Part 1 (Picard-Lindelöf, autonomous equations, explicit resolution in dimension 1): 5 to 6 weeks.

The program for the exercise sessions is the following. Week 1: Exercises 1, 3, 4, 6. Week 2: 7, 8, 15, 16; Week 3: 17, (13, 14). Week 4: 18, 19, 23 i)-iii), 25, Week 5: 31, 32a, 35, 36 ii)-iii), Week 6: 37 A or B, 39.

Students are encouraged to do more exercises, e.g., 3, 4, 9 10, 13, 20, 21, 22, 27, 32b), 33, 36 iii)-iv), 37 B or A, 38. Note also that there are exercises in the course's textbook. Most of those are relatively easy and intended to allow students to check their understanding of the course.

Existence and uniqueness of solutions. We state here a particular version of the theorems seen in the notes Chapter 1. Let $d \in \mathbb{N}$, $F : I \times \mathbb{R}^d \to \mathbb{R}^d$, $t_0 \in I$, $X_0 \in \mathbb{R}^d$, where I is a nonempty open interval of $\mathbb{R}$ (using the notation of the notes, we are considering the particular case $\Omega = I \times \mathbb{R}^d$).

A solution of the differential equation

$$X'(t) = F(t, X(t)) \tag{1}$$

is given by a nonempty open interval $J \subseteq I$ and a differentiable function $X : J \to \mathbb{R}^d$ satisfying (1) for all t in J. A solution of the initial value problem (Cauchy problem)

$$\begin{cases} X'(t) &= F(t, X(t)) \\ X(t_0) &= X_0 \end{cases} \tag{2}$$

is a solution of (1) such that $t_0 \in J$ and $X(t_0) = X_0$. This solution is *maximal* if it cannot be extended to a larger time interval. It is *global* if $J = I$, in this case the solution is trivially maximal.

The following is a corollary of Picard-Lindelöf Theorem and of the Characterization of maximal solutions Theorem stated in the notes.

Theorem 1 *If F is C^1 or Lipschitz on $I \times \mathbb{R}^d$, then (2) has exactly one maximal solution $(J, X(\cdot))$. Moreover, if $\sup J < \sup I$, then $||X(t)|| \to_{t\to\sup J} +\infty$, and if $\inf J > \inf I$, then $||X(t)|| \to_{t\to\inf J} +\infty$.*

As a particular case, Theorem 1 is also valid if F is defined on $\mathbb{R} \times \mathbb{R}^d$, in this case:

a) $t_0 \in \mathbb{R}$, $J \subset \mathbb{R}$ are trivially satisfied in the definition of a solution,

b) the theorem modifies in this way:

If F is C^1 or Lipschitz on $\mathbb{R}^{d+1}$, then (2) has exactly one maximal solution $(J, X(\cdot))$. Moreover, if $\sup J < +\infty$, then $||X(t)|| \to_{t\to\sup J} +\infty$, and if $\inf J > -\infty$, then $||X(t)|| \to_{t\to\inf J} +\infty$.

c) the solution is global if $J = \mathbb{R}$.

As a particular case, Theorem 1 is also valid if the equation is of the form $X'(t) = F(X(t))$ with $F : \mathbb{R}^d \to \mathbb{R}^d$ of class C^1 or Lipschitz on $\mathbb{R}^d$.

Importance of the assumptions of Theorem 1

Exercise 1 *(Assumptions of Theorem 1)* How many maximal solutions do the following problems have? Why is this consistent with Theorem 1? Check the assumptions and identify I and J in order to check if the Theorem applies.

a) $x'(t) = -\sin t$ and $x(0) = 2$; b) $x'(t) = -\sin t$ and $x(0) = x(2\pi)$; c) $x'(t) = -\sin t$ and $x''(0) = 0$; d) $tx'(t) = 2x(t)$ and $x(0) = 1$; e) $x''(t) = 0$ and $x(0) = 0$.

Exercise 2 (2015-I-1) *(Integration problems are differential equations; an initial value is a specific condition)* a) Solve (= find all maximal solutions of) the equation $x'(t) = -\sin t$.

b) Find all maximal solutions of the initial value problem $x'(t) = -\sin t$ and $x(0) = 1$.

c) Find all maximal solutions of the problem $x'(t) = -\sin t$ and: c1) $x(0) = x(2\pi)$; c2) $x"(0) = 0$; c3) $x(0)x(\pi) = 0$. How many solutions do you find? Why is this consistent with Theorem 1?

