still be hard to do in some case!).*

Exercise 3.2.6. *Solve (= find explicitly all solutions of) the equation $x'(t) = -x^2(t)$. Similarly, solve the logistic equation $x'(t) = x(t)(1-x(t))$. Compared to the qualitative resolution of Chapter 2, what additional information do we get?*

We conclude with a remark which will be used in Chapter 4.

Remark 3.2.7. *Consider the initial value problem $x'(t) = K|x|^\alpha$ and $x(t_0) = x_0$, with $K > 0$, $\alpha > 0$. The method of separation of variables allows us to compute explicitly the solutions. If $0 < \alpha \leq 1$, then solutions are global, while if $\alpha > 1$, solutions explodes: forward in time if $x_0 > 0$ (they are defined on an interval $]-\infty, T[$ with $T < +\infty$, and go to $+\infty$ as $t \to T$), and backward in time if $x_0 < 0$ (they are defined on an interval $]T, +\infty[$ with $T > -\infty$, and go to $-\infty$ as $t \to T$).*

3.3 Linear differential equations

In this section, I is a nonempty open interval, $t_0 \in I$ and x_0 are real numbers, $a(\cdot)$ and $b(\cdot)$ are continuous functions from I to $\mathbb{R}$, and $A(\cdot)$ is a primitive of $a(\cdot)$.

We want to solve explicitly the nonhomogeneous linear equation

$$x'(t) = a(t)x(t) + b(t). \tag{NH}$$

To do so, we first solve the *homogeneous equation associated* to (NH): [3]

$$x'(t) = a(t)x(t). \tag{H}$$

Proposition 3.3.1. *1) The maximal solutions of (H) are the functions $x : I \to \mathbb{R}$ given by*

$$x(t) = \lambda e^{A(t)} \tag{3.5}$$

with $\lambda \in \mathbb{R}$. Thus, they are global and the set of these solutions is a real vector space of dimension 1.

2) The initial value problem $x'(t) = a(t)x(t)$ and $x(t_0) = x_0$ has a unique maximal solution:

$$x(t) = x_0 \exp\left(\int_{t_0}^t a(s)ds\right).$$

Proof. 1) Let $(J, x(\cdot))$ be a potential solution and let $z(t) = e^{-A(t)}x(t)$. Then, recalling $a = A'$,

$$x' = ax \Leftrightarrow x' - A'x = 0 \Leftrightarrow e^{-A}(x' - A'x) = 0 \Leftrightarrow z' = 0.$$

Thus, $x(\cdot)$ is solution if and only if there exists $\lambda \in \mathbb{R}$ such that, for all t in J, $z(t) = \lambda$, that is, $x(t) = \lambda e^{A(t)}$, and since this defines a solution on the whole of I, maximal solutions are global. Moerover the set of these solutions is a real vector space of dimension 1 generated by the nonzero function $e^{A(t)}$.

2) Let $A(t) = \int_{t_0}^t a(s)ds$. By 1), maximal solutions of (H) are global and given by (3.5), that is $x(t) = \lambda e^{A(t)}$. Moreover, $x(t_0) = x_0$ if and only if $\lambda = x_0$. ∎

[3] Eq. (H) may be written as $x'(t) - a(t)x(t) = 0$ and may be called in French "equation linéaire sans second membre". By contrast, Eq (NH) may be written as $x'(t) - a(t)x(t) = b(t)$ and is thus "avec second membre".

Remark 3.3.2. *Eq.* (H) *may also be seen (and solved) as an equation with separated variables.*

Proposition 3.3.3. *1) Maximal solutions of (NH) are global and they are given by Duhamel's formula:*

$$x(t) = \left(\lambda + \int_{t_0}^{t} b(s)e^{-A(s)}ds\right)e^{A(t)} = \lambda e^{A(t)} + \int_{t_0}^{t} b(s)\exp\left(\int_s^t a(u)du\right)ds \tag{DF}$$

where λ is a real number.

2) The initial value problem $x'(t) = a(t)x(t) + b(t)$ and $x(t_0) = x_0$ has a unique solution, given by Duhamel's formula with $\lambda = x_0$ and $A(t) = \int_{t_0}^{t} a(s)ds$.

3) Let $x_p(\cdot)$ be a solution of (NH). Then the solutions of (NH) are the functions $x : I \to \mathbb{R}$ of the form

$$x(t) = x_p(t) + \lambda e^{A(t)}$$

That is, $x(\cdot)$ is solution of (NH) if and only if there exists a solution $y(\cdot)$ of (H) such that $x(\cdot) = x_p(\cdot) + y(\cdot)$. We say that the general solution of (NH) is the sum of a particular solution of (NH) and of the general solution of the homogenous equation (H) *and we write:*

$$S_{NH} = x_p(\cdot) + S_H$$

where S_{NH} and S_H are the sets of solutions of (NH) *and* (H)*, respectively.*

4) The solutions of (NH) form an affine subspace of dimension 1 *whose direction is the set of solutions of the homogeneous equation* (H)*.*

Proof. To prove 1), we start from the general solution of the associated homogeneous equation: $x(t) = \lambda e^{A(t)}$ and *we let the constant λ vary*! That is, letting J be a subinterval of I and $x : J \to \mathbb{R}$ be differentiable, we define $\lambda(t) = x(t)e^{-A(t)}$ so that

$$x(t) = \lambda(t)e^{A(t)} \tag{3.6}$$

and we look for conditions on $\lambda(\cdot)$ so that (3.6) is a solution of (NH): an example of *change of variables.* We find that:

$$x' = ax + b \Leftrightarrow \lambda' e^A + \lambda a e^A = a\lambda e^A + b \Leftrightarrow \lambda' e^A = b \Leftrightarrow \lambda' = be^{-A}.$$

Thus, letting $t_0 \in J$, $x(\cdot)$ is solution if and only if

$$\lambda(t) = \lambda(t_0) + \int_{t_0}^{t} b(s)e^{-A(s)}ds \qquad \forall t \in J.$$

Multiplying by $e^{A(t)}$ implies that $x(\cdot)$ is solution if and only if it is of the form (DF). Finally, since (DF) is defined on the whole I, maximal solutions are global. Note that the constant λ appearing in the statement of Duhamel's formula corresponds to $\lambda(t_0)$ in the proof.

Proof of 2). Let $A(t) = \int_{t_0}^{t} a(s)ds$. Then by 1), solutions of (NH) are given by (DF). Moreover, $x(t_0) = x_0$ if and only if $\lambda = x_0$.

Proof of 3) and 4). Let $x_p : I \to \mathbb{R}$ be solution of (NH), let $x : I \to \mathbb{R}$ be differentiable and let $y(\cdot) = x(\cdot) - x_p(\cdot)$ so that $x(\cdot) = x_p(\cdot) + y(\cdot)$. Then $x' = x_p' + y' = ax_p + b + y'$. Therefore:

$$x' = ax + b \Leftrightarrow ax_p + b + y' = a(x_p + y) + b \Leftrightarrow y' = ay.$$

This proves 3), and 3) implies 4). □

Remark 3.3.4.

a) If we let $x(\cdot)$ be given by (DF) *and add a solution $y(\cdot) = \mu e^{A(\cdot)}$ of (H), then $z(\cdot) = x(\cdot) + y(\cdot)$ is still of the form* (DF)*, with λ replaced by $\tilde{\lambda} = \lambda + \mu$. Thus Duhamel's formula is coherent with 3).*

b) You are allowed to used Duhamel's formula directly, but you do not have to know it by heart (the author of these lines does not know it). However, you should be able to find it by searching solutions of the form $x(t) = \lambda(t)e^{A(t)}$, as we did: a method called "variation of constants".

c) The fact that (NH) has a unique maximal solution such that $x(t_0) = x_0$ also follows from Theorem 1.5.4 (the strong version of Picard-Lindelöf).

d) The above results are valid for linear *equations. An equation such as $x'(t) = a(t)x^2(t)+b(t)$ is not linear, because of the term $x^2(t)$. Thus, the above results do not say anything on this equation.*

Remark 3.3.5. *To solve an initial value problem, it may be useful to proceed in two steps: first, find all solutions of the underlying differential equation (without taking into account the initial condition). This solution depends on one or several constants. Second, determine the value of these constants in order to satisfy the initial condition.*

Superposition principle A first way to solve (NH) is to apply (DF) or, equivalently, to recover it from (3.6). A second way is to guess a particular solution $x_p(\cdot)$ and then to add to this solution the general solution of (H). We will see later how to guess particular solutions when $b(\cdot)$ has a simple form (polynomial, exponential,...), however note that when $b(\cdot)$ is the sum of two functions for which we know a particular solution, we may apply the following trick:

Proposition 3.3.6 (Superposition principle)**.** *If $x_1(\cdot)$ is solution of $x' = ax + b_1$ and $x_2(\cdot)$ of $x' = ax + b_2$, then $x = x_1 + x_2$ is solution of $x' = ax + (b_1 + b_2)$.*

Proof. Under the assumptions of the proposition, $x' = x_1' + x_2' = (ax_1 + b_1) + (ax_2 + b_2) = a(x_1 + x_2) + (b_1 + b_2) = ax + (b_1 + b_2)$. □

We end this section with some worked-out exercises. Note that since we know that maximal solutions of (NH) are global, we are only interested in the expression of $x(t)$. If this were not the case, we would need to specify the interval over which the solutions are defined.

Example 3.3.7. *Solve (1) $x'(t) = -x(t)+t$; (2) $x'(t) = -x(t)+e^t$, and (3) $x'(t) = -x(t)+t+e^t$.*

Solution: The general solution of $x'(t) = -x(t)$ is $x(t) = \lambda e^{-t}$. Moreover, $x_1(t) = \alpha t + \beta$ is solution of (1) if and only if $\alpha = -(\alpha t + \beta) + t$, hence if $\alpha = 1$ and $\beta = -1$. Thus, a particular solution of (1) is $x_1(t) = t - 1$ and the general solution of (1) is $x(t) = t - 1 + \lambda e^{-t}$.

Similarly, $x_2(t) = \lambda e^t$ is solution of (2) if and only if $\lambda = -\lambda + 1$, that is, $\lambda = 1/2$, thus $x_2(t) = \frac{e^t}{2}$ is a particular solution of (2) and the general solution of (2) is $x(t) = \frac{e^t}{2} + \lambda e^{-t}$.

Finally, by the superposition principle, $x_p(t) = x_1(t) + x_2(t)$ is solution of (3), hence the general solution of (3) is $x(t) = t - 1 + \frac{1}{2}e^t + \lambda e^{-t}$.

Example 3.3.8. *Solve the initial value problem (IVP): $x'(t) = 2tx(t)$ and $x(1) = 1$.*

Solution: a primitive of $a(t) = 2t$ is $A(t) = t^2$, thus the general solution of $x'(t) = 2tx(t)$ is $x(t) = \lambda e^{t^2}$. It satisfies the initial condition $x(1) = 1$ if and only if $\lambda e = 1$, that is, $\lambda = 1/e$. Thus the solution of (IVP) is $x(t) = \frac{1}{e}e^{t^2} = e^{t^2-1}$.

Example 3.3.9. *Let a and b be real numbers (not functions!), with $a \neq 0$. Solve the initial value problem (IVP): $x'(t) = ax(t) + b$ and $x(t_0) = x_0$.*

Solution 1: let us first solve $x'(t) = ax(t)+b$, without the initial condition. A particular solution is the stationary solution $x_p(t) = -b/a$. The general solution of the associated homogeneous equation is $y(t) = \lambda e^{at}$. Thus the general solution of $x'(t) = ax(t) + b$ is $x(t) = -\frac{b}{a} + \lambda e^{at}$. It satisfies $x(t_0) = x_0$ if and only if $-\frac{b}{a} + \lambda e^{at_0} = x_0$, that is $\lambda = (x_0 + \frac{b}{a})e^{-at_0}$. Thus the solution of (IVP) is

$$x(t) = -\frac{b}{a} + \left(x_0 + \frac{b}{a}\right) e^{-at_0}e^{at} = -\frac{b}{a} + \left(x_0 + \frac{b}{a}\right) e^{a(t-t_0)}.$$

Solution 2: the general solution of $x'(t) = ax(t)$ is $x(t) = \lambda e^{at}$, thus we look for a solution of (IVP) of the form $x(t) = \lambda(t)e^{at}$. We find that:

$$x'(t) = ax(t) + b \Leftrightarrow \lambda'(t)e^{at} + a\lambda(t)e^{at} = a\lambda(t)e^{at} + b \Leftrightarrow \lambda'(t) = be^{-at}$$

Thus $x(\cdot)$ is solution of the equation if and only if there exists $\lambda_0 \in \mathbb{R}$ such that

$$\lambda(t) = \lambda_0 + \int_{t_0}^{t} be^{-as}ds = \lambda_0 - \frac{b}{a}\left[e^{-as}\right]_{t_0}^{t} = \lambda_0 - \frac{b}{a}(e^{-at} - e^{-at_0}) \tag{3.7}$$

Hence

$$x(t) = \lambda(t)e^{at} = \left[\lambda_0 - \frac{b}{a}(e^{-at} - e^{-at_0})\right] e^{at}$$

Moreover it satisfies $x(t_0) = x_0$ if and only if $\lambda_0 e^{at_0} = x_0$, that is, $\lambda_0 = x_0 e^{-at_0}$. Thus, we get that the solution of (IVP) is

$$x(t) = x_0 e^{a(t-t_0)} - \frac{b}{a} + \frac{b}{a}e^{a(t-t_0)} = -\frac{b}{a} + \left(x_0 + \frac{b}{a}\right) e^{a(t-t_0)}.$$

Solution 3: According to point 2) of Proposition 3.3.3, the solution is given by

$$x(t) = \left(x_0 + \int_{t_0}^{t} be^{-A(s)}ds\right) e^{A(t)}$$

with $A(t) = \int_{t_0}^{t} ads = a(t - t_0)$. Thus,

$$x(t) = \left(x_0 + \int_{t_0}^{t} be^{-a(s-t_0)}ds\right) e^{a(t-t_0)} = x_0 e^{a(t-t_0)} + \left(\int_{t_0}^{t} be^{-as}ds\right) e^{at}.$$

Moreover,

$$\int_{t_0}^{t} be^{-as}ds = -\frac{b}{a}\left[e^{-as}\right]_{t_0}^{t} = -\frac{b}{a}(e^{-at} - e^{-at_0})$$

Therefore

$$x(t) = x_0 e^{a(t-t_0)} - \left[\frac{b}{a}(e^{-at} - e^{-at_0})\right] e^{at} = -\frac{b}{a} + \left(x_0 + \frac{b}{a}\right) e^{a(t-t_0)}.$$

Note that if $a < 0$, then for all solutions, $x(t) \to -b/a$ as $t \to +\infty$.

Chapter 4

Comparison principles

Introduction. This chapter shows that we can learn a great deal by comparing the behavior of solutions of equations, we do not know how to solve, to the behavior of solutions we can solve. Let $f : \Omega \to \mathbb{R}$, where Ω is a nonempty open in $\mathbb{R} \times \mathbb{R}$, and consider the differential equation

$$x'(t) = f(t, x(t)), \tag{4.1}$$

A function $u : J \to \mathbb{R}$ is a *subsolution* of (4.1) if, for all t in J, $u'(t) \leq f(t, u(t))$, and a *supersolution* if, for all t in J, $u'(t) \geq f(t, u(t))$. Except for times t such that $u(t) = v(t)$, the fact that u is a subsolution and v a supersolution of (4.1) does not imply that $u'(t) \leq v'(t)$. Indeed:

Example 4.0.1. *$u(t) = e^t$, $x(t) = 0$ and $v(t) = -e^t$ are respectively subsolution, solution and supersolution of $x'(t) = 2x(t)$. But at all times t in $\mathbb{R}$, $u'(t) > x'(t) > v'(t)$.*

Nevertheless, we will prove that under some regularity conditions, if a subsolution starts below a supersolution, then it remains below at all later times. We will use such comparison principles as follows: assume that we do not know how to solve (4.1) but we know how to solve

$$x'(t) = g(t, x(t)) \tag{4.2}$$

for some function $g \geq f$. Let $u : J \to \mathbb{R}$ be solution of (4.1). Then for all t in J,

$$u'(t) = f(t, u(t)) \leq g(t, u(t)),$$

thus u is subsolution of (4.2). Therefore, under some regularity conditions, for all $t \geq t_0$, $u(t) \leq v(t)$ where v is the solution of (4.2) such that $u(t_0) = v(t_0)$. Since we assumed that we can compute v explicitly, this provides an explicit upper bound for u, which allows to better understand its qualitative behavior. Similarly, for $t \geq t_0$, lower bounds for u may be obtained by using that u is a supersolution of a differential equation.

4.1 Comparison principles

Conventions: I and $J \subset I$ are nonempty open intervals of $\mathbb{R}$, the function $f : I \times \mathbb{R} \to \mathbb{R}$ is at least continuous (we need at least existence of solutions!), t_0 is an element of I, and all functions are defined on I or a subinterval of I containing t_0. When we say that a function v is greater than u for all $t \geq t_0$, we mean that this holds for all times $t \geq t_0$ such that $u(t)$ and $v(t)$ are well defined. Finally, when we consider subsolutions or supersolutions of a differential equation, we implicitly assume that these functions are differentiable on the interior of the interval on which they are defined.

Proposition 4.1.1 (Linear comparison principle)**.** *Let $a(\cdot)$ and $b(\cdot)$ be continuous. Let u be a subsolution and v a supersolution of the linear equation $x'(t) = a(t)x(t) + b(t)$. If $u(t_0) \leq v(t_0)$ then $u(t) \leq v(t)$ for all $t \geq t_0$.*

Proof. Assume $u(t_0) \leq v(t_0)$. Then $w = v - u$ satisfies $w(t_0) \geq 0$. Moreover:

$$w' = v' - u' \geq (av + b) - (au + b) = a(v - u) = aw$$

hence $w' - aw \geq 0$. Therefore, if $A(\cdot)$ is a primitive of $a(\cdot)$ and $z(t) = e^{-A(t)}w(t)$:

$$z'(t) = e^{-A(t)}(w'(t) - a(t)w(t)) \geq 0.$$

Thus, for all $t \geq t_0$, $e^{-A(t)}w(t) \geq e^{-A(t_0)}w(t_0) \geq 0$, hence $w(t) \geq 0$. That is, $u(t) \leq v(t)$. □

Remark 4.1.2. *A solution is both a subsolution and a supersolution, so Proposition 4.1.1 also allows to compare solutions to subsolutions (or solutions to supersolutions). Besides, the condition $u(t_0) \leq v(t_0)$ is of course satisfied when $u(t_0) = v(t_0)$.*

Remark 4.1.3 (An important particular case, classically known as Gronwall's Lemma)**.** *If $u'(t) \leq a(t)u(t)$, then $u(t) \leq u(0)e^{\int_0^t a(s)ds}$ for all $t \geq 0$. Indeed, the function defined by $v(t) = u(0)e^{\int_0^t a(s)ds}$ is the unique solution of $x'(t) = a(t)x(t)$, and $v(0) \geq u(0)$.*

Exercise 4.1.4. *Under the same assumptions of Proposition 4.1.1, show that if $u(t_0) = v(t_0)$, then $u(t) \leq v(t)$ for all $t \geq t_0$, and $u(t) \geq v(t)$ for all $t \leq t_0$. That is, in a weak sense, u and v cross at t_0.*

Proposition 4.1.5 (Comparison principle)**.** *Let f be C^1. Let u be a subsolution and v a supersolution of $x'(t) = f(t, x(t))$. If $u(t_0) \leq v(t_0)$, then $u(t) \leq v(t)$ for all $t \geq t_0$.*

Proof. The proof relies on the linear comparison principle. Let $w(t) = v(t) - u(t)$. Let

$$a(t) = \begin{cases} \frac{f(t,v(t)) - f(t,u(t))}{v(t)-u(t)} & \text{if } u(t) \neq v(t) \\ \frac{\partial f}{\partial x}(t, u(t)) & \text{if } u(t) = v(t). \end{cases} \tag{4.3}$$

Note that for all t, $f(t, v(t)) - f(t, u(t)) = a(t)w(t)$. Therefore,

$$w'(t) = v'(t) - u'(t) \geq f(t, v(t)) - f(t, u(t)) = a(t)w(t)$$

hence w is supersolution of $x'(t) = a(t)x(t)$. We claim that $a(\cdot)$ is continuous. Therefore, by the linear comparison principle (with $b(\cdot) = 0$), we get that for all $t \geq t_0$, $w(t) \geq x(t)$ where $x(\cdot)$ is the solution of $x'(t) = a(t)x(t)$ such that $x(t_0) = w(t_0)$, that is, $x(t) = w(t_0)\exp(\int_{t_0}^t a(s)ds)$. Note that $x(t_0) \geq 0$ since $w(t_0) = v(t_0) - u(t_0) \geq 0$ by assumption. Therefore, for all $t \geq t_0$, $w(t) \geq x(t) \geq 0$ hence $v(t) \geq u(t)$.

We still need to prove that $a(\cdot)$ is continuous at all times t^* of its domain. If $u(t^*) \neq v(t^*)$, this is clear because in the neighborhood of t^*, $a(\cdot)$ is given by (4.3). If $u(t^*) = v(t^*)$, the proof is as follows: for all t such that $u(t) \neq v(t)$, $a(t)$ is the average slope of $f(t, \cdot)$ between $u(t)$ and $v(t)$, thus there exists $c(t)$ between $u(t)$ and $v(t)$ such that

$$a(t) = \frac{\partial f}{\partial x}(t, c(t))$$

By (4.3), this is also the case if $u(t) = v(t)$ (with $c(t) = u(t)$). Since $c(t)$ is between $u(t)$ and $v(t)$, and $u(t^*) = v(t^*)$, it follows that $c(t) \to u(t^*)$ as $t \to t^*$. Since f is C^1, this implies that:

$$a(t) = \frac{\partial f}{\partial x}(t, c(t)) \to_{t \to t^*} \frac{\partial f}{\partial x}(t^*, u(t^*)) = a(t^*)$$

hence $a(\cdot)$ is continuous at t^*. This concludes the proof. □

Remark 4.1.6. *The assumption f of class C^1 is not fully needed: we only need that the partial derivative with respect to the second variable (that is, x) is jointly continuous.*

Exercise 4.1.7 (Variants). *Under the same assumptions of Proposition 4.1.5, show that:*

i) if $u(t_0) \geq v(t_0)$, then $u(t) \geq v(t)$ for all $t \leq t_0$;

ii) if $u(t_0) = v(t_0)$, then $u(t) \leq v(t)$ for all $t \geq t_0$ and $u(t) \geq v(t)$ for all $t \leq t_0$;

iii) if $u(t_0) < v(t_0)$, then $u(t) < v(t)$ for all $t \geq t_0$. Hint: introduce the solutions $x(\cdot)$ and $y(\cdot)$ of $x'(t) = f(t, x(t))$ such that, respectively, $x(t_0) = u(t_0)$ and $y(t_0) = v(t_0)$, and use both Picard-Lindelöf Theorem and Proposition 4.1.5.