Exercise 3 (2015-I-2, with solution) *(an initial value problem with no solution)* Define $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $f(t, x) = 1$ if $x \leq 0$ and $f(t, x) = -1$ if $x > 0$. Show that the differential equation $x'(t) = f(t, x(t))$ has no solution such that $x(0) = 0$. Why is this consistent with Theorem 1?

Exercise 4 (2015-I-6, with solution) *(an initial value problem with multiple solutions)*

Consider the initial value problem $x'(t) = 3[x(t)]^{\frac{2}{3}}$ with $x(0) = 0$.

a) Show that the functions defined on $\mathbb{R}$ by $x(t) = 0$ and $y(t) = t^3$, respectively, are two maximal solutions of this problem. Why is this consistent with Theorem 1?

b) Let a and b be in $\bar{\mathbb{R}}$, with $a \leq 0 \leq b$. For all $t \in \mathbb{R}$, let $u_{a,b}(t) = (t - a)^3$ if $t \leq a$, $u_{a,b}(t) = 0$ if $t \in [a, b] \cap \mathbb{R}$, and $u_{a,b}(t) = (t - b)^3$ if $t \geq b$.

b1) Sketch the graph of $u_{a,b}$ and show that $u_{a,b}$ is a maximal solution.

b2) ($*$) Show that all maximal solutions are of this type. Hints in this footnote.[1]

Using Theorem 1: some simple examples

Exercise 5 (variant of 2015-I-3, done in Chapter 3) *(a maximal solution need not be global)* Call (IVP) the initial value problem $u'(t) = u^2(t)$ and $u(0) = 1$. Consider the function $u :]-\infty, 1[\to \mathbb{R}$ defined by $u(t) = 1/(1 - t)$. Sketch its graph. Check that it is solution of (IVP). Why is it maximal? Why is is the unique maximal solution of (IVP)? Is it global?

Exercise 6 *(a simple use of Theorem 1 in dimension 1)* Consider the initial value problem $x'(t) = \sin(tx(t))$ and $x(0) = 1$.

1. Show that it has a unique maximal solution and denote it by $(J, x(\cdot))$.
2. Show that for all t in J, $|x(t)| \leq 1 + |t|$.
3. Assume $\sup J < +\infty$. Show that $x(\cdot)$ is bounded on $[0, \sup J[$. Conclude that $|x(t)|$ does not go $+\infty$ as $t \to \sup J$ and find a contradiction based on Theorem 1.
4. Conclude that $\sup J = +\infty$ and show similarly that $\inf J = -\infty$, so that the solution is global.

Exercise 7 (variant of 2015-II-23, non-explosion) *(a simple use of Theorem 1 in dimension 2)* Let $(x_0, y_0) \in \mathbb{R}^2$. Consider the initial value problem:

$$\begin{cases} x'(t) = \sin\left(x(t) + y(t)\right), & x(0) = x_0 \\ y'(t) = e^{x(t)-1}, & y(0) = y_0. \end{cases} \tag{3}$$

[1] Consider a maximal solution $(J, x(\cdot))$. Show that $x(\cdot)$ is continuous and increasing, so that there are elements a and b in $\bar{\mathbb{R}}$ such that $x(t) < 0$ on $J_- =]-\infty, a[$, $x(t) = 0$ on $J_0 = [a, b] \cap \mathbb{R}$ and $x(t) > 0$ on $J_+ =]b, +\infty[$. Show that if $J_+ \neq \emptyset$, there exists a constant K such that, for all $t \in J_+$, $x(t) = (t - K)^3$, and show that $K = b$. Prove a similar result on J_- and conclude.

1. Show that the initial value problem (3) has a unique maximal solution $(J, (\bar{x}, \bar{y}))$.

2. Show that for all $t \in J$, $|\bar{x}(t) - x_0| \leq |t|$.

3. Show that for all $t \in J$, $|\bar{y}(t) - y_0| \leq C|t|e^{|t|}$ for some constant C.

4. Show that if $\sup J < +\infty$, then $X(t) = (\bar{x}(t), \bar{y}(t))$ is bounded on $[0, \sup J[$. Conclude that $\sup J = +\infty$.