Proposition 4.1.5 implicitly requires that the functions u and v are differentiable[1], but often we want to have information on functions that are only continuous. Thus, we would like to have a version of the comparison principle for functions that are only continuous. To do so, recall from the proof of the Picard-Lindelöf Theorem that the initial value problem

$$x'(t) = f(t, x(t)) \text{ and } x(t_0) = x_0 \tag{4.4}$$

may be written in integral form:

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \tag{4.5}$$

More precisely, the reader can check that (we already gave some details in the proof of Picard-Lindelöf Theorem):

Claim 4.1.8. *A function $x : J \to \mathbb{R}$ is differentiable and solve (4.4) if and only if it is continuous and a solution of (4.5).*

The idea of the following result is to replace the assumption that u is subsolution of $x'(t) = f(t, x(t))$ by a similar condition on the integral form (4.5).

Proposition 4.1.9 (Integral form of the comparison principle). *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be C^1 and increasing in its second variable. Let $u : J \to \mathbb{R}$ be continuous. If for all $t \geq t_0$,*

$$u(t) \leq K + \int_{t_0}^{t} f(s, u(s)) ds \tag{4.6}$$

then for all $t \geq t_0$, $u(t) \leq v(t)$ where $v(\cdot)$ is the solution of $v'(t) = f(t, v(t))$ such that $v(t_0) = K$.

Proof. Let $U(t) = K + \int_{t_0}^{t} f(s, u(s))\, ds$. For all $t \geq t_0$, $U(t) \geq u(t)$ and

$$U'(t) = f(t, u(t)) \leq f(t, U(t))$$

since f is increasing in its second variable. Therefore, U is a subsolution of $x'(t) = f(t, x(t))$. Moreover, $U(t_0) = K = v(t_0)$. Thus, by the comparison principle and by definition of v, for all $t \geq t_0$, $U(t) \leq v(t)$, hence $u(t) \leq v(t)$. □

Remark 4.1.10. *Taking $t = t_0$ in (4.6) shows that, necessarily, $K \geq u(t_0)$. In most applications, we actually take $K = u(t_0)$.*

Note that, if u is subsolution of $x'(t) = f(t, x(t))$, then u satisfies (4.6) for all $K \geq u(t_0)$, this proof is left to the reader. Thus, the assumptions of the integral form of the comparison principle are stronger than those of Proposition 4.1.5 in two ways: first, for a differentiable function, (4.6) is more demanding than $u'(t) \leq f(t, u(t))$; second, and more importantly, f must be increasing in its second variable. The upside is that the integral form applies to continuous functions u, while Proposition 4.1.5 does not.

Finally, a variant of the proof of Proposition 4.1.9 shows that the result still holds if $f(t, x) = a(t)x + b(t)$, with $a(\cdot)$ and $b(\cdot)$ continuous, and $a(\cdot)$ nonnegative. We thus get:[2]

[1] Indeed, the definition of u subsolution and v supersolution requires that these functions are differentiable.

[2] The reason why this is not a direct corollary of Proposition 4.1.9 is that f is not C^1.

Proposition 4.1.11 (Integral form of the linear comparison principle)**.** *Let $a(\cdot)$ and $b(\cdot)$ be continuous, with $a(\cdot)$ nonnegative. Let $u : J \to \mathbb{R}$ be continuous. If for all $t \geq t_0$,*

$$u(t) \leq K + \int_{t_0}^{t} (a(s)u(s) + b(s))ds \tag{4.7}$$

then for all $t \geq t_0$, $u(t) \leq v(t)$ where v is the solution of $v' = av + b$ such that $v(t_0) = K$.

Remark 4.1.12 (Integral form of the classical Gronwall's Lemma)**.** *In many applications, $b(\cdot) = 0$. We then get: $u(t) \leq v(t) = K \exp(\int_{t_0}^{t} a(s)ds)$.*

Vocabulary: comparisons principles are also known as *Gronwall's Inequalities* (or *Gronwall's Lemmas*), in honor of the Swedish mathematician Thomas Hakon Grönwall (1877-1932).

4.2 Applications to blow-up and to perturbation analysis

Explosion and non-explosion

A corollary of the integral form of the linear comparison principle is that if, asymptotically, $||F(t, X)||$ grows less than linearly in $||X||$, then all solutions of $X'(t) = F(t, X(t))$ are global:

Corollary 4.2.1. *Let $a(\cdot)$ and $b(\cdot)$ be continuous functions from I to $\mathbb{R}$, with $a(\cdot)$ nonnegative.*

a) Let $X : J \to \mathbb{R}^d$ be C^1. If for all $t \geq t_0$,

$$||X'(t)|| \leq a(t)||X(t)|| + b(t)$$

then for all $t \geq t_0$, $||X(t)|| \leq v(t)$ where v is solution of $v'(t) = a(t)v(t) + b(t)$ and $v(t_0) = ||X(t_0)||$.

b) Let $F : I \times \mathbb{R}^d \to \mathbb{R}^d$ be C^1 or Lipschitz, and such that for all $(t, X) \in I \times \mathbb{R}^d$,

$$||F(t, X)|| \leq a(t)||X|| + b(t).$$

Then all solutions of $X'(t) = F(t, X(t))$ are global.

Proof. a) Since $X(t) = X(t_0) + \int_{t_0}^{t} X'(s)ds$, for all $t \geq t_0$,

$$||X(t)|| \leq ||X(t_0)|| + \int_{t_0}^{t} ||X'(s)||ds \leq ||X(t_0)|| + \int_{t_0}^{t} [a(s)||X(s)|| + b(s)]ds$$

Thus, the result follows from Proposition 4.1.11 applied to $u(t) = ||X(t)||$ and $K = ||X(t_0)||$.

b) Let $X = J \to \mathbb{R}^d$ be solution of $X'(t) = F(t, X(t))$. For all t, $||X'(t)|| = ||F(t, X(t))|| \leq a(t)||X(t)|| + b(t)$. Therefore, by a), for all $t \geq t_0$, $||X(t)|| \leq v(t)$, where v is solution of $v'(t) = a(t)v(t) + b(t)$ and $v(t_0) = ||X(t_0)||$. But $v(\cdot)$ is global, thus the non-explosion criterion (with $g(t) = v(t)$) implies that $\sup J = \sup I$. The proof that $\inf J = \inf I$ is similar and we omit it (the idea is to first prove an analog of a) backward in times, see the exercise below, and then to apply the same reasoning). □

Remark 4.2.2. *Note that, if for all $(t, X) \in I \times \mathbb{R}^d$ we have $||F(t, X)|| \leq a(t)||X|| + b(t)$, then $a(t)$ is necessarily nonnegative for all $t \in I$. Indeed, suppose there exists $t \in I$ such that $a(t) < 0$, then taking the limit for $\|X\| \to +\infty$ we should have $0 \leq \lim_{\|X\| \to +\infty} ||F(t, X)|| \leq -\infty$ a contraddiciton.*

Remark 4.2.3. *Theorem 1.5.10 is a particular case of the above corollary (where $a(\cdot), b(\cdot)$ are taken constant).*

Exercise 4.2.4. *Under the same assumptions of point a) of Corollary 4.2.1, but for $t \leq t_0$, show that for all $t \leq t_0$, $||X(t)|| \leq w(t)$ where w is solution of $w'(t) = -a(t)w(t) - b(t)$ and $w(t_0) = ||X(t_0)||$. Hint: let $Y(t) = X(-t)$ and $\tau_0 = -t_0$; apply a) to $Y(\cdot)$ to show that for all $t \geq \tau_0$, $||Y(t)|| \leq v(t)$ where v is solution of $v'(t) = a(-t)v(t) + b(t)$ and $v(\tau_0) = ||Y(\tau_0)||$; then let $w(t) = v(-t)$ and find the differential equation satisfied by w.*

In dimension 1, comparison principles may also be used to show that solutions of some differential equations explode.

Corollary 4.2.5 (Blow-up in dimension 1)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Let $x : J \to \mathbb{R}$ be a maximal solution of $x'(t) = f(x(t))$ such that $x(t) \to +\infty$ as $t \to \sup J$. If there exists $\alpha > 1$, and $\bar{x} > 0$ such that $f(x) \geq x^\alpha$ for all $x \geq \bar{x}$, then $\sup J < +\infty$.*

Proof. Since $x(t) \to +\infty$ as $t \to \sup J$, there exists $t_0 \in J$ such that for all $t \in [t_0, \sup J[$, $x(t) \geq \bar{x}$, hence $x'(t) = f(x(t)) \geq x^\alpha(t)$. Therefore, by the comparison principle, for all $t \geq t_0$ such that these functions are well defined, $x(t) \geq v(t)$ where $v(\cdot)$ is solution of $v(t) = v^\alpha(t)$ and $v(t_0) = x(t_0) \geq \bar{x} > 0$. But it follows from Remark 3.2.7 that $v(\cdot)$ goes to $+\infty$ in finite time. Therefore, $\sup J < +\infty$.[3] □

Exercise 4.2.6. *Prove an analogue of this result for non autonomous differential equations.*

Exercise 4.2.7. *Prove an analogue of Corollary 4.2.5 backward in time.*

Example 4.2.8 (Forward in time blow-up)**.** *Let $x_0 > 1$. Let $(J, v(\cdot))$ be a solution of the initial value problem $x' = x^2 - 1$ and $x(0) = x_0$. We have: $\sup J < +\infty$.*

Proof. Let $f(x) = x^2 - 1$. Since $f(x_0) > 0$ and since there are no equilibria greater than x_0, it follows from Proposition 2.3.1 (on autonomous equations in dimension 1) that $v(t) \to +\infty$ as $t \to \sup J$. Moreover, in $+\infty$, $f(x) \sim x^2$, therefore, for all x large enough, $f(x) \geq x^\alpha$ for $\alpha = 3/2 > 1$. By Corollary 4.2.5, this implies that $\sup J < +\infty$. □

Example 4.2.9 (Backward in time blow-up)**.** *Let $x_0 > 1$. Let $(J, v(\cdot))$ be the solution of the logistic equation $x' = x(1-x)$ and $x(0) = x_0$. We have: $\inf J > -\infty$.*

Proof. Write $J =]a, b[$. Since we want to use Corollary 4.2.5, which is a result forward in time, define $y : \tilde{J} =]-b, -a[\to \mathbb{R}$ by $y(t) = x(-t)$. We have:

$$y'(t) = -x'(-t) = -x(-t)(1 - x(-t)) = -y(t)(1 - y(t)) = y^2(t) - y(t)$$

and $y(0) = x_0 > 1$. The same reasoning as in Example 4.2.8 shows that $\sup \tilde{J} < +\infty$. Therefore $-a < +\infty$, hence $a > -\infty$, that is, $\inf J > -\infty$. □

Perturbation analysis.

Comparison principles are useful to show that a perturbed equation behaves asymptotically as the underlying unperturbed equation.

Example 4.2.10 (Vanishing perturbation)**.** *Let $(J, x(\cdot))$ be a solution of $x'(t) = -x(t) + \varepsilon(t)$ with the perturbation term $\varepsilon : \mathbb{R} \to \mathbb{R}$ continuous and such that $\varepsilon(t) \to 0$ as $t \to +\infty$. Show that $x(t) \to 0$ as $t \to +\infty$, as in the unperturbed equation $x'(t) = -x(t)$.*

[3] Remark 3.2.7 considers the equation $x'(t) = |x(t)|^\alpha$, not $x'(t) = x^\alpha(t)$, but since we are interested in a solution that is positive for all large enough times, this is not an issue.

Proof. First note that $\sup J = +\infty$ since the equation is linear. Now let $\eta > 0$. Since $\varepsilon(t) \to 0$ as $t \to +\infty$, there exists $T \in \mathbb{R}$ such that, for all $t \geq T$, $|\varepsilon(t)| \leq \eta$, hence:

$$\forall t \geq T, \quad -x(t) - \eta \leq x'(t) \leq -x(t) + \eta \tag{4.8}$$

Therefore, on $[T, +\infty[$, $x(\cdot)$ is subsolution of $x' = -x + \eta$ and supersolution of $x' = -x - \eta$. Hence, by the linear comparison principle, for all $t \geq T$,

$$y(t) \leq x(t) \leq z(t) \tag{4.9}$$

where $y(\cdot)$ is the solution of $x' = -x - \eta$ such that $y(T) = x(T)$ and $z(\cdot)$ the solution of $x' = -x + \eta$ such that $z(T) = x(T)$. By Proposition 2.3.1 (on autonomous equations in dimension 1), in $+\infty$, $y(t) \to -\eta$ and $z(t) \to \eta$. Thus, taking liminf and limsup in (4.9), we get:

$$-\eta \leq \liminf_{t \to +\infty} x(t) \leq \limsup_{t \to +\infty} x(t) \leq +\eta$$

Since this holds for all $\eta > 0$, it follows that $x(t) \to 0$ as $t \to +\infty$. □

Exercise 4.2.11 (Bounded perturbation)**.** *Let $(J, x(\cdot))$ be a solution of $x'(t) = -x(t) + \varepsilon(t)$ with $\varepsilon : \mathbb{R} \to \mathbb{R}$ continuous and such that $\limsup_{t \to +\infty} |\varepsilon(t)| \leq K$. Show that $\limsup_{t \to +\infty} |x(t)| \leq K$.*

Exercise 4.2.12 (Vanishing perturbation for a nonlinear equation)**.** *(∗) Let $(J, x(\cdot))$ be a solution of $x'(t) = -x^3(t) + \varepsilon(t)$ with $\varepsilon : \mathbb{R} \to \mathbb{R}$ continuous and such that $\varepsilon(t) \to 0$ as $t \to +\infty$. Show that $\sup J = +\infty$ and $x(t) \to 0$ as $t \to +\infty$. Hint in this note.*[4]

In the above examples and exercises, all solutions of the unperturbed equation converge to the same limit, irrespective of the initial condition. Otherwise, the method is still useful but does not allow to pinpoint precisely the asymptotic behavior of $x(t)$.

Exercise 4.2.13 (unperturbed equation with several possible limit points)**.** *(∗)*
Let $(J, x(\cdot))$ be solution of $x'(t) = x(t)(1 - x(t)) + \varepsilon(t)$ with $\varepsilon : \mathbb{R} \to \mathbb{R}$ continuous and such that $\varepsilon(t) \to 0$ as $t \to +\infty$. Let $t_0 \in J$. We want to show that as $t \to \sup J$, $x(t)$ has a finite or infinite limit, which is $-\infty$, 0 or 1, and that moreover, $\sup J < +\infty$ in the first case, and $\sup J = +\infty$ in the two latter cases.

1. Assume $x(t_0) > 0$. Why can't we be sure that $x(t) \geq 0$ for all $t \geq t_0$? More generally, why knowing $x(t_0)$ is not helpful?

2a. Show that for all t in J, $x'(t) \leq x(t) + \varepsilon(t)$.

2b. Let (J_V, v) denote the solution of $v'(t) = v(t) + \varepsilon(t)$ and $v(t_0) = x(t_0)$. Show that $J_v = \mathbb{R}$ and that for all t in $[t_0, \sup J[$, $x(t) \leq v(t)$.

2c. Show that if $\sup J < +\infty$, $x(t) \to -\infty$ as $t \to \sup J$.

3. In questions 3), 4) and 5), we assume $\sup J = +\infty$. Let $\delta \in]0, 1/4[$.

3a. Show that there exists $T_\delta \geq t_0$ such that for all $t \geq T_\delta$,

$$x(t)(1 - x(t)) - \delta \leq x'(t) \leq x(t)(1 - x(t)) + \delta$$

3b. Denote by x_δ^- and x_δ^+ the smallest and largest roots of $x(1 - x) + \delta$, and give the phase portrait of $w'(t) = w(t)(1 - w(t)) + \delta$.

3c. Show that as $t \to +\infty$, $\limsup x(t) \leq x_\delta^+$, and then that $\limsup x(t) \leq 1$ (hint: $x_\delta^- \to 1$ as $\delta \to 0$).

3d. Show that if $\liminf x(t) < x_\delta^-$, then $x(t) \to -\infty$ as $t \to +\infty$. Deduce that if $\liminf x(t) < 0$, then $x(t) \to -\infty$ as $t \to +\infty$ (hint: $x_\delta^- \to 0$ as $\delta \to 0$).

[4] The difficulty is that the equation is no longer linear. Note however that when $x(t) \geq 0$, $x'(t) \leq \varepsilon(t)$ and if $x(t) \leq 0$, then $x'(t) \geq \varepsilon(t)$. Use this to prove that $\sup J = +\infty$. Then replace the linear comparison principle with the comparison principle and Remark 4.1.6.

4a. Denote by y_δ^- and y_δ^+ the smallest and largest roots of $x(1-x)+\delta$, and give the phase portrait of $u'(t) = u(t)(1-u(t)) - \delta$.

4b. Show that if $\liminf x(t) > y_\delta^-$, then $\liminf x(t) \geq y_\delta^+$. Deduce that if $\liminf x(t) > 0$, then $\liminf x(t) \geq 1$.

5. Show that as $t \to +\infty$, $x(t) \to 0$ or $x(t) \to 1$

6. Show that if $x(t) \to -\infty$ as $t \to \sup J$, then $\sup J < +\infty$ and conclude.

Chapter 5

Systems of linear differential equations

In Chapter 3, we saw how to solve linear differential equations in dimension 1. This chapter goes a step further and studies systems of linear differential equations such as:

$$\begin{cases} x'(t) & = & tx(t) + 3y(t) + t^2 \\ y'(t) & = & e^t x(t) - 2ty(t) + 3 \end{cases}$$

or more generally

$$\begin{cases} x_1'(t) & = & a_{11}(t)x_1(t) + \ldots + a_{1d}(t)x_d(t) + b_1(t) \\ \vdots & & \vdots \\ x_d'(t) & = & a_{d1}(t)x_1(t) + \ldots + a_{dd}(t)x_d(t) + b_d(t) \end{cases}$$

which may be written in matrix form:

$$X'(t) = A(t)X(t) + B(t) \tag{NH}$$

where $X(t)$ and $B(t)$ are column vectors in $\mathbb{R}^d$, and $A(t)$ is a $d \times d$ square matrix. This corresponds to the particular case of the equation $X'(t) = F(t, X(t))$ where $F(t, X) = A(t)X + B(t)$. We will mostly focus on the case where $A(t)$ is a constant matrix, but we begin with some general properties of Eq. (NH). As usual, I denotes a nonempty open interval.

Proposition 5.0.1. *Let $A : I \to \mathcal{M}_d(\mathbb{R})$ and $B : I \to \mathbb{R}^d$ be continuous. Let $t_0 \in I$, $X_0 \in \mathbb{R}^d$. Let S_{NH} and S_H denote respectively the set of solutions of the nonhomogeneous equation (NH) and of the associated homogeneous equation*

$$X'(t) = A(t)X(t) \tag{H}$$

1. *Equations (H) and (NH) have a unique solution such that $X(t_0) = X_0$, and it is global.*

2. *S_H is a d-dimensional real vector space, and the function $\varphi : S_H \to \mathbb{R}^d$ that associates to a solution of (H) its value in t_0 is an isomorphism.*

3. *Let $(X_1(\cdot), ..., X_d(\cdot))$ be a family of solutions of (H). The following assertions are equivalent:*
   (a) $(X_1(\cdot), ..., X_d(\cdot))$ is a basis of S_H
   (b) $(X_1(t_0), ..., X_d(t_0))$ is a basis of $\mathbb{R}^d$.
   (c) For all times t in I, $(X_1(t), ..., X_d(t))$ is a basis of $\mathbb{R}^d$.

4. *The general solution of (NH) is the sum of a particular solution of (NH) and the general solution of (H). That is, if $X_p(\cdot) \in S_{NH}$, then*

$$S_{NH} = X_p(\cdot) + S_H = \{X : I \to \mathbb{R}^d, \exists Y \in S_H, X = X_p + Y\}.$$

Thus, S_{NH} is a d-dimensional real affine space with direction S_H.

Proof. 1) The fact that these initial value problems have a unique solution follows from the strong form of Picard-Lindelöf theorem (Theorem 1.5.4)[1], and it is global by Corollary 4.2.1 with $a(t) = ||A(t)||$ and $b(t) = ||B(t)||$.

2) S_H is nonempty by 1), and if $X(\cdot)$ and $Y(\cdot)$ are elements of S_H and λ is a real number, then $Z(\cdot) = \lambda X(\cdot) + Y(\cdot)$ satisfies

$$Z' = \lambda X' + Y' = \lambda AX + AY = A(\lambda X + Y) = AZ$$

hence $Z(\cdot) \in S_H$. Therefore, S_H is a real vector space. Moreover, let $\varphi : S_H \to \mathbb{R}^d$ be the function that associates to a solution of (H) its value in t_0. Then φ is obviously linear ($\varphi(\lambda X + Y) = \lambda X(t_0) + Y(t_0) = \lambda\varphi(X) + \varphi(Y)$ for all $\lambda \in \mathbb{R}$ and $X, Y \in S_H$), and it is one-to-one - that is, bijective - by 1). Indeed, 1) implies that for any element X_0 of $\mathbb{R}^d$, there is exactly one element of S_H with image X_0. Therefore, φ is an isomorphism, hence S_H has dimension d as $\mathbb{R}^d$.

3) The equivalence between (a) and (b) follows from the fact that φ is an isomorphism. Moreover (c) implies (b), which implies (a), and (a) implies (c) since the implication (a) $\Rightarrow$ (b) is valid for all t_0 in I.

4) The proof is the same as in dimension 1. □

Remark 5.0.2. *The fact that S_H is a d-dimensional vector space, implies that every solution $X(\cdot)$ of (H) is a linear combination of d linearly independent solutions $X_1(\cdot), \ldots, X_d(\cdot)$ of (H), i.e.*

$$X(\cdot) = \lambda_1 X_1(\cdot) + \cdots + \lambda_d X_d(\cdot)$$

for some $\lambda_1, \ldots \lambda_d \in \mathbb{R}$. $X_1(\cdot), \ldots, X_d(\cdot)$ form a basis for S_H.

Remark 5.0.3. *From 3), it follows that d solutions $X_1(\cdot), \ldots, X_d(\cdot)$ of (H) are linearly independent if and only if for all $t \in I$ the vectors*

$$X_1(t), \ldots, X_d(t) \in \mathbb{R}^d$$

are linearly independent.