5. Show similarly that $\inf J = -\infty$, hence that the solution is global ($J = \mathbb{R}$).

Tools to show that solutions of differentials equations remain in a certain zone

When studying differential equations, it is often useful to show that a solution stays in a certain interval, when the solution evolves in $\mathbb{R}$, or more generally in a given subset of $\mathbb{R}^d$. The following exercises provide tools for doing so.

Exercise 8 (2015-I-8) *(first exit time from an open set, infinite sojourn in an open set)* Let $t_0 \in \mathbb{R}$, $K \in \mathbb{R}$. Let $g : [t_0, +\infty[\to \mathbb{R}$ be continuous and such that $g(t_0) < K$.

a) Assume that there exists $\tau > t_0$ such that $g(\tau) \geq K$. Show that there exists a time $t_1 > t_0$ such that $g(t) < K$ for all t in $[t_0, t_1[$ and $g(t_1) = K$. Hint: $t_1 = \min\{t \geq t_0 | g(t) \geq K\}$.

b) Assume that for all $t \in [t_0, +\infty[$: if $g(s) \leq K$ for all $s \in [t_0, t[$, then $g(t) < K$. Show that $g(t) < K$ for all $t \in [t_0, +\infty[$. What if g is not continuous?

Exercise 9 (2015-I-9) *(first exit time from a closed set, infinite sojourn in a closed set)* Let $g : [0, +\infty[\to \mathbb{R}$ be continuous and such that $g(0) \leq 1$.

a) Assume that there exists $\tau > 0$ such that $g(\tau) > 1$. Show that there exists $t_2 \geq 0$ such that $g(t) \leq 1$ for all t in $[0, t_2[$, $g(t_2) = 1$, and for all $\varepsilon > 0$, there exists $t \in]t_2, t_2 + \varepsilon[$ such that $g(t) > 1$. Indication : $t_2 = \max\{t \,|\, g(s) \leq 1 \text{ on } [0, t]\}$. [2]

b) Assume that for all $t \geq 0$, if $g(t) = 1$, then there exists $\varepsilon > 0$ such that $g(s) < 1$ on $]t, t + \varepsilon[$. Show that $g(t) \leq 1$ on $[0, +\infty[$.

c) Assume that $g(0) < 1$ and that for all $t \geq 0$: if $g(t) = 1$, then there exists $\varepsilon > 0$ such that $g(s) > 1$ on $]t - \varepsilon, t[$. Show that $g(t) < 1$ on $[0, +\infty[$.

Exercise 10 (2015-I-10) *(trap)* Let $x : [0, +\infty[\to \mathbb{R}$ be differentiable and such that $x(0) < 1$.

a) Show that if for all t such that $x(t) = 1$, $x'(t) < 0$, then $x(t) < 1$ on $[0, +\infty[$.

b) (link with differential equations) Assume that $x(0) \in]-1, 1[$ and that for all $t \geq 0$, $x'(t) = \sin(tx(t)) - 2x(t)$. Show that $x(t) \in]-1, 1[$ on $[0, +\infty[$.

Exercise 11 (2015-I-11) *(generalization of Exercise 8)* Let $n \in \mathbb{N}^*$, V an open subset of $\mathbb{R}^n$, and t_0 a real number. Let $g : [t_0, +\infty[\to \mathbb{R}^n$ be continuous and such that $g(t_0) \in V$. Assume that there exists $\tau > t_0$ such that $g(\tau) \notin V$. Show that there exists $t_1 > t_0$ such that $g(t) \in V$ for all t in $[t_0, t_1[$ and $g(t_1) \in \bar{V} \backslash V$ where $\bar{V}$ is the closure of V.

Exercise 12 (2015-I-12) *(generalization of Exercise 9)* Exercise 11 generalizes Exercise 8. Prove a similar generalization of Exercise 9.

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[2]Note: it may be that on every interval $]t_2, t_2 + \varepsilon[$, g takes values strictly lower than 1, because g may oscillate, e.g., $g(t) = t$ if $t \in [0, 1]$ and $g(t) = 1 + (t - 1)\sin(1/(t - 1))$ if $t > 1$.

Solution of Exercise 3. If $(J, x(\cdot))$ is a solution, then $x'(0) = f(x(0)) = f(0) = 1$. For ϵ small enough, $[0, \varepsilon] \subset J$ and since $x'(0) > 0$: for all $t \in [0, \varepsilon]$, $x(t) > 0$, hence $x'(t) = f(x(t)) = -1$. Thus, $x(\varepsilon) = x(0) + \int_0^\varepsilon x'(t)dt = 0 - \varepsilon < 0$. This contradicts the fact that $x(t) > 0$ for all $t \in [0, \varepsilon]$.

Solution of Exercise 4

a) Differentiating these functions shows that they are indeed solutions. They are global, hence maximal. The existence of two maximal solutions to this initial value problem does not contradict Picard-Lindelöf theorem, as the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x^{2/3}$ is not C^1 nor Lipschitz.

b1) Again, differentiating these functions shows that they are solutions (the only difficulty is to check that these functions are differentiable in a and in b, when a and b are in $\mathbb{R}$). Moreover, these solutions are global hence maximal.

b2) We want to prove that all maximal solutions are of this kind. Let $(J, x(\cdot))$ be a maximal solution. We will show that:

(i) $\mathbb{R}_+ \subset J$ and either $x(t) = 0$ on $\mathbb{R}_+$, or there exists $b \in \mathbb{R}_+$ such that $x(t) = 0$ on $[0, b]$ and $x(t) = (t - b)^3$ on $[b, +\infty[$.

It may be shown similarly that: (ii) $\mathbb{R}_- \subset J$, and either $x(t) = 0$ on $\mathbb{R}_-$, or there exists $a \in \mathbb{R}_-$ such that $x(t) = 0$ on $[a, 0]$ and $x(t) = (t - a)^3$ on $]-\infty, a]$, which completes the proof.

Proof of (i): Assume for now that the solution is defined for all $t \geq 0$, that is, $\sup J = +\infty$. Since $x'(t) = 3x^{2/3}(t) \geq 0$ for all $t \in J$, the function $x(\cdot)$ is increasing. Since $x(0) = 0$, this implies $x(t) \geq 0$ for all $t \geq 0$. Therefore, either $x(t) = 0$ on $[0, +\infty[$ (hence (i) is satisfied) or there exists $b \in [0, +\infty[$ such that $x(t) = 0$ on $[0, b]$ and $x(t) > 0$ on $]b, +\infty[$ (to see that, let $b = \sup\{t \geq 0, x(t) = 0\}$ and note that the sup is a max by continuity of $x(\cdot)$).

In the latter case, let $J_+ =]b, +\infty[$. For any $t \in J_+$, $x(t) \neq 0$, hence we may divide both sides of $x'(t) = 3x^{2/3}(t)$ by $3x^{2/3}(t)$ to obtain:

$$\forall t \in J_+, \frac{x'(t)}{3x^{2/3}(t)} = 1$$

Integrating shows that there exists a constant K such that, for all $t \in J_+$, $x^{1/3}(t) = t + K$ hence $x(t) = (t + K)^3$. Since $x(b) = 0$ and $x(\cdot)$ is continuous, $(b + K)^3 = 0$, hence $K = -b$. Therefore $x(t) = (t - b)^3$ on $]b, +\infty[$, and (i) is satisfied.

It remains to prove that $\sup J = +\infty$. Assume not. The same arguments as above show that either $x(t) = 0$ on $[0, \sup J[$ (case 1) or there exists $b \in J \cap \mathbb{R}_+$ such that $x(t) = 0$ on $[0, b]$ and $x(t) = (t - b)^3$ on $[b, \sup J[$ (case 2). Since $\sup J < +\infty$, the solution may be extended by letting for all $t \in [\sup J, +\infty[$, $x(t) = 0$ in case 1 and $x(t) = (t - b)^3$ in case 2. This contradicts the maximality of the initial solution. Therefore, $\sup J = +\infty$. This concludes the proof.

How to find all possible solutions? First note that $x(\cdot)$ is continuous increasing, and that by separation of variables, $x(\cdot)$ behaves as $(t + K)^3$ on the intervals where $x(\cdot)$ does not take the value 0. Second, note that maximal solutions are defined on the whole real line as otherwise they could be extended. Third, note that the case in which $x(t) = 0$ for all $t \geq 0$ should be distinguished from the case in which $x(\cdot)$ takes positive values, and similarly in backward time, so that there are 4 kinds of solutions. It remains to find a way to explain this in a concise way, and to prove it. This leads, for instance, to the phrasing of the exercise and the above solution.