As in dimension 1, the superposition principle may help us to find particular solutions. The proof is the same as is dimension 1.

Proposition 5.0.4. *Let $A : I \to \mathcal{M}_d(\mathbb{R})$, and let B_1, B_2, X and Y go from I to $\mathbb{R}^d$. If $X' = AX + B_1$ and $Y' = AY + B_2$, then $Z = X + Y$ satisfies $Z' = AZ + (B_1 + B_2)$.*

We now focus on the case of constant coefficients, that is, when the matrix $A(t)$ does not depend on t. We first study the homogeneous equation.

5.1 The equation $X'(t) = AX(t)$

We denote by (H) the equation

$$X'(t) = AX(t) \tag{H}$$

[1] Indeed, the function defined by $F(t, X) = A(t)X + B(t)$ is continuous. Moreover, if C is a compact subset of $I \times \mathbb{R}^d$, then its projection C_I on I is also compact, and for any (t, X, Y) such that (t, X) and (t, Y) are in C, $||F(t, X) - F(t, Y)|| \le ||A(t)|| \times ||X - Y|| \le K||X - Y||$ where $K = \max_{t \in C_I} ||A(t)||$. Thus F is locally Lipschitz with respect to X.

5.1.1 The case A diagonalizable in $\mathbb{R}$

Explicit solutions.

Assume first that $d = 2$. If A is diagonal:

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

then denoting solutions by $X(t) = (x(t), y(t))^T$, the system reads:

$$\left\{ \begin{array}{lcl} x'(t) & = & \lambda_1 x(t) \\ y'(t) & = & \lambda_2 y(t) \end{array} \right. . \tag{5.1}$$

Hence the system decouples into two equations that can be solved independently. The solutions are:

$$\left\{ \begin{array}{lcl} x(t) & = & x_0 e^{\lambda_1 t} \\ y(t) & = & y_0 e^{\lambda_2 t} \end{array} \right. \tag{5.2}$$

which, thinking to what comes later, can also be written as

$$X(t) = x_0 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y_0 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Similarly, for a generic d, if $A = diag(\lambda_1, ..., \lambda_d)$, then the d equations may be solved independently. The i^{th} line leads to $x_i(t) = \mu_i e^{\lambda_i t}$ and the general solution of the system is:

$$X(t) = \begin{pmatrix} \mu_1 e^{\lambda_1 t} \\ \vdots \\ \mu_d e^{\lambda_d t} \end{pmatrix} = \sum_{i=1}^{d} \mu_i e^{\lambda_i t} V_i \tag{5.3}$$

where V_i is the i^{th} vector of the canonical basis and μ_i the i^{th} coordinate of $X(0)$ in this basis. When A is diagonalizable, we get exactly the same formula, but in the eigenvector basis.

Proposition 5.1.1. *Let $B = (W_1, ..., W_d)$ be a basis of eigenvectors of A, associated to real eigenvalues λ_1,...,λ_d. Let $X_i(t) = e^{\lambda_i t} W_i$. Then $(X_1(\cdot), ..., X_d(\cdot))$ is a basis for the set of solutions of (H). Thus, the general solution is*

$$X(t) = \sum_{i=1}^{d} \mu_i^B e^{\lambda_i t} W_i, \quad (\mu_1^B, ..., \mu_d^B) \in \mathbb{R}^d$$

Moreover, μ_i^B is the i^{th} coordinate of $X(0)$ in the basis B.

Proof. The proof is based on the following fundamental Lemma:

Lemma 5.1.2. *Let V be a real or complex eigenvector of A associated to the real or complex eigenvalue λ. Then $X(t) = e^{\lambda t} V$ satisfies (H) and $X(0) = V$.*

Proof. $X'(t) = \lambda e^{\lambda t} V = e^{\lambda t} \lambda V = e^{\lambda t} AV = A e^{\lambda t} V = AX(t)$, and $X(0) = V$.[2] □

We now prove the proposition. By Lemma 5.1.2, $X_i(\cdot)$ is solution. Moreover, $(X_1(0), ..., X_d(0)) = (W_1, ..., W_d)$ is a basis of $\mathbb{R}^d$. Therefore, by point 3) in Proposition 5.0.1, $(X_1(\cdot), ..., X_d(\cdot))$ is a basis of the set S_H of solutions of (H). Finally, to see that μ_i^B is the i^{th} coordinate of $X(0)$ in the basis B, take $t = 0$ in the formula. □

[2] We let the reader check that when $\lambda \in \mathbb{C} \backslash \mathbb{R}$, the derivative of the application $\mathbb{C} \to \mathbb{C}$ defined by $t \to e^{\lambda t}$ is still given by $t \to \lambda e^{\lambda t}$, so that the previous computation is correct.

We may also adopt a matricial point of view to get the same proof. If A is diagonalizable, there exists an invertible matrix P such that $A = P^{-1}DP$ and $D = diag(\lambda_1, ..., \lambda_d)$. Let $Y = PX$. Then

$$X' = AX \Leftrightarrow X' = P^{-1}DPX \Leftrightarrow PX' = DPX \Leftrightarrow Y' = DY.$$

From (5.3) we get $Y(t) = \sum_{i=1}^{d} \mu_i e^{\lambda_i t} V_i$, where V_i is the i^{th} vector of the canonical basis, hence

$$X(t) = P^{-1}Y(t) = \sum_{i=1}^{d} \mu_i e^{\lambda_i t} W_i$$

with $W_i = P^{-1}V_i$. Note that W_i is an eigenvector of A associated to λ_i. Indeed,

$$AW_i = P^{-1}DPP^{-1}V_i = P^{-1}DV_i = P^{-1}\lambda_i V_i = \lambda_i W_i$$

Thus, we obtained exactly the same formula as in Proposition 5.1.1. Note that the matricial point of view is computationally more expensive: indeed, it requires, among other things, the computation of P^{-1}, whose columns are the eigenvectors W_i. However, once we know these eigenvectors, we can directly apply Proposition 5.1.1.

5.1.2 Phase portraits for A diagonalizable in $\mathbb{R}$.

General Introduction: trajectories. Let $X : J \to \mathbb{R}^d$ be a solution of a differential equation, modeling a system whose state at time t is $X(t)$. Recall that the *trajectory* (also called *orbit*) associated to this solution is the set $T = \{X(t), t \in J\}$ of successive states of the system as t describes J (*both forward and backward in times*).[3] The trajectory may also be defined as the projection of the graph of $X(\cdot)$ on $\mathbb{R}^d$. Indeed, the projection of $(t, X(t)) \in \mathbb{R} \times \mathbb{R}^d$ on $\mathbb{R}^d$ is $X(t)$. Thus the projection of the graph $\Gamma = \{(t, X(t)), t \in J\}$ is $\{X(t), t \in J\} = T$.

We would like to have an idea of trajectories associated to solutions of (H). We first note that, as seen in Proposition 2.1.5, for a C^1 or Lipschitz *autonomous equation*, the trajectory associated to a solution satisfying an initial condition $X(t_0) = X_0$ depends only on the initial position X_0, not on the initial time t_0. Thus, there are several solutions, but a unique trajectory going through X_0. (As seen in Chapter 2, if $X(\cdot)$ and $Y(\cdot)$ are two solutions taking the same value X_0 at different times, then $Y(\cdot)$ is just a translation in time of $X(\cdot)$, so both solutions visit exactly the same states, one following the other.[4])

It follows that, it is in principle possible to draw the trajectories associated to the solutions of an autonomous equation (though for practical reasons, we will only do it in dimension 2!). Such a drawing is called *phase portrait* - or phase line in dimension 1. For a non autonomous equation, there are typically several trajectories going through the same point X_0, possibly an infinity, so such a drawing would be a mess.

Let us give a few examples of trajectories in dimension 1 (not only in the linear case). Consider the equation $x' = x$. Since 0 is an equilibrium, any solution taking the value 0 is stationary and its trajectory is simply $\{0\}$. The solution such that $x(0) = 1$ is $x(t) = e^t$, defined on $\mathbb{R}$. The associated trajectory is $T = \{e^t, t \in \mathbb{R}\} =]0, +\infty[$. The solution such that $x(0) = -1$ is $-e^t$. The associated trajectory is $\{-e^t, t \in \mathbb{R}\} =]-\infty, 0[$. We let the reader check that similarly, for any $x_0 > 0$, the unique trajectory going through x_0 is $]0, +\infty[$, and for any

[3] You may also see the trajectory as the set of positions occupied successively by a particle whose position at time t is $X(t)$.

[4] Formally, if $X :]a, b[\to \mathbb{R}$ is the solution of $X'(t) = G(X(t))$ such that $X(0) = X_0$, then the solution such that $X(t_0) = X_0$ is the function $Y :]a + t_0, b + t_0[\to \mathbb{R}^d$ defined by $Y(t) = X(t - t_0)$. The trajectory T_Y associated to this solution is

$$T_Y = \{Y(t), t \in]a+t_0, b+t_0[\} = \{X(t-t_0), t \in]a+t_0, b+t_0[\} = \{X(t-t_0), t-t_0 \in]a, b[\} = \{X(\tau), \tau \in]a, b[\} = T_X.$$

Thus this trajectory does not depend on t_0

$x_0 < 0$, the unique trajectory going through x_0 is $]-\infty, 0[$. Note that, as should be, trajectories associated to two different solutions are either equal or disjoint. Note also that for $x_0 \neq 0$, solutions lead away from the equilibrium; that is, as time unfolds, the trajectories $]0, +\infty[$ and $]-\infty, 0[$ are described from 0 to $\pm\infty$. This is what the arrows indicate when we draw the phase line of this equation.

Exercise 5.1.3. *Draw the graph of three representative solutions of $x' = x$ (in the plane with an horizontal time axis and a vertical position axis), and draw arrows on these graphs indicating the direction of time. Project these graphs - including the arrow - on the position axis. Check that this gives you the phase line of $x' = x$, but drawn vertically.*

Exercise 5.1.4. *Show that the trajectories for $x' = -x$ are the same as for $x' = x$, but traveled in opposite directions. Do the same exercise as above for this equation.*

Now consider the logistic equation $x' = x(1-x)$. Since 0 and 1 are equilibria, the trajectories going through 0 and 1 are respectively $\{0\}$ and $\{1\}$. If a solution takes the value $x_0 \in]0, 1[$, then in the sense of Proposition 2.3.1, it comes from 0 and goes to 1, taking all the intermediate values by continuity, but not the values 0 and 1. So the associated trajectory is $]0, 1[$. Similarly, solutions starting at $x_0 > 1$ come from $+\infty$ and go to 1, and the associated trajectory is $]1, +\infty[$, traveled from $+\infty$ toward 1, as indicated by the arrow pointing towards 1 on the phase line. Finally, the trajectory going through $x_0 < 0$ is $]-\infty, 0[$, traveled from 0 to $-\infty$.

Here are examples in dimension 2. Consider first the function defined by $X(t) = (x(t), y(t))$ where $x(t) = \cos t$ and $y(t) = \sin t$. It is solution of $x' = -y$ and $y' = x$. As t describes $\mathbb{R}$, $X(t)$ describes the unit circle of $\mathbb{R}^2$, that is, $U = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 = 1\}$. Thus the trajectory associated to $X(\cdot)$ is U. Note that the graph of $X(t)$ would be an helicoid in $\mathbb{R} \times \mathbb{R}^2 = \mathbb{R}^3$, but its trajectory (projection of the graph on $\mathbb{R}^2$) is a circle.

Now consider the function $X(t) = (e^t \cos t, e^t \sin t) = e^t(\cos t, \sin t)$. As t increases, $X(t)$ does not only rotate counter-clockwise, it also goes away from 0. Indeed, $||X(t)|| = e^t$ increases. The graph would be an helicoid with circles of increasing length, and the trajectory is a spiral leading away from $(0, 0)$. Similarly, for $X(t) = (e^{-t} \cos t, e^{-t} \sin t)$, the graph is a spiral but now leading *towards* $(0, 0)$.

As a last example, consider the function $X(t) = (x(t), y(t))$ where $x(t) = e^t$ and $y(t) = e^{-t}$. Note that $y(t) = 1/x(t)$, so that $X(t) = (x(t), 1/x(t))$. As t describes $\mathbb{R}$, $x(t)$ describes $]0, +\infty[$, so the trajectory associated to $X(\cdot)$ is

$$T = \{(x(t), y(t)), t \in \mathbb{R}\} = \{(x(t), 1/x(t)), t \in \mathbb{R}\} = \{(x, 1/x), x \in]0, +\infty[\}$$

Thus, T is the graph of the function $x \to 1/x$ restricted to $]0, +\infty[$, hence a hyperbola.

Exercise 5.1.5. *Show that the trajectory associated to $X(t) = (e^{-t}, e^t)$ is also $\{(x, 1/x), x \in]0, +\infty[\}$.*

Phase portraits. We now come back to Eq. (H) in dimension 2. Recall that when A is diagonal:

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

the solutions are given by

$$\begin{cases} x(t) &= x_0 e^{\lambda_1 t} \\ y(t) &= y_0 e^{\lambda_2 t} \end{cases} \tag{5.4}$$

For any value of the eigenvalues, there is an equilibrium at the origin, and solutions that start on one of the axis remain on this axis. But for $x_0 \neq 0$, whether $x(t)$ goes from 0 to $\pm\infty$, $x(t)$ is constant, or $x(t)$ goes from $\pm\infty$ to 0, depends on the sign of λ_1. Similarly, the fate of $y(t)$

depends on the sign of λ_2. We thus have to consider all the possible cases.

The case $\lambda_1 < 0 < \lambda_2$.

A **diagonal.** As mentioned above, solutions that start on the x-axis ($y_0 = 0$) remain on the x-axis ($y(t) = 0$ for all t). Moreover, if $x_0 \neq 0$, then $x(t)$ comes from $+\infty$ or $-\infty$ (depending on the sign of x_0) and goes to zero, since $\lambda_1 < 0$. Thus, all solutions starting on the x-axis converge to the equilibrium $(0, 0)$. The x-axis is called the *stable line*. Similarly, solutions that start on the y-axis remain on the y-axis, but, except if they start at the equilibrium, they tend away from it: the y-axis is called the *unstable line*.

If $x_0 \neq 0$, $y_0 \neq 0$, then the sign of $x(t)$ and $y(t)$ remains constant; moreover, $x(t) \to 0$ and $|y(t)| \to +\infty$ as $t \to +\infty$, while $|x(t)| \to +\infty$ and $y(t) \to 0$ as $t \to -\infty$. Thus all solutions that do not start on the stable or unstable line come from the stable line and go towards the unstable line. Their trajectories look like portions of hyperbolas. We say that the equilibrium is a *saddle*.

We may be more precise (though we will never ask you to!). Assume for concreteness $x_0 > 0$ and $y_0 > 0$ (the other cases are similar). From $x(t) = x_0 e^{\lambda_1 t}$, we get $t = \frac{1}{\lambda_1} \ln \frac{x(t)}{x_0}$, hence

$$y(t) = y_0 e^{\lambda_2 t} = y_0 \exp\left(\frac{\lambda_2}{\lambda_1} \ln \frac{x(t)}{x_0}\right) = y_0 \exp\left(\ln\left[\frac{x(t)}{x_0}\right]^{\frac{\lambda_2}{\lambda_1}}\right) = C x^\gamma(t) \tag{5.5}$$

with $C = y_0/x_0^{\lambda_2/\lambda_1} > 0$ and $\gamma = \lambda_2/\lambda_1 < 0$. The trajectory associated to this solution is thus

$$T = \{(x(t), y(t)), t \in \mathbb{R}\} = \{(x(t), Cx^\gamma(t), t \in \mathbb{R}\} = \{(x, Cx^\gamma), x \in]0, +\infty[\}$$

since $x(t)$ describes $]0, +\infty[$ as t describes $\mathbb{R}$. This trajectory is thus the graph of the function $x \to Cx^\gamma$ restricted to $]0, +\infty[$. Since $\gamma < 0$, it indeed looks like a branch of hyperbola (it *is* a branch of hyperbola when $\gamma = -1$).

If $y_0 < 0$, $x_0 > 0$, we get the same formula but now $C < 0$. If $x_0 < 0$, we get

$$T = \{(x, C|x|^\gamma), x \in] - \infty, 0[\}$$

where $C = y_0/|x_0|^{\lambda_2/\lambda_1}$ has the sign of y_0. Thus, in all cases, the trajectory is a portion of the graph of $x \to C|x|^\gamma$, the portion in $]0; +\infty[$ if $x_0 > 0$, and the portion in $] - \infty; 0[$ if $x_0 < 0$, and with C of the sign of y_0.

These observations allow to draw the *phase portrait* of the equation: that is, a drawing of representative trajectories and the direction in which they are traveled. This drawing is made in the *phase plane* $\mathbb{R}^2$ (for $d \geq 3$, $\mathbb{R}^d$ is called the *phase space*). See Fig. 5.1.

A **diagonalizable.** Now consider the case where A has still two eigenvalues λ_1, λ_2 such that $\lambda_1 < 0 < \lambda_2$ but is not diagonal. Because A has two distinct real eigenvalues, it is diagonalizable in $\mathbb{R}$. Let W_1 and W_2 be eigenvectors associated to λ_1 and λ_2, respectively. We know from Proposition 5.1.1 that the general solution of (H) is

$$X(t) = x_0^B e^{\lambda_1 t} W_1 + y_0^B e^{\lambda_2 t} W_2 \tag{5.6}$$

where x_0^B and y_0^B are the coordinates of $X(0)$ in the eigenvector basis $B = (W_1, W_2)$:

$$X(0) = x_0^B W_1 + y_0^B W_2$$

The coordinates of the solution in the eigenvector basis are thus $(x^B(t), y^B(t)) = (x_0^B e^{\lambda_1 t}, y_0^B e^{\lambda_2 t})$. These are exactly the same equations as (5.4) but for the coordinates in the basis (W_1, W_2). We thus obtain the same kind of behavior as in the case A diagonal, except that the stable line

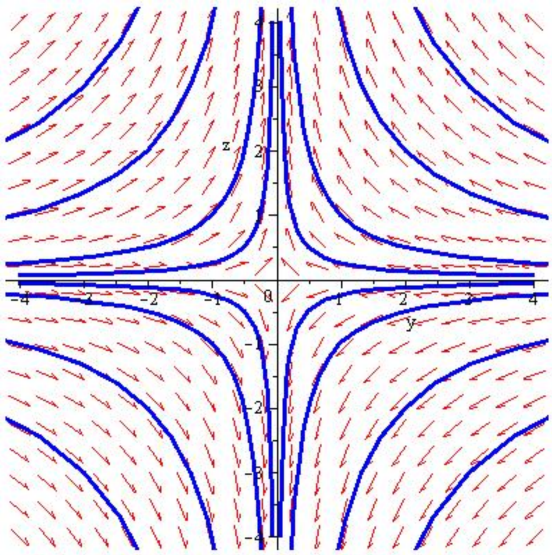

Figure 5.1: A diagonal, $\lambda_1 < 0 < \lambda_2$

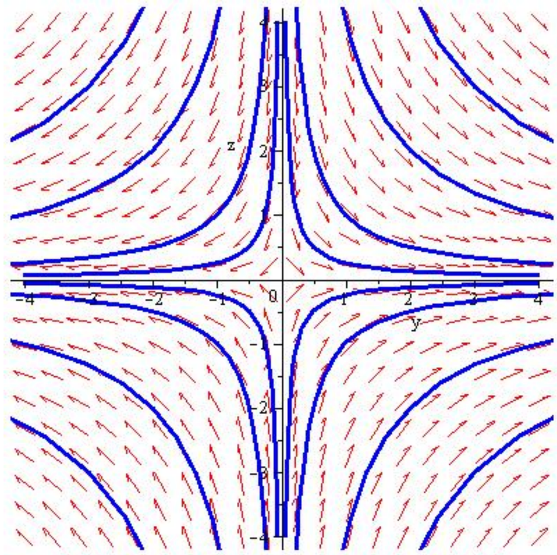

Figure 5.2: A diagonal, $\lambda_2 < 0 < \lambda_1$

Two saddles with A diagonal, respectively $A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Red arrows indicate the direction of movement at each point in space, blue curves are trajectories. Left, the stable line is the x-axis and the unstable line the y-axis. Right, this is the opposite. There are actually 9 kinds of trajectories (depending on the sign of x_0 and y_0), but the picture shows only the four cases corresponding to $x_0 \neq 0$ and $y_0 \neq 0$. When drawing the picture by hand, you do not need to put the red arrows, but you should indicate the other kinds of trajectories (the equilibrium and the trajectories on the axis, which are opened half-lines), and put arrows on the blue curves. See drawings made in the course.

is now the span of W_1 and the unstable line the span of W_2. In particular, all solutions that do not start on one of these lines come from the stable line and go towards the unstable line. Examples of phase portraits in this case will be given in the course.

Exercise 5.1.6. *Sketch the phase portrait if $\lambda_1 < 0 < \lambda_2$ and: a) $W_1 = (0,1)^T$, $W_2 = (1,0)^T$; b) $W_1 = (1,0)^T$ and $W_2 = (1,1)^T$.*

The case $\lambda_2 < 0 < \lambda_1$.

The phase portrait is the same as in the case $\lambda_1 < 0 < \lambda_2$. The stable line is now the y-axis (or more generally *span*(W_2) and the unstable line the x-axis (more generally, *span*(W_1)). See Fig. 5.2 for the case A diagonal.

The case $\lambda_1 < 0$, $\lambda_2 < 0$.

A diagonal. If A is diagonal with eigenvalues $\lambda_1 < 0$, $\lambda_2 < 0$, then all solutions tend towards the equilibrium $(0,0)$. We say that the equilibrium is a sink (“puits” in French, though “sink” means “évier”).