Applications of Picard-Lindelöf theorem, explosion, qualitative analysis

Exercise 13 (2015-I-13) *(qualitative resolution of the logistic equation)* The logistic equation

$$x'(t) = rx(t)(1 - x(t)/K) \tag{4}$$

models the evolution of the density $x(t)$ of a species depending on a limited resource, e.g., sheeps on an island, with no predators, but needing to find grazing fields. When the density of the population is low, the per-capita growth-rate is close to $r > 0$. When the density increases, the growth-rate decreases. It becomes negative if the density is higher than the carrying capacity K of the environment. This equation can be solved explicitly, e.g., by separation of variables, but the aim of this exercise is to show that its qualitative behavior may be understood without any computation.

Let $x_0 \in \mathbb{R}$. Let $(J, \bar{x}(\cdot))$ denote the maximal solution of (4) such that $x(0) = x_0$.

1. What happens if $x_0 = 0$ or $x_0 = K$?
2. Assume $0 < x_0 < K$.
   (a) Show that $0 < \bar{x}(t) < K$ for all t in J and that $J = \mathbb{R}$. Show that $\bar{x}(\cdot)$ is strictly increasing.
   (b) Show that $\bar{x}(t)$ has a limit in $[x_0, K]$ as $t \to +\infty$ and in $[0, x_0]$ as $t \to -\infty$.
   (c) Show that if $\bar{x}(t) \to x^* \in \mathbb{R}$ as $t \to +\infty$, then $f(x^*) = 0$, where $f(x) = rx(1 - x/K)$.
   (d) Show that $\bar{x}(t) \to_{t\to+\infty} K$ and $\bar{x}(t) \to_{t\to-\infty} 0$.
3. Assume $x_0 > K$. Show that $\bar{x}(t) > K$ for all t in J, that $\bar{x}(\cdot)$ is decreasing, that $\sup J = +\infty$ and that $\bar{x}(t) \to_{t\to+\infty} K$. Show that $\bar{x}(t) \to_{t\to\inf J} +\infty$.
4. Assume $x_0 < 0$ (a case with limited ecological interest...). Show that $\bar{x}(t) < 0$ for all t in J, that $\bar{x}(\cdot)$ is decreasing, that $\inf J = -\infty$ and $\bar{x}(t) \to_{t\to-\infty} 0$. Show that $\bar{x}(t) \to_{t\to\sup J} -\infty$.
5. Sketch the graph of representative solutions and the phase line.

Exercise 14 (2015-I-14) *(Application of Picard-Lindelöf theorem; dimension 1 is special)* Let (E) denote the autonomous differential equation $x'(t) = f(x(t))$ with $f : \mathbb{R} \to \mathbb{R}$ assumed C^1. Let $(J, x(\cdot))$ be a maximal solution of (E). This is a *stationary solution* if the function $x(\cdot)$ is constant.

1) Let $t_0 \in \mathbb{R}$ and $x_0 = x(t_0)$. Show that if $f(x_0) = 0$ then $x(\cdot)$ is stationary and $J = \mathbb{R}$. Show that if $f(x_0) \neq 0$ then $x(\cdot)$ is not stationary.

2) Show that if there exists $t_0 \in J$ such that $x'(t_0) = 0$, then $x(\cdot)$ is stationary and $J = \mathbb{R}$.

3) Show that all solutions of (E) are stationary or strictly monotone.

4) Show that all periodic solutions of (E) are stationary. Is this the case for nonautonomous differential equations (hint: recall Exercise 2)? And for autonomous differential equations in dimension 2, that is, with $f : \mathbb{R}^2 \to \mathbb{R}^2$? Hint in this note.[3]

Exercise 15 (2015-I-15) *(Picard-Lindelöf theorem and criterion of maximality).*

Consider a differential equation $x'(t) = f(t, x(t))$ with $f : \mathbb{R}^2 \to \mathbb{R}$ of class C^1. Let $(J_x, x(\cdot))$ and $(J_y, y(\cdot))$ be two maximal solutions. Assume that there exists $t_0 \in J_x \cap J_y$ such that $x(t_0) < y(t_0)$.

a) (seen in the course) Show that for all $t \in J_x \cap J_y$, $x(t) < y(t)$.

[3] Check that $X(t) = (\cos t, \sin t)$, defined on $\mathbb{R}$, is a solution of $X'(t) = F(X(t))$ with $F(x, y) = (-y, x)$.

b) Show that if $x(t) \to +\infty$ as $t \to \sup J_x$ then $\sup J_y \leq \sup J_x$ and $y(t) \to +\infty$ as $t \to \sup J_y$. Does $|x(t)| \to +\infty$ as $t \to \sup J_x$ implies $\sup J_y \leq \sup J_x$? Does $\sup J_x < +\infty$ implies $\sup J_y < +\infty$? Same questions as the two before if we assume in addiction that $x(\cdot)$ is increasing.

c) Without proof: what can we say if $y(t) \to -\infty$ as $t \to \sup J_y$? if $x(t) \to +\infty$ as $t \to \inf J_x$? if $y(t) \to -\infty$ as $t \to \inf J_y$?

d) Assume that the solutions $x(\cdot)$ and $y(\cdot)$ are global (that is, $J_x = J_y = \mathbb{R}$). Let $(J, z(\cdot))$ be a maximal solution. Show that if there exists $t_1 \in \mathbb{R}$ such that $z(t_1) \in]x(t_1), y(t_1)[$ then $z(\cdot)$ is global.

Exercise 16 (variant of 2015-II-24) *(a difference between dimension 1 and dimension 2)*

a) Let (E) denote the differential equation: $x'(t) = x(t)\phi(t, x(t))$ with $\phi : \mathbb{R}^2 \to \mathbb{R}$ of class C^1. Show that this equation has the stationary solution $(\mathbb{R}, t \to 0)$. Let $x_0 \in \mathbb{R}$. Let $(J, \bar{x}(\cdot))$ denote the unique maximal solution of (E) such that $x(0) = x_0$. Show that if $x_0 > 0$, then $\bar{x}(t) > 0$ for all $t \in \mathbb{R}$.

b) Consider the differential equation

$$\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t) \end{cases} \tag{5}$$

Let $(x_0, y_0) \in \mathbb{R}^2$ and let $(J, (\overline{x}, \overline{y}))$ denote the unique maximal solution of (5) such that $(x(0), y(0)) = (x_0, y_0)$. Why is this solution unique?

1/. Show that $(\mathbb{R}, t \to (0, 0))$ is solution of (5).

2/. Does this imply that if $x_0 > 0$ and $y_0 > 0$, then $\overline{x}(t) > 0$ and $\overline{y}(t) > 0$ for all $t \in J$.

3/. Note that $X(t) = (\cos t, \sin t)$ is solution of (5). In the plane $\mathbb{R}^2$, draw the associated trajectory, that is, the set $\{X(t), t \in \mathbb{R}\}$. Does this change your answer to question **2/.**? What happens geometrically?

Exercise 17 (2015-II-25, with solution) *(constant sign in dimension 2)*

Let $g = (g_1, g_2) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be of class C^1. Consider the initial value problem:

$$\begin{cases} x'(t) = x(t)g_1(t, x(t), y(t)) & x(0) = x_0 > 0, \\ y'(t) = y(t)g_2(t, x(t), y(t)) & y(0) = y_0 > 0. \end{cases} \tag{6}$$

1/. Show that (6) has a unique maximal solution $(J, (\overline{x}, \overline{y}))$.

2/. Show that for all $t \in J$, $\overline{x}(t) > 0$ and $\overline{y}(t) > 0$.

Solution of Exercise 17

1/. The initial value problem (6) may be written as $X'(t) = F(t, X(t))$ and $X(0) = (x_0, y_0)$ with $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(t, x, y) = (xg_1(t, x, y), yg_2(t, x, y))$. Since g is of class C^1, so if F, hence by Picard-Lindelöf theorem, (6) has a unique maximal solution.

2/. The fact that $t \to (0, 0)$ is a stationary solution does not imply in itself that the sign of $\overline{x}(t)$ and $\overline{y}(t)$ remains constant, because the solution might spiral (see Exercise 16). Thus, we need to use the specific form of the equation. The argument is more subtle that most students think, and should be studied with care.

Define $f : J \times \mathbb{R} \to \mathbb{R}$ by $f(t, x) = xg_1(t, \bar{x}(t), \bar{y}(t))$. Then $(J, \bar{x}(\cdot))$ and $(J, t \to 0)$ are two global hence maximal solutions of $x'(t) = f(t, x(t))$. Thus, if f is C^1, they cannot cross by a corollary of Picard-Lindelöf, hence $\bar{x}$ has a constant sign. But since F is of class C_1, $(\bar{x}(\cdot), \bar{y}(\cdot))$ is C^2 hence C^1. Therefore, f is C^1. This concludes the proof.