The way solutions tend toward the equilibrium depends on which eigenvalue is the strongest, in the sense of having the largest absolute value. If the strongest eigenvalue is λ_2, that is, if $\lambda_2 < \lambda_1 < 0$, then

$$\frac{y(t)}{x(t)} = \frac{y_0 e^{\lambda_2 t}}{x_0 e^{\lambda_1 t}} = \frac{y_0}{x_0} e^{(\lambda_2 - \lambda_1)t} \to_{t \to +\infty} 0$$

and solutions tend toward $(0,0)$ tangentially to the x-axis. As $t \to -\infty$, $|y(t)|/|x(t)| \to +\infty$, hence backward in time, solutions have asymptotically the direction of the y-axis, as in a

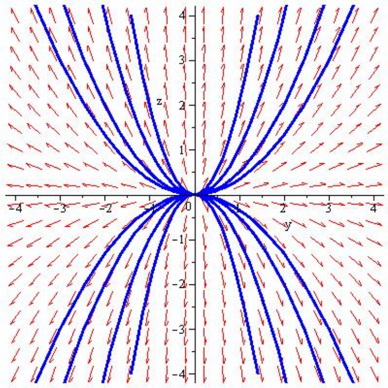

Figure 5.3: $\lambda_2 > \lambda_1 > 0$

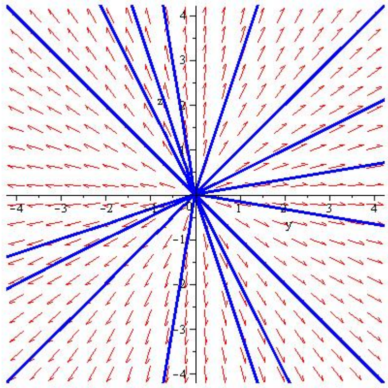

Figure 5.4: $\lambda_1 = \lambda_2 > 0$

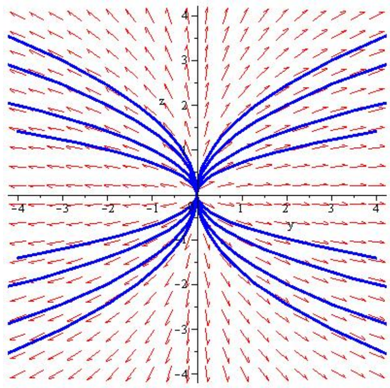

Figure 5.5: $\lambda_1 > \lambda_2 > 0$

Three sources with A diagonal. Only trajectories with $x_0 \neq 0$, $y_0 \neq 0$ are shown. The matrices used are:

$$A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad A_5 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

parabola. By contrast, if $\lambda_1 < \lambda_2 < 0$, that is, if λ_1 is the strongest eigenvalue, then trajectories look like portions of the graph of $x \to C\sqrt{|x|}$. If $\lambda_1 = \lambda_2$, then typical trajectories are opened half-lines.

To check that solutions really behave in that way, we may compute $y(t)$ as a function of $x(t)$. In the case $x_0 > 0$, $y_0 > 0$, we find as in (5.5) that along the trajectory, $y = Cx^\gamma$ with $\gamma = \lambda_2/\lambda_1$, but now $\gamma > 0$. If $\gamma > 1$, the trajectory looks like a portion of parabola; if $\gamma = 1$, this is a portion of straight line, and if $\gamma < 1$, it looks like a portion of the graph of $x \to \sqrt{x}$. Similar computations for the other possible signs of x_0 and y_0 lead to phase portraits as in Figures 5.3 to 5.5, but reversing all the arrows [see drawings made in the course].

A diagonalizable. If A is diagonalizable, then the solutions are given by (5.6). Since we assume that the eigenvalues λ_1 and λ_2 are both negative, solutions behaves as we just discussed, except that the x-axis is replaced by span(W_1) and the y-axis by span(W_2). Thus, if $\lambda_2 < \lambda_1 < 0$, solutions tend towards the origin tangentially to the direction of W_1 and in backward time, go to infinity with a direction tending towards the direction of W_2. If $\lambda_1 < \lambda_2 < 0$, the reverse thing happens [see drawings made in the course].

Exercise 5.1.7. *Why didn't we discuss the case A diagonalizable with $\lambda_1 = \lambda_2$? Answer here.*[5]

The case $\lambda_1 > 0$, $\lambda_2 > 0$. Solutions now go away from the equilibria: we say that the equilibrium is a *source*. Moreover, it may be seen (see this note[6]) that this case is the exact "opposite" of the case $\lambda_1 < 0$, $\lambda_2 < 0$ in the following sense: if $\lambda_2 > \lambda_1 > 0$, the phase portrait is exactly the same as in the case $\lambda_2 < \lambda_1 < 0$, except that we need to reverse the arrows, that is, the trajectories are the same but travelled in opposite directions. Similarly, the case $\lambda_1 > \lambda_2 > 0$ is the opposite of the case $\lambda_1 < \lambda_2 < 0$, and the case $\lambda_1 = \lambda_2 > 0$ is the opposite of the case $\lambda_1 = \lambda_2 < 0$. This leads to Fig. 5.3 to 5.5.

[5]Because if A is diagonalizable with a single eigenvalue λ, then A is diagonal! Indeed, $P^{-1}AP = \lambda I$ gives $A = P\lambda I P^{-1} = \lambda I$.

[6]If $X :]a,b[\to \mathbb{R}^d$ is solution of $X' = G(X)$, then the function $Y :]-b,-a[\to \mathbb{R}^d$ defined by $Y(t) = X(-t)$ satisfies for all $t \in]-b,-a[$: $Y'(t) = -X'(-t) = -G(X(-t)) = -G(Y(t))$, hence Y is solution of $Y' = -G(Y)$, and the reader will check that it is maximal if and only if $X(\cdot)$ is maximal. Since $X(\cdot)$ and $Y(\cdot)$ visit the same points in space, but at opposite times, it is easily seen that they define the same trajectory:

$$T_Y = \{Y(t), t \in]-b,-a[\} = \{X(-t), t \in]-b,-a[\} = \{X(-t), -t \in]-b,-a[\} = \{X(\tau), \tau \in]a,b[\} = T_X$$

but traveled in opposite directions.

The cases $\lambda_1 \neq 0$, $\lambda_2 = 0$ and $\lambda_1 = \lambda_2 = 0$. If $\lambda_1 \neq 0$, $\lambda_2 = 0$, then there is a line of equilibria: the span of W_2. We let the reader check that if $\lambda_1 < 0$, $\lambda_2 = 0$, then all solutions converge to this line of equilibria following trajectories parallel to the span of W_1, while if $\lambda_1 > 0$, $\lambda_2 = 0$, solutions go away from the line of equilibria, again with trajectories parallel to the span of W_1. In the even more special case $\lambda_1 = \lambda_2 = 0$, any point is an equilibrium, and solutions do not move.

5.1.3 The case A diagonalizable in $\mathbb{C}$

The case $d = 2$.

Proposition 5.1.8. *Let $A \in \mathcal{M}_2(\mathbb{R})$ be diagonalizable in $\mathbb{C}$. Let W be a complex eigenvector of A associated to the eigenvalue λ. Let $Z(t) = e^{\lambda t} W$. Then the real and imaginary part of $Z(t)$ form a basis of S_H. Thus, the general solution is*

$$X(t) = \mu_1 Re(Z(t)) + \mu_2 Im(Z(t)), \quad (\mu_1, \mu_2) \in \mathbb{R}^2$$

Proof. From Lemma 5.1.2, we know that $Z' = AZ$. Denoting by $X_z(t)$ and $Y_z(t)$ the real and imaginary part of $Z(t)$, this implies that $X_z' + iY_z' = A(X_z + iY_z) = AX_z + iAY_z$. Therefore, $X_z' = AX_z$ and $Y_z' = AY_z$. Thus, $X_z(\cdot)$ and $Y_z(\cdot)$ are real solutions of (H). Moreover, we claim that, $(X_z(0), Y_z(0))$ is a basis of $\mathbb{R}^2$. By point 3 in Proposition 5.0.1, it follows that $(X_z(\cdot), Y_z(\cdot))$ is a basis of S_H, proving the proposition.

It remains to prove the claim. In what follows, $\mathbb{C}^2$ is thought of as a complex vector space. Let $W = X_w + iY_w$. Since A is a real matrix, $\bar{W} = X_w - iY_w$ is another eigenvector of A, and $(W, \bar{W})$ is a basis of $\mathbb{C}^2$ (indeed, an eigenvector basis). Since both, W and $\bar{W}$ are complex linear combinations of X_w and Y_w, it follows that (X_w, Y_w) generates $\mathbb{C}^2$, hence is a basis of $\mathbb{C}^2$. Thus, X_w and Y_w are independent in $\mathbb{C}^2$: for all (μ_1, μ_2) in $\mathbb{C}^2$, $\mu_1 X_w + \mu_2 Y_w = 0 \Rightarrow \mu_1 = \mu_2 = 0$. This holds a fortiori for all (μ_1, μ_2) in $\mathbb{R}^2$. Therefore, X_w and Y_w are independent in $\mathbb{R}^2$, hence (X_w, Y_w) is a basis of $\mathbb{R}^2$. But $Z(0) = W$ hence $(X_z(0), Y_z(0)) = (X_w, Y_w)$, hence $(X_z(0), Y_z(0))$ is a basis of $\mathbb{R}^2$. □

Let us describe more precisely how solutions behave. Let $W = X_w + iY_w$ be an eigenvector of A associated to $\lambda = a + ib$. Let $Z(t) = e^{(a+ib)t} W = e^{at}(\cos bt + i \sin bt)(X_w + iY_w)$. We have:

$$Z(t) = e^{at}(\cos bt + i \sin bt)(X_w + iY_w) = e^{at}\left[(\cos bt\, X_w - \sin bt\, Y_w) + i(\cos bt\, Y_w + \sin bt\, X_w)\right].$$

Thus, the general solution is

$$X(t) = \mu_1 e^{at}(\cos bt\, X_w - \sin bt\, Y_w) + \mu_2 e^{at}(\cos bt\, Y_w + \sin bt\, X_w).$$

Let $X^B(t) = (X^B(t), Y^B(t))^T$ denote the vector of coordinates of $X(t)$ in the basis $B = (X_w, Y_w)$. Grouping the terms in X_w and in Y_w in the previous equation, we obtain:

$$X^B(t) = e^{at}\begin{pmatrix} \mu_1 \cos bt + \mu_2 \sin bt \\ -\mu_1 \sin bt + \mu_2 \cos bt \end{pmatrix} = e^{at}\begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

Note that $X^B(0) = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$. Therefore, letting $M(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, we obtain:

$$X^B(t) = e^{at} M(-bt) X^B(0) \tag{5.7}$$

As you may know, $M(t)$ is the rotation matrix of angle t, hence $M(-bt)$ the rotation matrix of angle $-bt$. Therefore, the interpretation is as follows: in the basis B, the coordinates of the solution are obtained from their value at time 0 by combining a dilatation of factor e^{at} and a rotation of angle $-bt$. When $a = 0$, they describe a circle. When $a > 0$, an expanding spiral.

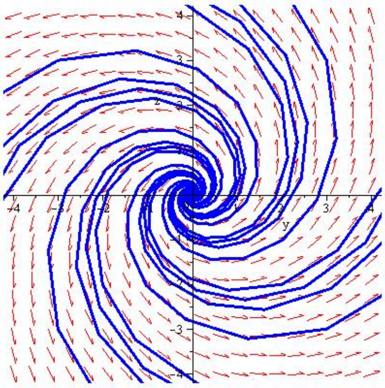

Figure 5.6: A spiral source.

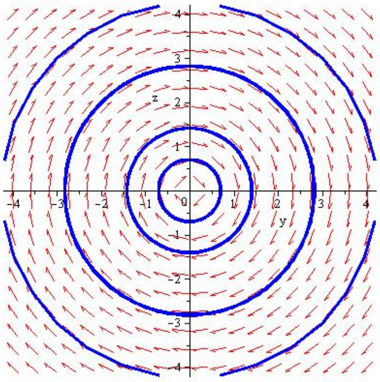

Figure 5.7: A center

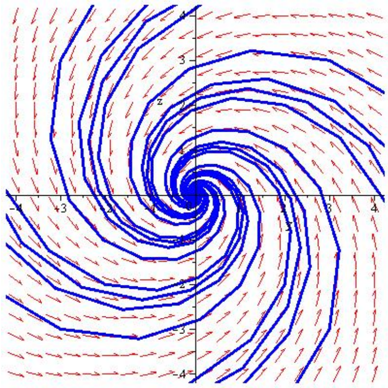

Figure 5.8: A spiral sink

The case A diagonalizable in $\mathbb{C}$ when $B = (X_w, Y_w)$ is the canonical basis. The matrices used are:

$$A_6 = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \qquad A_7 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad A_8 = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}$$

Note that A_8 is not the opposite of A_6, otherwise trajectories would be the same (with arrows reversed).

When $a < 0$, a contracting spiral. Whether the rotation is clockwise or counterclockwise depends on the sign of b (note that $b \neq 0$, since $\lambda = a + ib \notin \mathbb{R}$).

Phase portraits. We can now draw the phase portraits. There are two cases.

Case 1: if $B = (X_w, Y_w)$ is the canonical basis of $\mathbb{R}^2$ (that is, if $W = (1,0)^T + i(0,1)^T = (1,i)$).

In that case, the coordinates in the basis B are just the standard coordinates, so if $a = 0$, $X(t)$ describes a circle, if $a > 0$, an expanding spiral, if $a < 0$ a contracting spiral. We then say that the equilibrium $(0,0)$ is respectively a *center*, a *spiral source* and a *spiral sink*. See Fig. 5.6 to 5.8.

Case 2: if B is not the canonical basis, then $X(t)$ describes what could be called a "circle in the basis B", that is, a set defined by

$$C = \{xX_w + yY_w, x^2 + y^2 = r\}.$$

If X_w and Y_w are orthogonal, this is an ellipse whose axis have the directions of X_w and Y_w. Otherwise, this is a curve which is similar to an ellipse (whose axis no longer have the directions of X_w and Y_w), but the author of these lines did not check whether it is precisely an ellipse or not (the willing student can check!). Thus, when $a = 0$, solutions describes ellipses or similar curves. When $a > 0$ and $a < 0$, solutions still describe expanding or contracting spirals, respectively, but these spirals are "ellipsoidal". Again, we say that the equilibrium is a center, a spiral source or a spiral sink, respectively. The phase portraits will be drawn in the course.

Remark 5.1.9. *Matricially speaking, letting*

$$A_{ref} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \tag{5.8}$$

the first case corresponds to $A = A_{ref}$ and the second case to $A = P^{-1}A_{ref}P$, where P is the matrix whose i^{th} column give the coordinates of i^{th} vector in the canonical basis $B = (X_w, Y_w)$. Indeed, the real and imaginary parts of the equality $A(X_w + iY_w) = (a + ib)(X_w + iY_w)$ imply that the endomorphism with matrix A in the canonical basis has matrix A_{ref} in the basis B.

The case A diagonalizable in $\mathbb{C}$ in dimension d.

Summary: The case A diagonalizable in $\mathbb{C}$ in dimension d is similar to the case $d = 2$. Each real eigenvector leads to a real solution of $X' = AX$ and each complex eigenvector to a complex solution. The real and imaginary parts of this complex solution are real solutions. If we group the real solutions associated in this way to the real and complex eigenvectors, we again obtain a basis of the set of real solutions of $X' = AX$, hence a formula for the general solution.

Let us be a bit more precise. If A is diagonalizable in $\mathbb{C}$, then it has a basis of eigenvectors consisting of q pairs of complex conjugate eigenvectors $(W_i, \bar{W}_i)$, and possibly $d - 2q$ real eigenvectors W_{2q+1}, ..., W_d. Due to Lemma 5.1.2, we may associate to these eigenvectors a family of d complex solutions, of which the last $d - 2q$ are real.

$$(Z_1(\cdot), \bar{Z}_1(\cdot), ..., Z_q(\cdot), \bar{Z}_q(\cdot), Z_{2q+1}(\cdot), ..., Z_d(\cdot))$$

Denoting by $X_k(\cdot)$ and $Y_k(\cdot)$ the real and imaginary part of $Z_k(\cdot)$, for $1 \leq k \leq q$, we obtain as in the case $d = 2$ that $X_k(\cdot)$ and $Y_k(\cdot)$ are real solutions. Thus we obtain a family B_H of d real solutions:

$$B_H = (X_1(\cdot), Y_1(\cdot), ..., X_q(\cdot), Y_q(\cdot), Z_{2q+1}(\cdot), ..., Z_d(\cdot))$$

It may be shown, as in the case $d = 2$, that taking the values of these solutions at $t = 0$ gives a basis of $\mathbb{R}^d$. This shows by Proposition 5.0.1 that B_H is a basis of S_H. Thus, solutions are linear combinations of these basic solutions.

Matricially, A may be written in the form $P^{-1}M_{ref}P$ where M_{ref} is a block-diagonal matrix such that each block is either a 2×2 block of form (5.8) - corresponding to a pair $\lambda = a + ib$, $\bar{\lambda} = a - ib$ of complex eigenvalues - or a 1×1 block containing a real eigenvalue. Each of this block corresponds to an invariant plane or line, on which the solutions behaves as we described for the invariant planes, and as on the eigenaxis in the case A diagonalizable in $\mathbb{R}$ for the invariant lines. This is somewhat difficult to visualize, except in dimension 3, where when A is diagonalizable in $\mathbb{C}$ but not in $\mathbb{R}$, M_{ref} necessarily takes the form:

$$\begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

Exercise 5.1.10. *(∗)* Sketch the phase portrait for the equation $X' = M_{ref}X$, when M_{ref} is the above 3×3 matrix. Note that there are 9 such phase portraits (three choices for the sign of a, three choices for λ), or even 18 if we care about whether the rotation is clockwise or counterclockwise (which depends on the sign of b). So just draw a couple of cases.

5.1.4 The general case through triangularization

Consider the following system, where A is a 2×2 triangular matrix:

$$\left\{ \begin{array}{lcccc} x'(t) & = & x(t) & + & y(t) \\ y'(t) & = & & & y(t) \end{array} \right. . \tag{5.9}$$

The last line leads to $y(t) = y_0 e^t$, so the first line becomes $x'(t) = x(t) + y_0 e^t$. This is nonhomogeneous linear equation in dimension 1. We may solve it by the method of variation of constants, that is, by looking for solutions of the form $x(t) = \lambda(t)e^t$. This leads to $x(t) = (x_0 + y_0 t)e^t$. Thus the general solution of (5.9) is

$$X(t) = \begin{pmatrix} x_0 e^t + y_0 t e^t \\ y_0 e^t \end{pmatrix}$$

Any triangular system in $\mathbb{R}^d$ may be dealt with, in a similar fashion. We first find $x_d(\cdot)$. This allows to find $x_{d-1}(\cdot)$, which allows to find $x_{d-2}(\cdot)$, etc. If A is not triangular, but triangularizable, that is of the form $A = P^{-1}TP$ with T triangular, we let $Y = PX$ and note that:

$$X' = AX \Leftrightarrow X' = P^{-1}TPX \Leftrightarrow PX' = TPX \Leftrightarrow Y' = TY.$$

The system $Y' = TY$ is triangular, so we may solve it, and deduce from the solutions of this system the solutions of $X' = AX$. Note that in $\mathcal{M}_d(\mathbb{C})$, every matrix is triangularizable. So if we accept without proof that triangular systems in $\mathcal{M}_d(\mathbb{C})$ may be dealt with as triangular systems in $\mathcal{M}_d(\mathbb{R})$ (just make the same formal computation), this gives a method to solve any linear system of the form $X'(t) = AX(t)$ (with A in $\mathcal{M}_d(\mathbb{R})$, and even A in $\mathcal{M}_d(\mathbb{C})$!). We now show that there is another way to solve such systems in the general case.

5.1.5 The equation $X'(t) = AX(t)$ and the exponential of a matrix

If a is a real number, the solution of $x'(t) = ax(t)$ and $x(0) = x_0$ is $x(t) = e^{ta}x_0$. This section shows that this formula can be generalized to the equation $X'(t) = AX(t)$, but replacing the term e^{ta} with the *exponential of the matrix* tA, a notion we now define.

Definition 5.1.11. *Let $A \in \mathcal{M}_d(\mathbb{K})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The exponential of the matrix A is defined by*

$$\exp(A) = \sum_{k=0}^{+\infty} \frac{A^k}{k!} = \lim_{n\to+\infty} \sum_{k=0}^{n} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \dots + \frac{A^k}{k!} + \dots.$$

Remark 5.1.12. *As usual, we take the convention $A^0 = I$ where I is the $d \times d$ identity matrix.*

Remark 5.1.13. *The definition makes sense because the series is normally convergent. Indeed take the norm on the space $\mathcal{M}_d(\mathbb{K})$ such that $||A^k|| \le ||A||^k$. Such a norm is obtained by taking any norm $||\cdot||$ on $\mathbb{K}^d$ and taking on $\mathcal{M}_d(\mathbb{K})$ the* subordinate norm*:*

$$||A|| = \max_{X\neq 0} \frac{||AX||}{||X||} = \max_{\{X\in\mathbb{K}^d \,:\, ||X||=1\}} ||AX||.$$

Since $||A^k|| \le ||A||^k$ and $\sum_{k=0}^{+\infty} \frac{||A||^k}{k!} = \exp(||A||) < +\infty$, the series defining $\exp(A)$ converges.

Remark 5.1.14. *In dimension 1, when writing the solution of $x'(t) = ax(t)$ and $x(0) = x_0$, we can write indifferently ta or at and $\exp(ta)x_0$ or $x_0\exp(ta)$, because we are dealing with reals numbers. But in dimension d, we should write tA and not At, and more importantly, $\exp(tA)X_0$ and not $X_0\exp(tA)$: these are not the same vectors!*

Exponentials of matrices have properties similar to exponentials of real numbers, but the property $e^{A+B} = e^A e^B$ is only valid if A and B commute, that is, if $AB = BA$.

Proposition 5.1.15 (basic properties)**.** *Let A and B be in $\mathcal{M}_d(\mathbb{K})$. Let 0 and I denote the zero and identity matrix of $\mathcal{M}_d(\mathbb{K})$, respectively. Let t and s be real numbers. We have:*

1. *$e^0 = I$.*
2. *If P is an invertible matrix in $\mathcal{M}_d(\mathbb{K})$, then $\exp(P^{-1}AP) = P^{-1}(\exp A)P$.*
3. *e^A is a polynom in A (the polynom depending on A).*
4. *If $AB = BA$, then:*
   - (a) *$P(A)Q(B) = Q(B)P(A)$ for any polynoms P and Q.*
   - (b) *$P(A)e^B = e^B P(A)$ for any polynom P.*
   - (c) *$\exp(A + B) = \exp(A)\exp(B) = \exp(B)\exp(A)$*

5. *e^A is invertible with inverse e^{-A}.*

6. *$e^{(t+s)A} = e^{tA}e^{sA}$.*

Proof. 1) is an easy computation. To prove 2), recall that $(P^{-1}AP)^k = P^{-1}A^kP$. Thus:

$$\sum_{k=0}^{n} \frac{(P^{-1}AP)^k}{k!} = \sum_{k=0}^{n} \frac{P^{-1}A^kP}{k!} = P^{-1}\left(\sum_{k=0}^{n} \frac{A^k}{k!}\right)P$$

As $n \to +\infty$, the expression on the left goes to $e^{P^{-1}AP}$ while the expression on the right goes to $P^{-1}e^AP$. This proves 2).