An introduction to trajectories.

Let $X : J \to \mathbb{R}^d$ be solution of a differential equation, modeling a system whose state at time t is $X(t)$. The *trajectory* (also called *orbit*) associated to this solution is the set $T = \{X(t), t \in J\}$ of successive states of the system as t describes J (*both in forward and backward time*).[4] The trajectory may also be defined as the projection of the graph of $X(\cdot)$ on $\mathbb{R}^d$. Indeed, the projection of $(t, X(t)) \in \mathbb{R} \times \mathbb{R}^d$ on $\mathbb{R}^d$ is $X(t)$. Thus the projection of the graph $\Gamma = \{(t, X(t)), t \in J\}$ is $\{X(t), t \in J\} = T$.

Before trying to o understand trajectories associated to solutions of differential equations and we begin with the understanding of the notion of curves and trajectories.

Exercise 18 *(trajectories in dimension 1)* For each of the following functions, defined on $\mathbb{R}$, sketch its graph, then compute and draw its trajectory (indicating with arrows the direction in which it is traveled) : a) $x(t) = 0$; b) $x(t) = e^t$; c) $x(t) = e^{-t}$; d) $x(t) = -3e^{2t}$.

Exercise 19 *(spiraling trajectories in dimension 2)* For each of the following functions, from $\mathbb{R}$ to $\mathbb{R}^2$, sketch its graph and its trajectory, indicating the direction in which it is traveled:
a) $X(t) = (\cos t, \sin t)$; b) $X(t) = (e^t \cos t, e^t \sin t)$; c) $X(t) = (e^{-t} \cos t, e^{-t} \sin t)$

Exercise 20 *(other trajectories in dimension 2)* For each of the following functions, from $\mathbb{R}$ to $\mathbb{R}^2$, compute and draw its trajectory, indicating with arrows the direction in which it is traveled:
a) $X(t) = (e^t, e^{-t})$; b) $X(t) = (e^t; e^t)$; c) $X(t) = (e^t; e^{2t})$; d) $X(t) = (e^{2t}; e^t)$.

Exercise 21 *(trajectories and autonomous equations)* Consider an autonomous equation $X'(t) = g(X(t))$, with $g : \mathbb{R}^d \to \mathbb{R}^d$ of class C^1. Let times t_0 and $t_1 = t_0 + \Delta$ be in $\mathbb{R}$ and $X_0 \in \mathbb{R}^d$. Let $u : J \to \mathbb{R}^d$ be the unique maximal solution of $X'(t) = g(X(t))$ and $X(t_0) = X_0$. Let $J =]a, b[$, where a and b are in $\bar{\mathbb{R}}$ and define $v :]a + \Delta, b + \Delta[\to \mathbb{R}$ by $v(t + \Delta) = u(t)$.

a) Show that v is the unique maximal solution of $X'(t) = g(X(t))$ and $X(t_1) = X_0$.

b) Show that u and v define the same trajectory.

c) Let (J_w, w) denote another solution of $X'(t) = g(X(t))$. Show that the trajectories associated to u and w are either disjoint or equal.

Autonomous equations in dimension 1

Exercise 22 (2015-II-1) *(a simple stability test)* Let x^* be an equilibrium of $x'(t) = g(x(t))$, where $g : \mathbb{R} \to \mathbb{R}$, $g \in C^1$ or Lipschitz. Show that if there exists $\varepsilon > 0$ such that g is positive on $]x^* - \varepsilon, x^*[$ and negative on $]x^*, x^* + \varepsilon[$, then x^* is attracting. Conclude that if $g'(x^*) < 0$, then x^* is attracting. Show that if $g'(x^*) > 0$, then x^* is repelling.

Exercise 23 (2015-II-2) *(autonomous equations in dimension 1)* For each of the following equations: find the equilibria, draw the phase line, and say which equilibria are attracting, repelling, and neither attracting nor repelling. What does the phase line tell us on the interval on which the solutions are defined?

i) $x' = 2x + 1$; ii) $x' = -2x - 1$; iii) $x' = x - x^3$; iv) $x' = 1 + x^2$;
v) $x' = x^4 - x^2$; vi)$x' = \sin x$; vii) $x' = \sin^2 x$; viii) $x' = \max(0, x^3)$.