3) Let $F = \{P(A) | P \in \mathbb{K}[X]\} \subset \mathcal{M}_d(\mathbb{K})$ denote the set of polynoms in A. It is easily checked that F is a vector subspace of $\mathcal{M}_d(\mathbb{K})$, hence is closed in $\mathcal{M}_d(\mathbb{K})$. Thus, as a limit of elements of F, $\exp(A)$ is also in F.

4a) is a classical linear algebra property. We recall the proof for completeness. Assume that $AB = BA$. Since $A^0 = I$, we have $A^kB = BA^k$ for $k = 0$. Moreover, if $A^kB = BA^k$, then $A^{k+1}B = AA^kB = ABA^k = BAA^k = BA^{k+1}$, where we used successively $A^kB = BA^k$ and $AB = BA$. So by induction $A^kB = BA^k$ for all $k \in \mathbb{N}$. It follows immediately that for any polynom P, $P(A)B = BP(A)$. Now let $\tilde{A} = P(A)$. Since $\tilde{A}B = B\tilde{A}$, the same reasoning as above shows that $\tilde{A}$ commutes with any polynom in B.

4b) follows from 3) and 4a).

4c) The proof that $e^{A+B} = e^Ae^B$ is exactly the same as the proof that $e^{x+y} = e^xe^y$. This is because this proof involves only additions and multiplications (no division). Since when $AB = BA$, addition and multiplication of the matrices A and B have the same properties as addition and multiplication of real numbers, the proof for real numbers extends to matrices. We could also have used this argument for 4a) and 4b).

5) Since A and $-A$ commute, it follows that $e^Ae^{-A} = e^{A+(-A)} = e^0 = I$.

6) tA and sA commute. □

The following example shows that if $AB \neq BA$, then there is no reason to expect that $e^{A+B} = e^Ae^B$.

Example 5.1.16. *If $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then e^{A+B}, e^Ae^B and e^Be^A all differ. Indeed, let $C = A + B$. For all $k \geq 2$, $A^k = A$, $C^k = C$, and $B^k = 0$. It follows that: $e^A = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$, $e^B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $e^{A+B} = \begin{pmatrix} e & e \\ 0 & 0 \end{pmatrix}$, $e^Ae^B = \begin{pmatrix} e & e \\ 0 & 1 \end{pmatrix}$ and $e^Be^A = \begin{pmatrix} e & 1 \\ 0 & 1 \end{pmatrix}$.*

Proposition 5.1.17 (Continuity and derivability)**.** *Let $A \in \mathcal{M}_d(\mathbb{R})$. The application $g : \mathbb{R} \to \mathcal{M}_d(\mathbb{R})$ defined by $g(t) = e^{tA}$ is of class C^∞ and*

$$g'(t) = Ag(t) = Ae^{tA} = e^{tA}A \tag{5.10}$$

Proof. We first show that g is differentiable with derivative $g'(t) = Ae^{tA}$. We begin by showing that $g'(0) = A$. Let $h(t) = \frac{g(t)-g(0)}{t} - A$. We want to show that $h(t) \to 0$ as $t \to 0$. We have

$$h(t) = \frac{1}{t}\left(e^{tA} - I - tA\right) = \frac{1}{t}\sum_{k=2}^{+\infty} \frac{t^kA^k}{k!} = t\sum_{k=2}^{+\infty} \frac{t^{k-2}A^k}{k!}$$

Therefore, for $|t| < 1$,

$$||h(t)|| \leq |t| \sum_{k=2}^{+\infty} \frac{|t^{k-2}|\,||A^k||}{k!} \leq |t| \sum_{k=2}^{+\infty} \frac{||A||^k}{k!} \leq |t| \exp(||A||) \to_{t\to 0} 0.$$

Therefore, g is differentiable at $t = 0$ and $g'(0) = A$. We now prove the general formula. We have:

$$\frac{g(t_0+t)-g(t_0)}{t} = \frac{e^{(t_0+t)A}-e^{t_0A}}{t} = \left(\frac{e^{tA}-I}{t}\right)e^{t_0A} \to_{t\to 0} g'(0)e^{t_0A} = Ae^{t_0A}$$

due to the previous computation. Therefore, g is differentiable and $g'(t) = Ae^{tA} = Ag(t)$ for all t. Since A and tA commute, we also have $Ae^{tA} = e^{tA}A$, hence we proved (5.10).

It remains to check that g is C^∞. To see this, note that g is differentiable, hence C^0. Moreover, if g is C^k then, since $g'(t) = Ag(t)$ and matrix multiplication is a C^∞ operation, g' is C^k; therefore g is C^{k+1}. Thus, by induction, g is C^k for all k. This concludes the proof. □

Corollary 5.1.18. *Let $A \in \mathcal{M}_d(\mathbb{R})$ and let $X_0 \in \mathbb{R}^d$. The application $X : \mathbb{R} \to \mathbb{R}^d$ defined by $X(t) = e^{tA}X_0$ is differentiable with derivative $X'(t) = Ae^{tA}X_0 = AX(t)$.*

Proof. If $M : \mathbb{R} \to \mathcal{M}_d(\mathbb{R})$ is differentiable and $X_0 \in \mathbb{R}^d$, then the application $X : \mathbb{R} \to \mathbb{R}^d$ defined by $X(t) = M(t)X_0$ is differentiable with derivative $X'(t) = M'(t)X_0$. Indeed,

$$\frac{X(t+h)-X(t)}{h} = \left(\frac{M(t+h)-M(t)}{h}\right)X_0 = (M'(t)+\varepsilon(h))X_0 \to_{h\to 0} M'(t)X_0,$$

where $\varepsilon(h)$ is a matrix in $\mathcal{M}_d(\mathbb{R})$ such that $\varepsilon(h) \to 0$. This ensures that $\varepsilon(h)X_0 \to_{h\to 0} 0$ because taking a subordinate norm: $||\varepsilon(h)X_0|| \leq ||\varepsilon(h)||\,||X_0||$. The result then follows from Proposition 5.1.17. □

Theorem 5.1.19. *Let $A \in \mathcal{M}_d(\mathbb{R})$ and $X_0 \in \mathbb{R}^d$. The initial value problem $X'(t) = AX(t)$ and $X(0) = X_0$ has a unique maximal solution: the function $X : \mathbb{R} \to \mathbb{R}^d$ defined by*

$$X(t) = e^{tA}X_0$$

Proof. $X(\cdot)$ is solution by Corollary 5.1.18. It is global, hence maximal. Finally, the function $G : \mathbb{R}^d \to \mathbb{R}^d$ defined by $G(X) = AX$ is C^1, hence there is a unique maximal solution. Uniqueness may also be shown directly. Indeed, if $(J, \tilde{X}(\cdot))$ is another solution (maximal or not), then letting $Y(t) = e^{-tA}\tilde{X}(t)$ we get: $Y'(t) = e^{-tA}(\tilde{X}'(t) - A\tilde{X}(t)) = 0$ since $\tilde{X}' = A\tilde{X}$. So for all t in J, $Y(t) = Y(0) = \tilde{X}(0) = X_0$ hence $\tilde{X}(t) = e^{tA}\tilde{X}_0$. That is, $\tilde{X}(\cdot)$ is just a restriction of the solution $X(\cdot)$. □

5.1.6 Computing the exponential of a matrix

To make a good use of Theorem 5.1.19, we need to learn how to compute the exponential of a matrix. We begin with some simple cases.

Case 1: A diagonal. If $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, then $A^k = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}$, so that

$$e^A = \sum_{k=0}^{+\infty} \frac{1}{k!}\begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{+\infty}\frac{\lambda_1^k}{k!} & 0 \\ 0 & \sum_{k=0}^{+\infty}\frac{\lambda_2^k}{k!} \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$$

More generally, if $A = diag(\lambda_1, ..., \lambda_d)$, then $e^A = diag(e^{\lambda_1}, ..., e^{\lambda_d})$ and $e^{tA} = diag(e^{\lambda_1 t}, ..., e^{\lambda_d t})$.

Case 2: A diagonalizable (in $\mathbb{C}$). If $A = P^{-1}DP$, with D diagonal, then $e^A = P^{-1}e^DP$. Similarly, $e^{tA} = P^{-1}e^{tD}P$. Since computing e^{tD} is immediate, this allows to compute e^{tA}.

Case 3: *A* nilpotent. If A is nilpotent, that is, if $A^q = 0$ for some integer $q \geq 1$, then

$$e^A = \sum_{k \geq 0} \frac{A^k}{k!} = \sum_{k=0}^{q-1} \frac{A^k}{k!}$$

since $A^k = 0$ for all $k \geq q$. Thus the sum defining e^A is finite and relatively easy to compute when q is small. Note that by a standard linear algebra exercise, if a $d \times d$ matrix A is nilpotent, then $A^d = 0$, thus, checking that a matrix is nilpotent, is a computation that takes a finite time.[7]

Case 4: *A=D+N* with *D* diagonal, *N* nilpotent, and *DN=ND*. Since D and N commute, $e^{D+N} = e^D e^N$, and since D is diagonal and N nilpotent, e^D and e^N are easy to compute.

Case 5: $\mathbf{A = P^{-1}(D+N)P}$ with D diagonal, N nilpotent, and $\mathbf{DN = ND}$. Then $e^A = P^{-1}e^D e^N P$.

General case It may be shown that any matrix A in $\mathcal{M}_d(\mathbb{C})$ may be put in the form

$$A = P^{-1}(D+N)P$$

with P invertible, D diagonal, N nilpotent, and $DN = ND$. The matrices P, D and N may be computed explicitly. Thus, the previous case is the general case. Readers who want to know how to compute the matrices P, D, N are invited to look up "décomposition de Dunford" on Wikipedia or any good linear algebra textbook. See also "Jordan reduction".

Note that for 2×2 matrices, the usual case in the exercises we will ask you to solve, the decomposition is readily obtained. Indeed, either A is diagonalizable or it has a single eigenvalue λ. But in the latter case, its characteristic polynomial is $P(X) = (X - \lambda)^2$ so that by Cayley-Hamilton Theorem $(A - \lambda I)^2 = 0$. Thus, letting $D = \lambda I$ and $N = A - \lambda I$ we get that $A = D + N$ with D diagonal, N nilpotent (since $N^2 = 0$), and $DN = ND$ because D is not only diagonal, but scalar.

5.2 The equation $X'(t) = AX(t) + B(t)$

The notion of the exponential of a matrix allows us to generalize Duhamel's formula for nonhomogeneous systems of linear equations, when the matrix A is constant.

Proposition 5.2.1. *Let $A \in \mathcal{M}_d(\mathbb{R})$. Let $B : \mathbb{R} \to \mathcal{M}_d(\mathbb{R})$ be continuous. Solutions of $X'(t) = AX(t) + B(t)$ are global and given by*

$$Y(t) = e^{tA}\left(X_0 + \int_0^t e^{-sA}B(s)ds\right). \tag{DF2}$$

with X_0 in $\mathbb{R}^d$. This formula gives the solution with value X_0 at $t = 0$.

Proof. First note that by Proposition 5.0.1, we know that solutions are defined on the whole $\mathbb{R}$ (hence at $t = 0$) and that for each X_0 in $\mathbb{R}^d$, there is a unique solution with value X_0 at $t = 0$. Thus, it suffices to check that if $Y(\cdot)$ is given by (DF2), then it is a global solution with

[7] To prove that if a $d \times d$ matrix A is nilpotent, then $A^d = 0$, a possibility is to show that the sequence $KerA^k$, $k = 0, 1, ...$ is increasing for the inclusion relationship, that is, $KerA^k \subset KerA^{k+1}$, and that as soon as there is equality for some k, the sequence becomes stationary. Since each time $KerA^k$ is strictly included in $KerA^{k+1}$, the latter has a strictly higher dimension, and since these dimensions are at most equal to d, such a strict inclusion may happen at most d times. Thus, after d steps (starting at $k = 0$), the sequence must be stationary, that is, $KerA^d = KerA^k$ for all $k \geq d$. If $KerA^d = \mathbb{R}^d$, then $A^d = 0$ and the matrix is nilpotent. Otherwise, no power of A is equal to 0, hence the matrix is not nilpotent.

$Y(0) = X_0$. We'll let you do it. Of course, this perfectly correct proof feels like cheating, since this does not tell us where the formula comes from!

To get the formula, as in dimension 1, start with the general solution $t \to e^{tA}X_0$ of the associated homogeneous equation. Then let the constant vector X_0 vary, that is, look for solutions of the form $Y(t) = e^{tA}X(t)$. Formally, let $Y : \mathbb{R} \to \mathbb{R}^d$ be differentiable. Define $X : \mathbb{R} \to \mathbb{R}$ by $X(t) = e^{-tA}Y(t)$. Then $X(\cdot)$ is differentiable and:

$$Y'(t) = Ae^{tA}X(t) + e^{tA}X'(t) = e^{tA}(AX(t) + X'(t)) = AY(t) + e^{tA}X'(t)$$

Therefore, $Y'(t) = AY(t) + B(t)$ if and only if $e^{tA}X'(t) = B(t)$, that is, $X'(t) = \tilde{B}(t)$, where $\tilde{B}(t) = e^{-tA}B(t)$ is a column vector. Letting $x_i(t)$ and $\tilde{b}_i(t)$ denote the i^{th} coordinate of $X(t)$ and $\tilde{B}(t)$, respectively, this means that:

$$\begin{pmatrix} x_1'(t) \\ \vdots \\ x_d'(t) \end{pmatrix} = \begin{pmatrix} \tilde{b}_1(t) \\ \vdots \\ \tilde{b}_d(t) \end{pmatrix}$$

That is, for all i in $\{1, .., d\}$, $x_i'(t) = \tilde{b}_i(t)$. This is equivalent to $x_i(t) = x_{i0} + \int_0^t \tilde{b}_i(s)\,ds$, for some constant x_{i0}. In matrix form, this reads

$$X(t) = X_0 + \int_0^t \tilde{B}(s)\,ds.$$

where $X_0 = (x_{10}, ..., x_{d0})^T$. Thus, $Y(\cdot)$ is solution of $Y'(t) = AY(t) + B(t)$ if and only if $X(t) = X_0 + \int_0^t e^{-sA}B(s)ds$ for some X_0 in $\mathbb{R}^d$. This is equivalent to (DF2). □

Exercise 5.2.2. *Show that the solution with value X_0 at t_0 is $Y(t) = e^{tA}\left(e^{-t_0A}X_0 + \int_{t_0}^t e^{-sA}B(s)ds\right)$.*

Another approach to the equation

$$X'(t) = AX(t) + B(t) \tag{NH}$$

is to start from a basis $(X_1(\cdot), ..., X_d(\cdot))$ of the set of solutions of the associated homogeneous equation (H). The general solution of (H) is $X(t) = \sum_i \mu_i X_i(t)$. Now, let the constants μ_i vary. That is, search for a solution of (NH) of the form

$$X(t) = \sum_i \mu_i(t)X_i(t)$$

with $\mu_i(\cdot) : \mathbb{R} \to \mathbb{R}$ differentiable. We then have

$$X' = \sum_i(\mu_i'X_i + \mu_iX_i') = \sum_i(\mu_i'X_i + \mu_iAX_i) = \sum_i \mu_i'X_i + A\left(\sum_i \mu_iX_i\right) = AX + \sum_i \mu_i'X_i.$$

Thus, $X(\cdot)$ is solution of (NH) if and only if for all t in $\mathbb{R}$:

$$AX(t) + \sum_i \mu_i'(t)X_i(t) = AX(t) + B(t),$$

that is, if and only if

$$\sum_i \mu_i'(t)X_i(t) = B(t)$$

Letting $M(t)$ denote the matrix with columns $X_1(t), ...X_d(t)$, and $\mu(t) = (\mu_1(t), ..., \mu_d(t))^T$, this condition may be written $M(t)\mu'(t) = B(t)$. Moreover, the matrix $M(t)$ is invertible since by point 3 in Proposition 5.0.1, $(X_1(t), ...X_d(t))$ is a basis of X_H for any t in $\mathbb{R}$. Thus $X(t)$ is solution if and only if

$$\mu'(t) = M^{-1}(t)B(t).$$

It then suffices to integrate in order to find $\mu(t)$ (up to d integration constants which depend on the initial condition), hence $X(t)$. As the method used to derive (DF2), this is a generalization of the method of variation of constants in dimension 1. This method is natural when A is diagonalizable, so that we easily obtain a basis of S_H. The method using the exponential is natural when A is not diagonalizable. Note, however, that the exponential method is essentially a particular case of the basic method with $X_i(t) = e^{tA}V_i$, where V_i is the i^{th} vector of the canonical basis.

We end this chapter by coming back to our initial equation: $X'(t) = A(t)X(t) + B(t)$. The difficulty in solving this equation is not the term $B(t)$, but the fact that the matrix $A(t)$ is not constant. If for any times s and t, $A(t)$ and $A(s)$ commute (which boils down to the fact that $A(t)$ and $A'(t)$ commute when $A(\cdot)$ is differentiable), then it may be seen that Eq. (DF2) may be generalized, replacing as in dimension 1 the term tA by $\int_0^t A(s)ds$. If $A(t)$ does not commute with $A(s)$ for all t, s, then life is harder and the equation cannot always be solved explicitly.

Chapter 6

Nth order equations

Many differential equations are second-order equations, in particular those coming from physics, and higher order-equations may also arise. A way to study these equations is to show that they are equivalent to an auxiliary first-order equation. This is the topic of the following section. Some techniques however are specific to n^{th} order equations. This is the topic of Section 6.2.

6.1 How to reduce a n^{th} order equation to a first-order one?

An example. Let g denote the gravitational constant of the Earth and consider the equation

$$z''(t) = -g \tag{6.1}$$

with $z(t) \in \mathbb{R}$. This is a second-order differential equation: indeed it involves a second derivative. It models the evolution of the altitude of an object with unit mass subject to the earth gravitational force. Maximal solutions are easily found by integrating twice. These are the functions $z : \mathbb{R} \to \mathbb{R}$ such that $z(t) = -\frac{1}{2}g(t-t_0)^2 + v_0(t-t_0) + z_0$, where v_0 and z_0 correspond respectively to the ascending speed and the altitude at time t_0. Note that there is an infinite number of solutions such that $z(t_0) = z_0$. This does not contradict Picard-Lindelöf theorem, since the equation is not of first-order.

To apply Picard-Lindelöf theorem, we need to transform the equation into a first-order one. To do so, let $F(z_1, z_2) = (z_2, -g)$ and $Z(t) = (z_1(t), z_2(t))$. We then have:

$$Z'(t) = F(Z(t)) \Leftrightarrow \left\{ \begin{array}{c} z_1'(t) = z_2(t) \\ z_2'(t) = -g \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} z_2(t) = z_1'(t) \\ z_1''(t) = -g \end{array} \right.$$

Thus, $Z(\cdot)$ is solution of

$$Z'(t) = F(Z(t)) \tag{6.2}$$

if and only if its second coordinate is the derivative of its first coordinate, and its first coordinate is a solution of (6.1); that is, if and only if it is of the form $Z(t) = (z(t), z'(t))$ for some solution $z(\cdot)$ of (6.1). Moreover, it is easily checked that $Z(\cdot)$ is then maximal if and only if $z(\cdot)$ is maximal.

Exercise 6.1.1. *Show that solving (6.1) is equivalent to solving (6.2) in the sense that the mapping from the set of maximal solutions of (6.1) to the set of maximal solutions of (6.2) which maps the real valued function $z(\cdot)$ to $Z(\cdot) = (z(\cdot), z'(\cdot))$ is well defined and bijective.*

Applying Picard-Lindelöf to the first order equation (6.1) and coming back to (6.2) yields:

Proposition 6.1.2. *Let $t_0 \in \mathbb{R}$, let $(z_0, v_0) \in \mathbb{R}^2$. There is a unique maximal solution of (6.1) such that $z(t_0) = z_0$ and $z'(t_0) = v_0$.*

Proof. Let (IPV1) denote the problem $z''(t) = -g$ with $(z(t_0), z'(t_0)) = (z_0, v_0)$ and (IPV2) the problem $Z'(t) = F(Z(t))$ with $Z(t_0) = (z_0, v_0)$. Since F is C^1, Picard-Lindelöf Theorem tells us that (IPV2) has a unique maximal solution. Denote it by $(J, Z(\cdot))$ and let $Z(\cdot) = (z_1(\cdot), z_2(\cdot))$. Then, as we saw, $z_1(\cdot)$ is a maximal solution of (6.1) and $z_1'(\cdot) = z_2(\cdot)$ so that $(z_1(t_0), z_1'(t_0)) = (z_1(t_0), z_2(t_0)) = (z_0, v_0)$. Therefore, $z_1(\cdot)$ is a maximal solution of (IPV1). Similarly, if $(\hat{J}, z(\cdot))$ is another maximal solution of (IPV1), then letting $\hat{Z}(\cdot) = (z(\cdot), z'(\cdot))$, we get that $(\hat{J}, \hat{Z})$ is solution of (IPV2). By uniqueness of the maximal solution of (IPV2), $(\hat{J}, \hat{Z}(\cdot)) = (J, Z(\cdot))$ hence $(\hat{J}, z(\cdot)) = (J, z_1(\cdot))$. Thus, (IPV1) has a unique maximal solution. □

The key-point is that the initial condition does not involve just the initial position $z(t_0)$, but also the initial speed $z'(t_0)$. This shows that, even if we are only interested in the evolution of the position of the object, we need to describe its state as a vector (position, speed) and study the evolution of this vector in $\mathbb{R}^2$, called the phase space of this equation.

General case. More generally, an explicit n^{th} order equation is an equation of the form

$$x^{(n)}(t) = f(t, x(t), x'(t), ..., x^{(n-1)}(t)) \tag{6.3}$$

with $x(t) \in \mathbb{R}^d$ and $f : \Omega \to \mathbb{R}^d$ where Ω is an open subset of $\mathbb{R} \times \mathbb{R}^{nd}$. If $\Omega = I \times \mathbb{R}^{nd}$ where I is an open interval, then a solution of (6.3) is a couple $(J, x(\cdot))$ where J is a nonempty open subinterval of I and $x : J \to \mathbb{R}^d$ a differentiable function satisfying (6.3) for all t in J. The notions of maximal and global solutions are the same as for first-order equations. To simplify notation, we assume in what follows that $d = 1$, that is, $x(t) \in \mathbb{R}$. The case $x(t) \in \mathbb{R}^d$ is similar.

Assume thus that $d = 1$. The equation (6.3) may be reduced to a first-order equation in $\mathbb{R}^n$ by letting

$$F(t, x_1, ..., x_n) = (x_2, ..., x_n, f(t, x_1, ..., x_n)).$$

The function defined by $X(t) = (x_0(t), x_1(t), ..., x_{n-1}(t))$ is the solution of

$$X'(t) = F(t, X(t)). \tag{6.4}$$

if and only if it is of the form $X(t) = (x(t), x'(t), ..., x^{(n-1)}(t))$ for some function $x(\cdot)$ solution of (6.3). Note that F is as regular as f. We let the reader check that applying Picard-Lindelöf theorem to (6.4) and then coming back to the initial equation, we obtain:

Proposition 6.1.3. *If f is C^1 or Lipschitz, then for any initial time t_0 and vector $(x_0, x_1, ..., x_{n-1}) \in \mathbb{R}^n$, there is a unique maximal solution of (6.3) such that for all $k \in \{0, ..., n-1\}$, $x^{(k)}(t_0) = x_k$.*

The phase space is then $\mathbb{R}^n$. For the explosion alternative, we obtain:

Proposition 6.1.4. *If f is C^1 or Lipschitz and $z : J \to \mathbb{R}$ is a maximal solution of (6.3), then:*

1) $\sup J = \sup I$ *or (*$\sup J < \sup I$ *and* $||(z(t), z'(t), ..., z^{(n-1)}(t))|| \to +\infty$ *as* $t \to \sup J$*).*

2) $\inf J = \inf I$ *or (*$\inf J > \inf I$ *and* $||(z(t), z'(t), ..., z^{(n-1)}(t))|| \to +\infty$ *as* $t \to \inf J$*).*

The fact that $||(z(t), z'(t), ..., z^{(n-1)}(t))||$ goes to infinity does not imply that $|z(t)|$ goes to infinity. E.g., it might be that $z(\cdot)$ is bounded but that the absolute value $|z'(t)|$ of its derivative goes to infinity in $\sup J$. This is why $\sup J < \sup I$ does not imply that $|z(t)| \to +\infty$ as $t \to \sup J$.

6.2 Nth order linear equations with constant coefficients

Consider the second order linear equation with constant coefficients:

$$x''(t) + a_1 x'(t) + a_0 x(t) = b(t) \tag{NH2}$$

with a_0, a_1, $x(t)$, $b(t)$ in $\mathbb{R}$. Letting $A = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}$, $B(t) = \begin{pmatrix} 0 \\ b(t) \end{pmatrix}$, we let the reader check that (NH2) is equivalent in the sense of Section 6.1 to the linear system of equations

$$X'(t) = AX(t) + B(t) \tag{6.5}$$

More generally, the n^{th} order linear differential equation with constant coefficients

$$x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + \cdots + a_1x'(t) + a_0x(t) = b(t) \tag{6.6}$$

may be solved through the system of n first-order linear differential equations (6.5) with

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix} \text{ and } B(t) = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ b(t) \end{pmatrix}$$

There is however a more efficient method to solve (6.6), at least when $b(t)$ has a simple form.

Let S_{NH} and S_H denote respectively the sets of solutions of (6.6) and of the associated homogeneous equation

$$x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + \cdots + a_1x'(t) + a_0x(t) = 0 \tag{6.7}$$

The characteristic polynomial of this equation is $P = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$. We now explain that:

1) S_{NH} is an affine space with direction S_H;

2) a basis of S_H may be found through the decomposition of the characteristic polynomial;

3) a particular solution of (6.6) may be easily found when $b(t)$ has a simple form, which together with 1) and 2), allows to solve (6.6) .

The next proposition proves point 1).

Proposition 6.2.1. *Let $X_0 = (x_0, x_1, .., x_{n-1}) \in \mathbb{R}^n$*

1. *There is a unique solution of (6.6) such that $(x(t_0), x'(t_0), ..., x^{(n-1)}(t_0)) = X_0$ and it is global. The same holds for (6.7) as a particular case.*
2. *S_H is a real vector space of dimension n.*
3. *If $x_p(\cdot) \in S_{NH}$, then $S_{NH} = x_p(\cdot) + S_H$.*

Proof. 1. Existence and uniqueness of a solution follows from the version of Picard-Lindelöf theorem for n^{th} order differential equations. It is global because solutions of (6.5) are global.

2 & 3. The proof is the same as the proof of the corresponding results in Proposition 5.0.1. □

The next proposition proves point 2). In its statement, we make an abuse of notation and denote by $t^k e^{\lambda t}$ the function from $\mathbb{R}$ to $\mathbb{R}$ defined by $x(t) = t^k e^{\lambda t}$.

Proposition 6.2.2.

1. *Let $\lambda \in \mathbb{R}$. If $P(\lambda) = 0$, then $e^{\lambda t}$ is solution of (6.7). More generally, if λ is a root of P with multiplicity m then $t^k e^{\lambda t}$ is solution of (6.7) for all $k \in \{0, 1, ..., m-1\}$.*

2. *Let $\lambda = \alpha + i\beta$, with $\beta \neq 0$. If λ is a root of P with multiplicity m, then $t^k e^{\alpha t}\cos(\beta t)$ and $t^k e^{\alpha t}\sin(\beta t)$ are solutions of (6.7) for all $k \in \{0, 1, ..., m-1\}$.*

3. *The solutions obtained as above form a basis of S_H.*

Proof. For simplicity, we only prove the result in the case of second order equations ($n = 2$), which is the one which typically arises in applications. The proof in the general case is similar but requires more tedious computations.

1. Assume $n = 2$. Let $x(t) = e^{\lambda t}$. Then, the kth derivative of x is $x^{(k)}(t) = \lambda^k e^{\lambda t}$, therefore $x''(t) + a_1 x'(t) + a_0 x(t) = \lambda^2 e^{\lambda t} + a_1 \lambda e^{\lambda t} + a_0 e^{\lambda t} = P(\lambda)e^{\lambda t}$. Therefore, if $P(\lambda) = 0$ then $t \to e^{\lambda t}$ is solution of (H). The proof is the same for a general n.

Now, let λ be a root of P with multiplicity 2, so that $P(\lambda) = P'(\lambda) = 0$. We already know that $t \to e^{\lambda t}$ is a solution of (6.7). Moreover, letting $x(t) = te^{\lambda t}$ we have $x'(t) = e^{\lambda t}(\lambda t + 1)$ and more generally, $x^{(k)}(t) = e^{\lambda t}(\lambda^k t + k\lambda^{k-1})$. Thus,

$$x''(t) + a_1 x'(t) + a_0 x(t) = e^{\lambda t}[(\lambda^2 t + 2\lambda) + a_1(\lambda t + 1) + a_0 t] = e^{\lambda t}(P(\lambda)t + P'(\lambda)) = 0$$

Thus, $t \to te^{\lambda t}$ is indeed solution. In the case $n = 2$, a root of P has multiplicity at most 2, so this concludes the proof. For a general n, it may be shown as a corollary of Lemma 6.2.4 below that if $x(t) = te^{\lambda t}$, then we still have $x^{(n)}(t) + ... + a_1 x'(t) + a_0 x(t) = e^{\lambda t}(P(\lambda)t + P'(\lambda))$, and that more generally, if $x(t) = t^k e^{\lambda t}$, then $x^{(n)}(t) + ... + a_1 x'(t) + a_0 x(t) = e^{\lambda t}Q(t)$ where Q is a polynomial whose coefficients are all 0 if $P(\lambda) = ... = P^{(k)}(\lambda) = 0$. The result follows.

2. The same computation, but in $\mathbb{C}$, shows that if λ is a root of P with multiplicity q, then $t^k e^{\lambda t}$ is a complex solution of (6.7) for all $k \in \{1, ..., q-1\}$; but it is easily seen that the real and imaginary parts of a complex solution are real solutions. The result follows.

3. First note that a real root with multiplicity q gives q solutions. A complex root λ with multiplicity q gives $2q$ solutions, but its conjugate root $\bar{\lambda}$ gives the same solutions (or solutions that are proportional), so together, λ and $\bar{\lambda}$ give $2q$ potentially independent solutions. Thus, the number of solutions we obtained from the roots of P is equal to the sum of the multiplicity of these roots, that is, to n, which is the dimension of S_H. To prove that these solutions form a basis of S_H, it remains to check that they are independent. For simplicity, we do it only in the case $n = 2$. There are then three cases.

Case 1: If P has two distinct real roots λ_1, λ_2, then the two solutions we obtain are $s_1(t) = e^{\lambda_1 t}$, $s_2(t) = e^{\lambda_2 t}$. Let $(\mu_1, \mu_2) \in \mathbb{R}^2$ and assume that $x(\cdot) = \mu_1 s_1(\cdot) + \mu_2 s_2(\cdot) = 0$, that is, $x(t) = \mu_1 e^{\lambda_1 t} + \mu_2 e^{\lambda_2 t} = 0$ for all $t \in \mathbb{R}$. Without loss of generality, assume $\lambda_2 > \lambda_1$. Then as $t \to +\infty$, $x(t)e^{-\lambda_2 t} = \mu_1 e^{(\lambda_1 - \lambda_2)t} + \mu_2 \to_{t \to +\infty} \mu_2$. But $x(t) = 0$ for all t, therefore, $\mu_2 = 0$, and then $\mu_1 = 0$ as well. It follows that $s_1(\cdot)$ and $s_2(\cdot)$ are independent.[1]

Case 2: if P has a double real root λ, then the solutions we obtain are $s_1(t) = e^{\lambda t}$ and $s_2(t) = te^{\lambda t}$. Since $s_1(t)/s_2(t) \to 0$ as $t \to +\infty$, an argument similar to the one of Case 1 shows that s_1 and s_2 are independent.

Case 3: If P has two complex conjugate roots λ, $\bar{\lambda}$ with $\lambda = \alpha + i\beta$, $\beta \neq 0$. The solutions we obtain are then $s_1(t) = e^{\alpha t}\cos\beta t$ and $s_2(t) = e^{\alpha t}\sin\beta t$. If $\mu_1 s_1 + \mu_2 s_2 = 0$, then taking $t = 0$ gives $\mu_1 = 0$, and then $\mu_2 = 0$. So s_1 and s_2 are independent. □

To solve the nonhomogeneous equation (6.6), we still need to find a particular solution. In

[1] This is an analytic proof. A more algebraic proof is as follows: if $x(\cdot) = \mu_1 s_1(\cdot) + \mu_2 s_2(\cdot) = 0$, then we also have $x'(\cdot) = \lambda_1 \mu_1 s_1(\cdot) + \lambda_2 \mu_2 s_2(\cdot) = 0$. It follows that for all t, $AX(t) = 0$ where $A = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$ and $X(t) = \begin{pmatrix} \mu_1 s_1(t) \\ \mu_2 s_2(t) \end{pmatrix}$. Since $\det(A) = (\lambda_2 - \lambda_1) \neq 0$, the matrix A is invertible, and $AX(t) = 0$ implies $X(t) = 0$. It follows that the functions $\mu_1 s_1$ and $\mu_2 s_2$ are both zero, hence that $\mu_1 = \mu_2 = 0$.

general, this may be done by variation of parameters (but be careful, see this note)[2]. However, when $b(t)$ has a simple form, e.g., polynomial, then it is possible to find a particular solution directly, thanks to the following result.

Proposition 6.2.3. *Let Q be a polynomial of degree q and $\lambda \in \mathbb{R}$. Let m be the multiplicity of λ as a root of P ($m = 0$ if $P(\lambda) \neq 0$).*

1a). If $b(t) = e^{\lambda t}Q(t)$, then there is a solution of the form $e^{\lambda t}S(t)$ where S is a polynomial of degree $q + m$, and with valuation at least m (that is, all the terms of degree $m - 1$ or less are zero). In particular, if $P(\lambda) \neq 0$, then there is a solution of the above form with $\deg S = q$.

1b). If $b(t) = Q(t)$, then there is a polynomial solution of degree $q + m$ where m is the smallest integer such that $a_m \neq 0$.

1c). If $b(t) = e^{\lambda t}$, then there is a solution of the form $\mu t^m e^{\lambda t}$ where m is the multiplicity of λ as a root of P.

2) If $b(t) = e^{\alpha t}\cos(\beta t)Q(t)$ or $b(t) = e^{\alpha t}\sin(\beta t)Q(t)$, then there is a solution of the form $e^{\alpha t}[\cos(\beta t)S_1(t)+\sin(\beta t)S_2(t)]$, where S_1 and S_2 are polynomials such that $\max(\deg S_1, \deg S_2) = q + m$, where m is the multiplicity of $\lambda = \alpha + i\beta$ as a root of P.

We first need a lemma.

Lemma 6.2.4. *Let $P = a_nX^n + ... + a_1X + a_0$ and S be polynomials. Let $\lambda \in \mathbb{C}$, and $x(t) = e^{\lambda t}S(t)$. Let $\varphi_x(\cdot) = a_nx^{(n)}(\cdot) + ... + a_1x'(\cdot) + a_0x(\cdot)$. Then*

$$\varphi_x(t) = e^{\lambda t}\sum_{k=0}^{n} P^{(k)}(\lambda)\frac{S^{(k)}(t)}{k!} \tag{6.8}$$

Proof. Since both φ_x and the right-hand-side of (6.8) depend linearly on P, it suffices to prove the formula when $P = X^q$. We then have $P^{(k)}(\lambda) = \frac{q!}{(q-k)!}\lambda^{q-k}$ and

$$\varphi_x(t) = x^{(q)}(t) = \sum_{k=0}^{q}\frac{q!}{k!(q-k)!}(e^{\lambda t})^{(q-k)}S^{(k)}(t) = \sum_{k=0}^{q}\frac{q!}{k!(q-k)!}e^{\lambda t}\lambda^{q-k}S^{(k)}(t)$$

Using the previous expression for $P^{(k)}(\lambda)$ then yields (6.8). □

We now prove the proposition

Proof. 1a). Let m denote the multiplicity of λ as a root of P, so that $P^{(k)}(\lambda) = 0$ for all $k < m$ and $P^{(m)}(\lambda) \neq 0$. Let $f : \mathbb{R}_{q+m}[X] \to \mathbb{R}_q[X]$ be the linear map defined by $f(S) = \sum_{k=0}^{n} P^{(k)}(\lambda)\frac{S^{(k)}}{k!}$. Since $P^{(k)}(\lambda) = 0$ for all $k < m$, $f(S) = \sum_{k=m}^{n} P^{(k)}(\lambda)\frac{S^{(k)}}{k!}$; so if $deg(S) \leq q + m$ then $deg(f(S)) \leq q$, hence f is well defined.

We claim that $Ker f = \mathbb{R}_{m-1}[X]$. Indeed, if $S = 0$ or $deg\, S < m$, then $S^{(k)} = 0$ for all $k \geq m$, and $f(S) = 0$. Otherwise, since by definition of m, $P^{(m)}(\lambda) \neq 0$, $f(S)$ is of degree $(deg\, S) - m \geq 0$, hence $f(S) \neq 0$. It follows that the rank of f is $(q + m + 1) - m = q + 1 = \dim \mathbb{R}_q[X]$, hence that f is onto. Therefore, there exists S in $\mathbb{R}_{q+m}[X]$ such that $f(S) = Q$.

By Lemma 6.2.4, this implies that the function defined by $x(t) = e^{\lambda t}S(t)$ is solution of (6.6). Note that since $deg(f(S)) = deg(S) - m$, we have $deg(S) = q + m$. Moreover, if $\tilde{S}$ is

[2] Though $x(t) \in \mathbb{R}$, due to the fact that the equation is of order n, the equation is equivalent to a system of n linear equations, not to a first order equation in dimension 1. Thus the appropriate method of variation of constants is not to take a single solution $x(\cdot)$ of (6.7) and to search for solutions of the form $y(t) = \lambda_i(t)x(t)$. Rather, it consists in taking a basis of solutions of (6.7) $(x_1(\cdot), ..., x_n(\cdot))$, and to search for solutions of the form $y(t) = \sum_i \lambda_i(t)x_i(t)$ satisfying certain conditions. These conditions, which we do not write here, are those that arise from the method of variations of parameters for the equivalent system of linear equations. It is just as simple as finding a particular solution of this system, and then saying that its first component is a particular solution of (6.6).

the polynomial obtained from S by subtracting all the terms in S of degree $m-1$ or less, so that $deg(\tilde{S}-S) < m$, we have: $f(\tilde{S}) = f(S) + f(\tilde{S}-S) = f(S) = Q$, so that the function $t \to \tilde{x}(t) = e^{\lambda t}\tilde{S}(t)$ is also solution. Since $deg(\tilde{S}) = deg(S) = q+m$ and since $\tilde{S}$ has valuation at least m, this proves 1a).

1b) and 1c) are particular cases of 1a), with respectively $\lambda = 0$, and $Q = 1$.

2) Consider the equation in $\mathbb{C}$ with $b(t) = e^{\lambda t}Q(t)$ where $\lambda = \alpha + i\beta$. The same argument as for 1a) but in $\mathbb{C}[X]$ shows that there is a complex solution of the form $x(t) = e^{\lambda t}S(t)$, with $degS = q+m$. Write $S = S_r + iS_{im}$. Taking real and imaginary parts of the equality $x^{(n)}(t) + ... + a_0x(t) = e^{\lambda t}Q(t)$ shows that the real and imaginary parts of $t \to e^{\lambda t}S(t)$ are solutions of the equation with respectively $b(t) = e^{\alpha t}\cos(\beta t)Q(t)$, and $b(t) = e^{\alpha t}\sin(\beta t)Q(t)$, and they are both of the form $e^{\alpha t}[\cos(\beta t)S_1(t) + \sin(\beta t)S_2(t)]$ with $\max(degS_1, degS_2) = degS$. The result follows. □

Superposition principle. Finally, if $b(t)$ is the sum of two terms with a nice form, then as for first-order equations, we may use the superposition principle. That is, if $x_1(\cdot)$ is a solution of the equation with right-hand side $b_1(\cdot)$ and $x_2(\cdot)$ of the equation with right-hand side $b_2(\cdot)$, then $x_1(\cdot) + x_2(\cdot)$ is a solution of the equation with right-hand side $b_1(\cdot) + b_2(\cdot)$. The proof is the same as for Proposition 3.3.6.

Chapter 7

Stability and linearization

7.1 Definitions

Consider an autonomous differential equation $X'(t) = F(X(t))$ with $F : \mathbb{R}^d \to \mathbb{R}^d$ of class C^1. Let X^* be an equilibrium of this equation. We say that:

- X^* is *stable* (or *Lyapunov stable*), if for any neighborhood V of X^*, there is a neighborhood W of X^* such that any solution starting in W remains in V at all later times. That is, if $X(0) \in W$, then $X(t) \in V$ for all $t \geq 0$.

- X^* is *attracting*, if it has a neighborhood V such that any solution starting in V converges towards X^* as $t \to +\infty$. That is, $X(0) \in V \Rightarrow X(t) \to_{t \to +\infty} X^*$.

- X^* is *asymptotically stable*, if it is both stable and attracting.

- X^* is *unstable*, if it is not stable. That is, X^* is unstable if there exists a neighborhood V of X^* such that, for any neighborhood W of X^*, there exists X_0 in W such that the solution starting at X_0 eventually leaves V (forward in time).

Of course, in the definitions of stable and attracting, we implicitly require that any solution starting close enough to X^* be defined for all t in $[0, +\infty[$.

Equivalent definitions may be given using basis of neighborhoods of X^*. For instance, X^* is stable if and only if for any $\varepsilon > 0$, there exists $\alpha > 0$ such that, for any solution $X(\cdot)$: $||X(0) - X^*|| \leq \alpha \Rightarrow \forall t \geq 0, ||X(t) - X^*|| \leq \varepsilon$.

Exercise 7.1.1. *Give equivalent definitions of "attracting" and "unstable" in a similar way.*

Exercise 7.1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be C^1. Show that for the differential equation $x'(t) = f(x(t))$, attracting implies stable, so that attracting is equivalent to asymptotically stable. This is no longer the case in higher dimension.*

7.2 Stability of linear systems

Before stating the next proposition, we need some vocabulary. Let $A \in \mathcal{M}_d(\mathbb{R})$ and let λ be a (real or complex) eigenvalue of A. The *geometric multiplicity* of λ is the dimension of $Ker(A - \lambda I)$. The *algebraic multiplicity* of λ is its multiplicity as a root of the characteristic polynomial of A.

Exercise 7.2.1. *Show that the algebraic multiplicity is always weakly larger than the geometric multiplicity.*

Exercise 7.2.2. *Show that A is diagonalizable if and only if the geometric multiplicity of each eigenvalue of A is equal to its algebraic multiplicity.*

Exercise 7.2.3. *What are the geometric and algebraic multiplicity of* 2 *as an eigenvalue of the following matrix?*

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Consider a system of linear autonomous differential equations $X'(t) = AX(t)$, with $A \in \mathcal{M}_d(\mathbb{R})$. Note that the origin (the zero of $\mathbb{R}^d$) is an equilibrium for this system. Let $Sp(A)$ denote the spectrum of A, that is, its set of eigenvalues. Let $a = \max_{\lambda \in Sp(A)} Re(\lambda)$ be the maximal real part of the eigenvalues of A. The following result shows that when $a \neq 0$, the sign of a determines the stability of the origin.

Theorem 7.2.4. *(stability of linear systems)*

1. *If $a < 0$, then* 0 *is asymptotically stable.*
2. *If $a > 0$, then* 0 *is unstable.*
3. *If $a = 0$ then* 0 *is not asymptotically stable. It is stable if and only if the geometric multiplicity of each eigenvalue of A with zero real part is equal to its algebraic multiplicity. This is the case for instance if A is diagonalizable.*

Proof. (sketch) The proof of point 2. is simple: just consider solutions starting on an unstable eigenaxis (that is, the span of an eigenvector associated with a positive eigenvalue) or what could be called an unstable eigenplane (the span of the real and imaginary part of a complex eigenvector associated to a complex eigenvalue with a positive real part).

The proof of point 1. is based on the fact that any matrix A in $\mathcal{M}_d(\mathbb{R})$ may be written in the more convenient form $A = P^{-1}(D + N)P$ with P invertible, D diagonal, N nilpotent, $DN = ND$, moreover, N is triangular with zeros on the diagonal, which implies that D and A have the same eigenvalues (e.g., using the Jordan reduction of a matrix). Using the subordinate norm associated to the infinite norm on $\mathbb{R}^d$, we then have:

$$||e^{tA}|| \leq C||e^{tD}||\,||e^{tN}|| \text{ with } C = ||P^{-1}||\,||P||$$

Moreover, it is easily shown that $||e^{tD}|| \leq e^{at}$ where a is the maximal real part of the eigenvalues of D, hence of the eigenvalues of A. Besides, due to the fact that there is a finite number of term in the sum defining e^{tN}, $||e^{tN}|| \leq Q(|t|)$ for some polynomial Q of degree at most $d - 1$. It follows that:

$$||e^{tA}|| \leq e^{at}Q(|t|) \to_{t \to +\infty} 0$$

since $a < 0$. For each solution $X(\cdot)$, since $X(t) = e^{tA}X(0)$ and since we use a subordinate norm, we get $||X(t)|| \leq ||e^{tA}||\,||X(0)|| \to_{t \to +\infty} 0$. We say that 0 is globally attracting (it does not only attract solutions starting in a neighborhood of itself, but all solutions; in other words, we can take $V = \mathbb{R}^d$ in the definition of "attracting"). Moreover, since e^{tA} goes to zero as $t \to +\infty$, it follows that e^{tA} is bounded by some constant K on $[0; +\infty[$. Thus, for each $\varepsilon > 0$, if $||X(0)|| \leq \alpha := \varepsilon/K$, then for all $t \in [0, +\infty[$, $||X(t)|| \leq K||X(0)|| \leq \varepsilon$. Thus 0 is stable. Since it is also attracting, it is asymptotically stable.

The proof of point 3. is based on the fact that when $a = 0$, it may be shown that the norm of e^{tA} is bounded under the condition of the theorem, but grows till infinity at a polynomial speed if these conditions are not fulfilled, due to polynomial terms coming from e^{tN}. □

Exercise 7.2.5. *For the following matrices A, is the origin stable, asymptotically stable or unstable for the system $X' = AX$?*

$$A = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}; A' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}; A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}; A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}; \; A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Exercise 7.2.6. *Let A and P be $d \times d$ real matrices, with P invertible. Show that the origin has the same stability for $X' = AX$ and for $X' = P^{-1}APX$.*

Exercise 7.2.7. *Show that the following matrix is not diagonalizable but that for each of its eigenvalues with zero real part, the geometric multiplicity is equal to the algebraic multiplicity:*

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Exercise 7.2.8. *Consider the case of planar linear systems of differential equations. What is the stability of a saddle? a source? a spiral source? a sink? a spiral sink? a center?*

7.3 Linearization

Consider a nonlinear autonomous differential equation

$$X'(t) = F(X(t)) \tag{7.1}$$

with $F : \mathbb{R}^d \to \mathbb{R}^d$ of class C^1. Let X^* be an equilibrium and

$$A_{X^*} = \left(\frac{\partial F_i}{\partial x_j}\right)_{1 \leq i,j \leq d} (X^*).$$

be the Jacobian matrix of F at X^*. Then by definition of the Jacobian matrix,

$$F(X^* + h) = F(X^*) + A_{X^*}h + o(h),$$

where $h \in \mathbb{R}^d$. Since X^* is an equilibrium, $F(X^*) = 0$, hence

$$F(X^* + h) = A_{X^*}h + o(h) \tag{7.2}$$

Thus, if $X(\cdot)$ is a solution of (7.1) and $h(t) = X(t) - X^*$, we have:

$$h'(t) = X'(t) = F(X(t)) = F(X^* + h(t)) = A_{X^*}h(t) + o(h(t)).$$

That is,

$$h'(t) = A_{X^*}h(t) + o(h(t)) \tag{7.3}$$

The *linearized system* at X^* is the first-order approximation of (7.3), that is:

$$h'(t) = A_{X^*}h(t). \tag{7.4}$$

The[1] idea of introducing the linearized system is that close to X^*, the term $o(h)$ in (7.3) is small, so that the evolution of $h(t) = X(t) - X^*$ is almost given by (7.4). This suggests that the behavior of the linearized system close to 0 should provide information on the behavior of the initial system close to X^*. This should be the case if a condition that ensures that the first order approximation governs the local behavior of (7.3) is satisfied (like when, to use the derivative of a nonlinear function from $\mathbb{R}$ to $\mathbb{R}$ to know whether this function is locally increasing or decreasing, we need this derivative to be nonzero). This condition is that the equilibrium is hyperbolic, in the following sense:

Definition 7.3.1. *The equilibrium X^* is hyperbolic if the Jacobian matrix of F at X^* (that is, A_{X^*}) has no eigenvalues with zero real part.*

[1] Of course, we could write the linearized system as $X'(t) = AX(t)$: this is the same system and a more usual notation. We do not do so here, because we want to insist on the fact that the variable in the linearized system is meant to approximate the difference between the solution and the equilibrium in the initial, nonlinear system.

In that case, either all eigenvalues of A_{X^*} have negative real parts, and the origin is asymptotically stable under (7.4) by Theorem 7.2.4, or at least one eigenvalue of A_{X^*} has a positive real part and the origin is unstable under (7.4). Thus, for the linearized system associated to a hyperbolic equilibrium, the origin cannot be stable without being asymptotically stable. This explains why this case does not appear in the theorem below.

Theorem 7.3.2 (Linearization theorem)**.** *Assume that an equilibrium X^* of a nonlinear system (7.1) is hyperbolic. Then it has the same stability as the origin in the linearized system at X^*. That is, X^* is asymptotically stable for (7.1) if and only if 0 is asymptotically stable for (7.4), and unstable for (7.1) if and only if 0 is unstable for (7.4).*

Let us give some examples. Consider the equations in dimension 1: i) $x' = x^3$, ii) $x' = -x^3$, iii) $x' = 0$. In all three cases, the unique equilibrium is 0 and the linearized system is $h' = 0$. However, the equilibrium is unstable in case i), asymptotically stable in case ii), and stable but not asymptotically stable in case iii). This shows that we cannot deduce the stability of the equilibrium of the nonlinear equation from the linearized system. This is because, in this case, the Jacobian matrix at the equilibrium is a 1×1 matrix whose unique entry is zero, hence with a zero eigenvalue. Thus, the equilibrium is not hyperbolic and we cannot apply Theorem 7.3.2.

Now consider the nonlinear system:

$$\left\{ \begin{array}{lllll} x' & = & x^2 & + & y \\ y' & = & x & - & y \end{array} \right. \tag{7.5}$$

We let the reader check that there are two equilibria: $(0,0)$ and $(-1,-1)$, and that the Jacobian matrix at $X = (x,y)$ is $A_{(x,y)} = \begin{pmatrix} 2x & 1 \\ 1 & -1 \end{pmatrix}$. Thus, the Jacobian matrix at $X_1^* = (0,0)$ is $A_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ and the linearized system at $(0,0)$ is $h' = A_{(0,0)}h$, that is:

$$\left\{ \begin{array}{lllll} h_1' & = & & & h_2 \\ h_2' & = & h_1 & - & h_2 \end{array} \right. \tag{7.6}$$

Since $\det A_{X^*} = -1$, the eigenvalues of A_{X^*} are real and with opposite signs (in other words, the origin is a saddle for the linearized system). It follows that: a) X^* is hyperbolic (all eigenvalues of A_{X^*} have nonzero real parts) ; b) the origin is unstable for the linearized system (since at least one eigenvalue of A_{X^*} has a positive real part). Therefore, X^* is unstable for (7.5).

Similarly, at $X_2^* = (-1,-1)$, the Jacobian matrix is $A_{(-1,-1)} = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$. Its determinant is equal to 1. This implies that either the eigenvalues are real and with the same sign, or they are complex conjugates. In both cases, their real parts have the same sign, hence they are negative since $Tr(A_{(-1,-1)}) = -3 < 0$. It follows that X_2^* is hyperbolic, and that the origin is asymptotically stable for the linearized system at X_2^*. Therefore, by Theorem 7.3.2, X_2^* is asymptotically stable under (7.5). [2]

Remark 7.3.3. *The behavior of a smooth nonlinear system of differential equations near an hyperbolic fixed point is much more related to the behavior of the linearized system than stated in Theorem 7.3.2. This follows from the Hartman-Grobman Theorem, which is outside the scope of this course. In essence, it states that oriented trajectories of the smooth nonlinear system near an hyperbolic equilibrium are continuous deformations of oriented trajectories of the linearized system near 0. For a first approach, see, e.g., Section 8.1-8.3 of Hirsch, Smale and Devaney's textbook: "Differential Equations, Dynamical Systems and An Introduction to Chaos".*

[2] It is easily checked that the eigenvalues of $A_{(-1,-1)}$ are actually real, but this is not needed: knowing the sign of their real parts suffices.

Chapter 8

Geometric approach to differential equations

8.1 The slope field

Let $(J, X(\cdot)\ )$ be a solution of a non autonomous equation

$$X'(t) = F(t, X(t)) \tag{8.1}$$

with $F : \mathbb{R}^{d+1} \to \mathbb{R}^d$. Recall that, as usual, the graph of $X(\cdot)$ is the set $\Gamma = \{(t, X(t)), t \in J\}$. The aim of this section is to explain how to have a rough idea of this graph without explicitly solving the differential equation.

A *vector field* in $\mathbb{R}^n$ is a function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ seen as associating to each point X in $\mathbb{R}^n$ the vector starting from X and with coordinates $\varphi(X)$ (thus, going from X to $X + \varphi(X)$).

The *velocity field* associated to (8.1) is the vector field in $\mathbb{R}^{d+1}$ associating to the point (t, X) the vector starting from (t, X) and with coordinates $(1, F(t, X))$.

The *slope field* is a vector field colinear to the velocity field but where we consider normalized vectors for a better visualization.

The slope field allows to get a rough idea of the graph of the solutions. For simplicity, the result below is stated for $d = 1$, but it is actually general.

Proposition 8.1.1. *(Geometric characterization of solutions) Consider a differential equation*

$$x'(t) = f(t, x(t)) \tag{8.2}$$

with $f : \mathbb{R}^2 \to \mathbb{R}$. Let $x : J \to \mathbb{R}$ be differentiable. The following assertions are equivalent:

a) $(J, x(\cdot))$ is solution of (4.1); that is: $\forall t \in J$, $x'(t) = f(t, x(t))$.

b) For all t in J, the slope of the graph at $(t, x(t))$ is $f(t, x(t))$.

c) For all t in J, the tangent to the graph of $x(\cdot)$ at $(t, x(t))$ has the direction of the vector $(1, f(t, x(t)))$.

d) For all $t \in J$, the graph of $x(\cdot)$ is tangent at $(t, x(t))$ to the velocity field (or equivalently, to the slope field).

We leave the proof to the reader. In higher dimension, we similarly get that a differentiable function $X : J \to \mathbb{R}^d$ is solution of (8.1) if and only if for each t in J, the direction of the graph at $(t, X(t))$ is the direction of the slope field. Thus, a way to get a rough idea of the graph of solutions is to draw and "follow" the slope field. This is also the idea of numerical methods, such as Euler method, which we roughly explain below.

8.2 Idea of Euler method with a fixed step size

Let $X : J \to \mathbb{R}^d$ be a solution of (8.1) such that $X(t_0) = X_0$. We would like to approximate numerically this solution. To do so, fix a stepsize $\tau > 0$ and let $t_n = t_0 + n\tau$. Then consider the function Y_τ defined as follows:

a) $Y_\tau(t_0) = X_0$ (*we start from* X_0).

b) $Y_\tau(t_1) = Y_\tau(t_0) + \tau F(t_0, X_0)$, with $Y_\tau(t)$ affine between t_0 and t_1 (*from* t_0 *to* t_1*, we follow the slope field at* (t_0, X_0) *for* τ *units of time*).

c) $Y_\tau(t_2) = Y_\tau(t_1) + \tau F(t_1, Y_\tau(t_1))$ (*from* t_1 *to* t_2*, we follow the slope field at* $(t_1, Y_\tau(t_1))$ *for* τ *units of time*).

d) More generally $Y_\tau(t_{n+1}) = Y_\tau(t_n) + \tau f(t_n, Y_\tau(t_n))$, with $Y_\tau(t)$ affine between t_n and t_{n+1}.

Remark 8.2.1. *From* t_0 *to* t_1*, we follow the direction of the slope field at* (t_0, X_0)*; that is, we approximate the true solution by its tangent at* t_0*. From* t_1 *to* t_2*, we follow the direction of the slope field in* $(t_1, Y_\tau(t_1))$*, that is, the direction of the tangent to the solution of the equation with value* $Y_\tau(t_1)$ *at* t_1*. Since* $Y_\tau(t_1) \neq X(t_1)$*, this is not the solution that we try to follow. Thus there are two mistakes: first, approximating a curve by its tangent; second, using a wrong tangent, that is, a line with slope* $F(t_1, Y_\tau(t_1))$ *instead of* $F(t_1, X(t_1))$*.*

It is clear that these mistakes may accumulate, and that the further we get from X_0, the worse our guarantee (and our accuracy) on our numerical approximation will be. Nevertheless, it may be shown that on any compact subinterval of J, $Y_\tau(\cdot)$ converges uniformly towards $X(\cdot)$ as $\tau \to 0$, with an explicit bound on the approximation error.

Note that, if solutions of our differential equation are convex, then Euler method is bound to underestimate them, because the tangent is always below a convex curve. Similarly, we expect that Euler method will overestimate concave solutions. A way to deal with this problem is to use more refined methods, that try to estimate and take into account higher derivatives of F, for instance, the Runge-Kutta method which will be studied in Master 1.

8.3 The case of autonomous equations

For an autonomous differential equation $X'(t) = f(X(t))$, the slope field is invariant by translation through time. Together with Proposition 8.1.1, this gives a geometric proof to the fact that the set of solutions of an autonomous equation is invariant through time, a fact we already proved analytically.

Besides, let us define the direction field of this equation as the vector field in $\mathbb{R}^d$ associating to the point X the vector starting from X and with coordinates $f(X)$. It may be seen that, except for equilibria, trajectories of solutions are tangent to the direction field. Thus, for autonomous equations, following the direction field allows to get an idea of the phase portrait.[1]

Of course, we cannot draw the direction field at each and every point. So we will try to partition the phase space $\mathbb{R}^d$ (the state of possible states of the system) into several zones, such that in each of this zone trajectories go, roughly, qualitatively, in the same direction. A standard tool to do so are the nullclines. Given an autonomous equation $X'(t) = F(X(t))$, with $F = (F_1, ..., F_d)$, the i^{th}-nullcline is the set of points X in $\mathbb{R}^d$ such that $F_i(X) = 0$. In one of the regions separated by the nullclines, all the components of F have a constant sign, so that trajectories in this region go roughly in the same direction. This is best illustrated by an example.

[1] The expressions "velocity field", "slope field" and "direction field" may be used with a somewhat different meaning depending on authors, but the context should make the vocabulary clear.

Consider the linear system

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$

Here the nullclines are just the x-axis (on which $y' = 0$) and the y-axis (on which $x' = 0$). They separate $\mathbb{R}^2$ in four orthants. For any solution (x, y), as long as it lies in the open orthant $x > 0$, $y > 0$, we have $x' > 0$, $y' < 0$, so solutions move towards the "South-East" of $\mathbb{R}^2$. Similarly, solutions move South-West when $x > 0$, $y < 0$, North-West when $x < 0$, $y < 0$, and North-East when $x < 0$, $y > 0$. This suggests that, unless solutions escape to infinity, they keep rotating clockwise. Of course, we already knew that the system is linear and the origin a center, since it is easily checked. However, the same kind of arguments apply even to the nonlinear system

$$\begin{cases} x' = y + y^3 \\ y' = -x^3 \end{cases}$$

whose behavior cannot even be analyzed locally by linearization, since the unique equilibrium $(0, 0)$ is not hyperbolic (even if it were, this would only provide information on the behavior close to the origin, while the above geometrical arguments provide information on the behavior of the system globally, that is, in the whole $\mathbb{R}^2$). We will see in some exercises how to use these ideas to make rigorous statements, and sometimes fully understand qualitatively, the behavior of nonlinear systems.

For systems in $\mathbb{R}^2$, it may also be useful to draw the *isoclines*. These are the curves defined by $F_2/F_1 = k$ (or $y'/x' = k$), that is, the set of points X in $\mathbb{R}^2$ such that the slope of the trajectory at X is equal to k. The nullclines correspond to the particular case $k = 0$ (which corresponds to $y' = 0$) and $k = \pm\infty$ (which corresponds to $x' = 0$).

Chapter 9

Continuity of the flow

Consider an autonomous differential equation $X'(t) = F(X(t))$, with $F : \mathbb{R}^d \to \mathbb{R}^d$ of class C^1 or Lipschitz, and assume, to begin with, that all solutions are defined for all times. Let $\phi(t, X_0)$ denote the value at time t of the solution starting at X_0 (that is, satisfying $X(0) = X_0$). This defines a function $\phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ called the flow of this differential equation. The function $\phi_t : \mathbb{R}^d \to \mathbb{R}^d$ defined by $\phi_t(X_0) = \phi(t, X_0)$ is called the time t map of the flow. Applying the function ϕ_t amounts to following solutions for t units of time. When t is positive, this makes us travel into the future. When t is negative, into the past.

The flow is a fundamental tool to study differential equations, as it allows to consider solutions globally, and not only one at a time. For instance, if A is a subset of $\mathbb{R}^d$ filled at time 0 with imaginary particles whose law of motion is described by the differential equation, then $\phi_t(A)$ is the position of this set of particles t units of time later. It may be that each of these particles moves but that collectively they always occupy the same set of positions (think of a rotating circle), so that for all t, $\phi_t(A) = A$, in which case we say that A is *globally invariant*. It may be that for each subset A of $\mathbb{R}^d$, and for all times t, the volume of $\phi_t(A)$ is equal to the volume of A, in which case we say that the flow (or the underlying differential equation) is *volume-preserving*. These and other notions are very useful to study dynamical systems. Asking whether these properties hold requires a tool allowing to consider the behavior of sets of solutions, and this tool is the flow. Here, we will only study some of its basic properties. The most basic are the following.

Proposition 9.0.1 (basic properties of the flow). *1) $\phi_0(X_0) = X_0$ for all $X_0 \in \mathbb{R}^d$; in other words, $\phi_0 = Id$.*

2) $\phi_{s+t}(X_0) = \phi_s(\phi_t(X_0)) = \phi_t(\phi_s(X_0))$ for all X_0 in $\mathbb{R}^d$, and t, s in $\mathbb{R}$

3) For all t in $\mathbb{R}$, the time t map of the flow ϕ_t is invertible with inverse ϕ_{-t}.

4) $\frac{\partial \phi}{\partial t}(t, X_0) = F(\phi(t, X_0))$

Proof. There properties are all obvious once one understands the definition of the flow.

Point 1) just says that if we follow a solution for 0 units of time, we do not move!

Point 2) says that following a solution for $t + s$ units of time amounts to first following it for t units of time, and then again for s units of time starting for the point we had reached (or first s units of time, then t). The only subtle point is that this also holds if s, t or both are negative (e.g., if we follow a solution for, say, 10 units of time, and then go back in time for 1 unit of time, then it is as if we had followed the solution for 9 units of time.)

Point 3) follows from 1) and 2): by 2), $\phi_t \circ \phi_{-t} = \phi_{-t} \circ \phi_t = \phi_{t+(-t)} = \phi_0$ and by 1), $\phi_0 = Id$. The result follows.

For point 4), let $X_0 \in \mathbb{R}^d$, and denote by $X(t)$ the value at time t of the solution with initial condition $X(0) = X_0$. Thus, by definition of the flow, $\phi(t, X_0) = X(t)$. Therefore,

$\frac{\partial \phi}{\partial t}(t, X_0) = X'(t) = F(X(t)) = F(\phi(t, X))$. □

The following property implies that solutions of differential equations depend continuously on their initial position. We do not assume anymore that all solutions are defined for all times; thus, the time t map of the flow is now only defined on the set Ω_t of initial positions X_0 in $\mathbb{R}^d$ such that the solution satisfying $X(0) = X_0$ is defined until time t.

Proposition 9.0.2 (Continuity of solutions with respect to their initial position)**.** *Consider an autonomous differential equation $X'(t) = F(X(t))$ with $F : \mathbb{R}^d \to \mathbb{R}^d$ of class C^1 or Lipschitz. Let $t \in \mathbb{R}$. Fix X_0 in Ω_t. There exists $\varepsilon > 0$ and a constant K such that for any Y_0 in $\mathbb{R}^d$, if $||Y_0 - X_0|| < \varepsilon$, then $Y_0 \in \Omega_t$ and for all s between 0 and t:*

$$||\phi_s(Y_0) - \phi_s(X_0)|| \leq ||Y_0 - X_0||e^{K|s|}.$$

That is, if $X(\cdot)$ and $Y(\cdot)$ are the solutions such that $X(0) = X_0$ and $Y(0) = Y_0$, respectively,

$$||Y(s) - X(s)|| \leq ||Y_0 - X_0||e^{K|s|}. \tag{9.1}$$

Proof. We prove the result when F is K-Lipschitz. In this case, all solutions are defined for all times (check it!), so $\Omega_t = \mathbb{R}^d$ for all t. Moreoever, the result is then true for any ε, so that ε will not appear in the proof below. When F is C^1 but not Lipschitz, the proof uses the fact that F is still locally Lipschitz, hence the need to focus on a neighborhood of the trajectory of $X(\cdot)$. This is why we need to use ε in this case.

Assume thus that F is K-Lipschitz. Let $X(\cdot)$ and $Y(\cdot)$ be the solutions such that $X(0) = X_0$ and $Y(0) = Y_0$ respectively. We have:

$$X(t) = X(0) + \int_0^t X'(s)ds = X_0 + \int_0^t F(X(s))ds.$$

and similarly, $Y(t) = Y_0 + \int_0^t F(Y(s))ds$. Thus,

$$Y(t) - X(t) = (Y_0 - X_0) + \int_0^t [F(Y(s)) - F(X(s))]ds.$$

Letting $u(t) = ||Y(t) - X(t)||$, and using that

$$||F(Y(s)) - F(X(s))|| \leq K||Y(s) - X(s)||,$$

we obtain that for all $t \geq 0$,

$$u(t) \leq u(0) + \int_0^t Ku(s)\, ds.$$

Therefore, it follows from Gronwall's Lemma in the integral form, that for all $t \geq 0$, $u(t) \leq u(0)e^{Kt}$. A similar reasoning shows that for all $t \leq 0$, $u(t) \leq u(0)e^{K|t|}$, so that in general,

$$||Y(t) - X(t)|| \leq ||Y_0 - X_0||e^{K|t|}.$$

□

Remark 9.0.3. *Of course the above proposition holds even if we start from an initial time $t_0 \neq 0$.*

Corollary 9.0.4 (Continuously dependence from the initial data)**.** *Under the assumption of Proposition 9.0.2, if the initial condition Y_0 tends to X_0, then the solution $Y(\cdot)$ converges uniformly to $X(\cdot)$ over every bounded interval of time $[t_0 - T, t_0 + T]$.*

Proof. Form 9.1, we have

$$||Y(\cdot) - X(\cdot)||_\infty = \sup_{t \in [t_0 - T, t_0 + T]} ||Y(t) - X(t)|| \leq ||Y_0 - X_0||e^{KT}.$$

□

We can also use Proposition 9.0.2 to prove uniqueness of the solution.

Corollary 9.0.5 (Uniqueness)**.** *Under the assumption of Proposition 9.0.2, if* $Y_0 = X_0$ *then* $Y(\cdot) = X(\cdot)$*.*

Remark 9.0.6. *There are many variants of Proposition 9.0.2: the value at a fixed time* t *of the solution of* $X' = F(X)$ *such that* $X(t_0) = X_0$ *depends continuously not only on* X_0*, but also on* t_0 *and on the law of motion* F*, and on all of this jointly. This also holds for non autonomous equations.*

Chapter 10

Lyapunov functions

Consider an autonomous differential equation

$$X'(t) = F(X(t)) \tag{10.1}$$

with $F : \mathbb{R}^d \to \mathbb{R}^d$ of class C^1. We have seen in Chapter 7 that linearizing this system allows to get information on the stability of hyperbolic equilibria. Analyzing nonlinear equations, via the method of linearization close to equilibria, has however several drawbacks. First, the method does not apply to all equilibria. Second, it does not allow to detect setwise attractors (e.g., if a solution spirals and eventually approaches a whole circle instead of a single point). Third, it does not provide any information on the size of basins of attraction, where the basin of attraction of an equilibrium X^* is defined as the set of points X_0 in $\mathbb{R}^d$ such that the solution starting at X_0 is defined for all $t \geq 0$ and converges to X^* as $t \to +\infty$. The aim of this chapter is to introduce a method that allows, at least in some cases, to solve these problems. The method consists in studying quantities that decrease along trajectories. A good candidate in applications to physics is the total energy of the system which, in the absence of an exterior energy source, is constant or decreases, due to friction terms. If we can find such a quantity, we then know that solutions travel from areas of the phase plane where this quantity is large, to areas where it is small. Drawing the landscape obtained by taking this quantity as the altitude allows then to get a good understanding of the global behavior of the system.

10.1 Evolution of a quantity along trajectories

Let $U \subset \mathbb{R}^d$. Consider a solution $(J, X(\cdot))$ and a subinterval $\tilde{J}$ of J such that $X(t) \in U$ on $\tilde{J}$. Let $L : \mathbb{R}^d \to \mathbb{R}$ be differentiable and for all $X \in \mathbb{R}^d$, let

$$\dot{L}(X) = \nabla L(X) \cdot F(X).$$

Finally define $l : \tilde{J} \to \mathbb{R}$ by $l(t) = L(X(t))$. In this way, the evolution of l corresponds to the evolution of L as we follow the solution $X(\cdot)$.

Proposition 10.1.1. *If $\dot{L}(X) \leq 0$ on U, then the function l is decreasing. That is, L decreases along pieces of trajectories in U.*

Similarly, if $\dot{L}(X) < 0$ (resp. $= 0$) then l is strictly decreasing (resp. is constant). That is, L decreases strictly (resp. is constant) along pieces of trajectories in U.

Proof. For all t in $\tilde{J}$, $l'(t) = \nabla L(X(t)) \cdot X'(t) = \nabla L(X(t)) \cdot F(X(t)) = \dot{L}(X(t))$ with $X(t) \in U$ by definition of $\tilde{J}$. Thus, if $\dot{L}(X) \leq 0$ (resp. < 0, $= 0$) on U, then $l'(t) \leq 0$ (resp. < 0, $= 0$), for all t in $\tilde{J}$. $\square$

Remark 10.1.2. *Geometrically, the condition $\nabla L(X) \cdot F(X) \leq 0$ means that the angle between the gradient of L and the vector field defined by F is obtuse (that is, of at least 90 degrees). Since trajectories are tangent to this vector field, this means that trajectories go from high level curves of L to low level curves (or are tangent to level curves of L if $\nabla L(X) \cdot F(X) = 0$, remember that $\nabla L(X)$ gives the direction of maximal growth of L at X).*

10.2 Global Lyapunov function

Definition 10.2.1. *L is a global Lyapunov function for (10.1) if $\dot{L}(X) \leq 0$ on $\mathbb{R}^d$.*

By Proposition 10.1.1, a global Lyapunov function is a function that decreases along all trajectories. Moreover, if $\dot{L}(X) = 0$ on $\mathbb{R}^d$, then L is actually constant along trajectories. We then say that L is a constant of movement.

Proposition 10.2.2. *Let L be a global Lyapunov function for (10.1).*

1. *If L is coercive (i.e. $L(X) \to +\infty$ as $||X|| \to +\infty$), then all solutions of (10.1) defined at $t = 0$ are defined and bounded on $[0, +\infty[$.*
2. *If $X(\cdot)$ is a solution of (10.1) defined till $+\infty$, then any accumulation point X^* of $X(t)$ as $t \to +\infty$ satisfies $\dot{L}(X^*) = 0$. In other words, if there exists a sequence (t_n) such that $t_n \to +\infty$ and $X(t_n) \to X^*$ as $n \to +\infty$, then $\dot{L}(X^*) = 0$.*
3. *Let $E = \{X \in \mathbb{R}^d | \dot{L}(X) = 0\}$. Let $X(\cdot)$ be a solution defined and bounded on $[0, +\infty[$. Then $d(X(t), E) \to 0$ as $t \to +\infty$, where $d(X, E) = \inf_{Y \in E} ||X - Y||$. Moreover, if E consists of isolated points, then there exists $X^* \in E$ such that $X(t) \to X^*$ as $t \to +\infty$.*
4. *If L is coercive and E consists of isolated points, then for all solutions $X(\cdot)$, there exists X^* in E such that $X(t) \to X^*$ as $t \to +\infty$ (this limit X^* may depend on the solution.).*

Proof. 1) Assume that L is coercive. We claim that for all λ, $G_\lambda = \{X \in \mathbb{R}^d | L(X) \leq \lambda\}$ is bounded. Indeed, otherwise, there exists a sequence of points X_n in G_λ such that $||X_n|| \to +\infty$ as $n \to +\infty$. Since L is coercive, this implies that $L(X_n) \to +\infty$ as $n \to +\infty$. This contradicts the fact that $L(X_n) \leq \lambda$ for all n.

This being seen, let $(J, X(\cdot))$ be a solution of (10.1). Since L decreases along trajectories, it follows that for all $t \in J \cap \mathbb{R}_+$, $L(X(t)) \leq L(X(0))$. By the previous claim with $\lambda = L(X(0))$, this implies that $X(t)$ is bounded on $J \cap \mathbb{R}_+$. Thus, $\sup J = +\infty$ by the explosion alternative. The result follows.

2) Let X^* in $\mathbb{R}^d$. Assume that there exists a sequence (t_n) going to $+\infty$ such that $X(t_n) \to X^*$ as $n \to +\infty$. By contradiction, assume that $\dot{L}(X^*) \neq 0$, hence $\dot{L}(X^*) < 0$ (since $\dot{L} \leq 0$). Recall that if we follow a solution $X(\cdot)$ going through $X*$, then $\dot{L}(X^*)$ is the derivative of the function $l(t) = L(X(t))$ when $X(t) = X^*$. Thus, letting $\phi_t(X_0)$ denote the value at time t of the solution such that $X(0) = X_0$ and $l(t) = L(\phi_t(X^*))$ the value of L along the solution starting at X^*, we have:

$$l'(0) = \dot{L}(X^*) < 0.$$

Therefore, there exists $t > 0$ such that

$$l(t) = L(\phi_t(X^*)) < L(X^*).$$

Let $\varepsilon = L(X^*) - L(\phi_t(X^*)) > 0$. Since L and the time t map of the flow ϕ_t are continuous (see Proposition 9.0.2), the function $L \circ \phi_t$ is also continuous. Therefore, there exists a neighborhood V of X^* such that for all X in V,

$$L(\phi_t(X)) \leq L(\phi_t(X^*)) + \varepsilon/2 = L(X^*) - \varepsilon/2.$$

Since $X(t_n) \to X^*$, there exists $N \in \mathbb{N}$ such that $X(t_N) \in V$ (indeed, this holds as soon as N is large enough). Therefore, $L(\phi_t(X(t_N))) \leq L(X^*) - \varepsilon/2$, that is,

$$L(X(t_N + t)) \leq L(X^*) - \varepsilon/2.$$

Since L decreases along trajectories, for all $s \geq t_N + t$, $L(X(s)) \leq L(X(t_N + t)) \leq L(X^*) - \varepsilon/2$. Therefore,

$$\limsup_{s \to +\infty} L(X(s)) \leq L(X^*) - \varepsilon/2.$$

But since $X(t_n) \to L(X^*)$ as $n \to +\infty$, it follows that

$$\lim_{n \to +\infty} L(X(t_n)) = L(X^*) \leq L(X^*) - \varepsilon/2 < L(X^*),$$

a contradiction. Thus, $\dot{L}(X^*) = 0$.

3) Let $X(\cdot)$ be a solution defined and bounded on $[0, +\infty[$. Assume by contradiction that $d(X(t), E)$ does not go to 0 as $t \to +\infty$. Then there exists $\varepsilon \geq 0$ and a sequence (t_n) going to $+\infty$ such that $d(X(t_n), E) \geq \varepsilon$ for all n, hence $||X(t_n) - Y|| \geq \varepsilon$ for all n and for all $Y \in E$. Since the function $X(\cdot)$ is bounded on $[0, +\infty[$, it follows that the sequence $X(t_n)$ is also bounded. Therefore, up to considering a subsequence,[1] we may assume that $X(t_n)$ converges: $\exists X^* \in \mathbb{R}^d, X(t_n) \to_{n \to +\infty} X^*$. But then we both have $||X^* - Y|| \geq \varepsilon$ for all $Y \in E$, by taking limits in $||X(t_n) - Y|| \geq \varepsilon$, and, due to 2), $X^* \in E$, a contradiction. This shows that $d(X(t), E) \to 0$ as $t \to +\infty$.

Assume now that E consists of isolated points and let X^* be an accumulation point of $X(t)$ as $t \to +\infty$ (such an accumulation point exists since $X(t)$ is bounded in the neighborhood of $+\infty$). Recall that by 2), $X^* \in E$. We will show that $X(t) \to X^*$. Indeed, otherwise there exists $\varepsilon > 0$ such that:

$$\forall T \in \mathbb{R}, \exists t \geq T, ||X(t) - X^*|| \geq \varepsilon. \tag{10.2}$$

But since E consists of isolated points, there exists $\alpha > 0$ such that:

$$\forall X \in E, ||X - X^*|| \leq \alpha \Rightarrow X = X^*. \tag{10.3}$$

Let $\eta = \min(\varepsilon, \alpha)$ and let $T \in \mathbb{R}$. Due to (10.2), there exists $s_1 \geq T$ such that $||X(s_1) - X^*|| \geq \eta$. But since X^* is an accumulation point of $X(\cdot)$ as $t \to +\infty$, there also exists $s_2 \geq T$ such that $||X(s_2) - X^*|| \leq \eta$, hence, by continuity of $X(\cdot)$, a time t between s_1 and s_2 such that $||X(t) - X^*|| = \eta$. Note that $t \geq T$. We conclude that:

$$\forall T \in \mathbb{R}, \exists t \geq T, ||X(t) - X^*|| = \eta.$$

It follows that there exists a sequence (t_n) going to $+\infty$ such that $||X(t_n) - X^*|| = \eta$ for all n. Since $X(t_n)$ is bounded, up to considering a subsequence, we may assume that $X(t_n)$ converges: $\exists Y \in \mathbb{R}^d, X(t_n) \to_{n \to +\infty} Y$. Moreover, $||Y - X^*|| = \eta \leq \alpha$ and $Y \in E$. But by (10.3), this implies $Y = X^*$, contradicting the fact that $||Y - X^*|| = \eta$.

4) Simply put 1) and 3) together.

□

10.3 Lyapunov function for an equilibrium

Let U be an open subset of $\mathbb{R}^d$. Let $X^* \in U$ be an equilibrium of (10.1). Let $L : U \to \mathbb{R}$ be differentiable.

[1]The standard expression "up to considering a subsequence" should be understood as follows: there is a subsequence (τ_n) of (t_n) such that $X(\tau_n)$ converges. This is to this subsequence that what follows applies. But since we want to avoid to introduce new notation, we will forget about the initial sequence $X(t_n)$ and use the same piece of notation to denote its converging subsequence.

Definition 10.3.1. *L is a Lyapunov function for X^* if:*

(a) $L(X^*) = 0$ and $L(X) > 0$ on $U\backslash\{X^*\}$; and[2]

(b) $\dot{L}(X) \leq 0$ on U.

It is a strict *Lyapunov function for X^* if moreover:*

(c) $\dot{L}(X) < 0$ on $U\backslash\{X^*\}$.

In the next proposition, $Bd(K)$ denotes the boundary of the set K, that is the difference between the closure and the interior of K.

Proposition 10.3.2. *1) If L is a Lyapunov function for X^*, then X^* is stable.*

2) If L is a strict Lyapunov function for X^*, then X^* is asymptotically stable. Moreover, if $K \subset U$ is a compact set containing X^* in its interior and $\lambda = \min_{\{X \in Bd(K)\}} L(X)$, then the basin of attraction of X^* contains $\{X \in K \mid L(X) < \lambda\}$.

3) If L is a Lyapunov function for X^* and $\dot{L} = 0$ on U, then X^* is stable but not asymptotically stable.

Proof. Let us start with a lemma:

Lemma 10.3.3. *Assume that L is a Lyapunov function for X^*. Let $K \subset U$ be a nonempty compact set and let $\lambda = \min_{\{X \in Bd(K)\}} L(X)$. Let $A = \{X \in K \mid L(X) < \lambda\}$. For any solution $(J, X(\cdot))$ such that $X(0) \in A$, we have: $\sup J = \infty$ and $X(t) \in K$ for all $t \geq 0$.*

The proof of the lemma is as follows: since K is bounded, by the explosion alternative, it suffices to show that $X(t)$ never leaves K forward in time. Assume by contradiction that this is not the case. Then, since $X(\cdot)$ is continuous and K is closed, it can be shown that there is a time $T \geq 0$ such that $X(t) \in K$ on $[0, T]$ and $X(T) \in Bd(K)$. [3] Since $X(t) \in K \subset U$ on $[0, T]$ and $\dot{L} \leq 0$ on U, it follows that $L(X(t))$ decreases between 0 and T. Thus, $L(X(T)) \leq L(X(0)) < \lambda$. Since $X(T) \in Bd(K)$, this contradicts the definition of λ.

We now prove the proposition.

1) Assume that L is a Lyapunov function for X^* and let V be a neighborhood of X^*. We want to show that there exists a neighborhood W of X^* such that if a solution starts in W, then it remains in V forever. We first note that there exists $\varepsilon > 0$ such that $K := \bar{B}(X^*, \varepsilon) \subset V \cap U$ (where $\bar{B}(X^*, \varepsilon)$ is the closed ball of center X^* and radius ε). We have $Bd(K) \subset K \subset U$ and $X^* \notin Bd(K)$ (since $X^* \in int(K)$). By definition L is positive on $Bd(K)$. Since $Bd(K)$ is compact and L is continuous, this implies that $\lambda := \min\{L(X), X \in Bd(K)\}$ is positive. Since $L(X^*) = 0$ and L is continuous, it follows that there exists $\alpha > 0$ such that $L(X) < \lambda$ for all $X \in \bar{B}(X^*, \alpha)$. Letting $W = \bar{B}(X^*, \alpha)$, the lemma shows that if $X(0) \in W$, then for all $t \geq 0$, $X(t) \in K$ hence $X(t) \in V$. Therefore, X^* is stable.

2) X^* is stable by 1) so it suffices to show that X^* is attracting. For $\varepsilon > 0$ small enough, $K := \bar{B}(X^*, \varepsilon)$ is a subset of U. Fix such ε and let $\lambda := \min_{\{X \in Bd(K)\}} L(X)$. For the same reasons as in the proof of 1), $\lambda > 0$ and there exists $\alpha > 0$ such that $L(X) < \lambda$ for all $X \in V := \bar{B}(X^*, \alpha)$. Note that by definition of λ, this implies $\alpha < \varepsilon$. Now let $X(\cdot)$ be a solution such that $X(0) \in V$. It follows from the Lemma that $X(t) \in K$ for all $t \geq 0$. Thus, since K is closed, any accumulation point of $X(t)$ as $t \to +\infty$ is in K. Moreover, since $K \subset U$, the same

[2] Assuming $L(X^*) = 0$ is just a convention, to fix ideas, what is important is that X^* is a strict local minimum of the function L; but we stick to this convention because this is not a loss of generality (we can always replace L by $L - L(X^*)$, without affecting b) and c)), and because it is sometimes nicer to check that a quantity is positive rather than greater than a fixed level $L(X^*)$.

[3] Indeed, let $T = \sup \tilde{A}$ where $\tilde{A} = \{t \geq 0, \forall s \in [0, T], X(s) \in K\}$; note that $\tilde{A}$ is nonempty (contains 0) and upper bounded (by assumption), so that T is well defined; moreover $\tilde{A} = [0, +\infty[\cap X^{-1}(K)$ hence $\tilde{A}$ is closed because K is closed and $X(\cdot)$ is continuous, so $T \in \tilde{A}$ hence $X(t) \in K$ on $[0, T]$, and in particular, $X(T) \in K$. Finally, $X(T) \notin int(K)$. Otherwise, by continuity of $X(\cdot)$, we would have $X(s) \in K$ for all s in a neighborhood of T, contradicting the definition of T. Thus $X(T) \in K\backslash int(K) = cl(K)\backslash int(K) = Bd(K)$.

arguments as in the proof of Proposition 10.2.2 show that $X(t)$ converges to the set of points of K such that $\dot{L}(X) = 0$. Since L is a strict Lyapunov function for X^* and $K \subset U$, the only point of K such that $\dot{L}(X) = 0$ is X^*, therefore $X(t)$ converges to X^*. It follows that X^* is attracting, hence asymptotically stable. The above argument applied to any compact set $K \subset U$ proves the end of 2).

3) X^* is stable by 1). Thus, to show that it is not asymptotically stable, we must show that it is not attracting. Let V be any neighborhood of X^*. We need to show that there are solutions starting in V that do not converge to X^*. The proof of 1) shows that there exists $X_0 \in V \cap U \backslash \{X^*\}$ (in fact a whole open set of such points) such that the solution starting at X_0 is defined and in U for all $t \geq 0$. Since $X_0 \in U \backslash \{X^*\}$, $L(X_0) > 0$, hence $L(X_0) \neq L(X^*)$. Moreover, since $\dot{L} = 0$ on U and $X(t) \in U$ for all $t \geq 0$, it follows that for all $t \geq 0$, $L(X(t)) = L(X(0)) = L(X_0) \neq L(X^*)$. Therefore, as $t \to +\infty$, $L(X(t))$ cannot converge towards $L(X^*)$, hence by continuity of L, $X(t)$ cannot converge towards X^*. This concludes the proof.

□

10.4 Gradient systems

A gradient system is a system of differential equations of the form $X'(t) = -\nabla V(X(t))$ for some function V which we will assume to be C^2 so that the above system of differential equations is C^1. Note that V is defined only up to an additive constant, in the sense that for any constant C, the function $\tilde{V} = V + C$ generates the same gradient system.

Proposition 10.4.1. *Let $V : \mathbb{R}^d \to \mathbb{R}$ be of class C^2. For the gradient system $X'(t) = -\nabla V(X(t))$:*

1. *V is a global Lyapunov function. Moreover, $\dot{V}(X) = 0$ if and only if X is an equilibrium.*
2. *If X^* is an isolated equilibrium and a strict local minimum of V, then X^* is asymptotically stable.*

Proof. 1) Let $F(X) = -\nabla V(X)$, so that the equation reads $X'(t) = F(X(t))$. We have: $\dot{V}(X) = \nabla V(X) \cdot F(X) = -F(X) \cdot F(X) = -||F(X)||^2$, hence $\dot{V}(X) \leq 0$ with equality if and only if $F(X) = 0$, that is, if X is an equilibrium.

2) Assume first that $V(X^*) = 0$. Since X^* is an isolated equilibrium, there exists a neighborhood U of X^* in which X^* is the only equilibrium. Thus, it follows from 1) that for every X in U, we have $\dot{V}(X) \leq 0$, and $\dot{V}(X) < 0$ if $X \neq X^*$. Moreover, since X^* is a strict local minimum of V, we may also assume, up to considering a smaller neighborhood, that $V(X) > V(X^*) = 0$ on $U \backslash \{X^*\}$. Thus, V is a strict Lyapunov function for X^* and the result follows from point 2) in Proposition 10.3.2. If $V(X^*) \neq 0$, then apply the above reasoning to $\tilde{V} = V - V(X^*)$, which generates the same gradient system and satisfies $\tilde{V}(X^*) = 0$. □

10.5 Hamiltonian systems in $\mathbb{R}^2$

An Hamiltonian system in $\mathbb{R}^2$ is a system of the form:

$$\begin{cases} x'(t) &= \dfrac{\partial H}{\partial y}(x(t), y(t)) \\ y'(t) &= -\dfrac{\partial H}{\partial x}(x(t), y(t)) \end{cases}$$

with $H : \mathbb{R}^2 \to \mathbb{R}$ of class C^2. The function H is called the Hamiltonian (or Hamiltonian function) of the system. It is defined up to an additive constant.

Proposition 10.5.1. *Consider an Hamiltonian system in $\mathbb{R}^2$ with Hamiltonian function H.*

1. *H is a constant of motion, i.e. a quantity that is conserved throughout the motion.*
2. *If X^* is a strict local minimum of H, then X^* is stable but not asymptotically stable.*

Proof. 1) The system may be written $X'(t) = F(X(t))$ with $F(x,y) = \left(\frac{\partial H}{\partial y}(x,y), -\frac{\partial H}{\partial x}(x,y)\right)$. Thus,

$$\dot{H} = \nabla H \cdot F = \left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}\right) \cdot \left(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x}\right) = 0.$$

2) Assume first $H(X^*) = 0$. Since X^* is a strict local minimum of H, there exists a neighborhood U of X^* such that $H(X) > 0$ on $U \backslash \{X^*\}$. Moreover, $\dot{H} = 0$ on $\mathbb{R}^d$, hence on U. The result then follows from point 3 of Proposition 10.3.2. If $H(X^*) \neq 0$, apply the same reasoning to $\tilde{H} = H - H(X^*)$, which generates the same Hamiltonian system and satisfies $\tilde{H}(X^*) = 0$. $\square$