[4] You may also see the trajectory as the set of positions occupied successively by a particle whose position at time t is $X(t)$.

Exercise 24 (2015-II-3) *(phase line of opposite equation)* Compare the phase lines for $x' = f(x)$ and $x' = -f(x)$. Show that if $x :]a, b[\to \mathbb{R}$ is solution of $x' = f(x)$ then $y :] - b, -a[\to \mathbb{R}$ defined by $y(t) = x(-t)$ is solution of $x' = -f(x)$.

Exercise 25 (2015-II-4) *(fisheries)* We study the evolution of the perch stock $s(t)$ in a lake (a perch is a fish).

1. In the absence of fishing activity, $s(t)$ satisfies the logistic equation: $s'(t) = s(t)\,(1 - s(t))$. Recall the phase line for this equation and its qualitative behavior.
2. Assume now that a quantity p of perches are fished per unit of time. This leads to:
$$s'(t) = s(t)\,(1 - s(t)) - p, \quad (p > 0).$$
Draw the phase line for $0 < p < 1/4$, $p = 1/4$ and $p > 1/4$. Is this model meaningful for initial conditions $s(0) < 0$? For initial conditions $s(0) > 0$, what happens if the quantity p (say a fishing quota) is too large? Assume that you are in charge of fixing a fishing quota (that is, to fix p) for the next season. Roughly, what quota would you choose?
3. The fact that a quantity p of perches may be fished per unit of time independently of the current perch stock seems implausible. Thus, we consider another model in which the quantity fished is proportional to the current stock. This leads to:
$$s'(t) = s(t)\,(1 - s(t)) - ps(t), \quad (p > 0).$$
Compare the qualitative behavior of the solutions to those of the previous model.
4. Propose a model combining the previous models in that the quantity fished per unit of time is roughly constant as long as the stock is large enough, but goes to zero as $s(t)$ goes to zero.

Exercise 26 (2015-II-5) *(mating requires meeting)* Even in the absence of fishing activities, the balena population may decline if it falls below a certain threshold, because of the lack of mating opportunities (the population may be so small that males do not meet females often enough for the birth rate to be higher than the death rate). Propose a simple population growth model having this behavior, and behaving roughly as the logistic equation when the population is large.

Time rescaling

Exercise 27 *(time-change)* Let $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be C^1. Let $x_0 \in \mathbb{R}$. Consider the initial value problems: (IV1) $x'(t) = g(x(t))$ and $x(0) = x_0$; and (IV2) $x'(t) = h(t)g(x(t))$ and $x(0) = x_0$. Let (J_u, u) and (J_v, v) denote the unique maximal solutions of (IV1) and (IV2), respectively. Assume for simplicity that u is global, that is, $J_u = \mathbb{R}$.

a) Let $\tau(t) = \int_0^t h(s)\,ds$. Show that $v(t) = u(\tau(t))$.

b) Show that if $h(\cdot)$ is positive and $\int_0^{+\infty} h(t)\,dt = \int_{-\infty}^0 h(t)\,dt = +\infty$, then u and v define the same trajectory. What happens if, for instance, $h(t) = 1/(1 + t^2)$?

Exercise 28 (2015-II-6, $*$) *(equations with the same phase line)* Let $x_0 \in \mathbb{R}$. Let $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ be C^1. Let $(J_1, x_1(\cdot))$ and $(J_2, x_2(\cdot))$ be, respectively, the solutions of $x'(t) = g_1(x(t))$ and $x'(t) = g_2(x(t))$ such that $x(0) = x_0$. Show that if g_1 and g_2 have always the same sign, then there exists an increasing function $\tau : J_2 \to J_1$ (called a rescaled time) such that $x_2(t) = x_1(\tau(t))$. How does this relate to the fact that the sign of g determines the phase line of $x'(t) = g(x(t))$?

Hint: in the case $g_1(x_0) \neq 0$, differentiate the relation $x_2(t) = x_1(\tau(t))$, prove that necessarily $g_2(x_2(t)) = g_1(x_1(\tau(t)))\tau'(t)$, and find a function $\tau(\cdot)$ that works.

Exercise 29 (2015-II-7) *(explicit solution : extra lucid)*
Solve the initial value problem $x'(t) = e^{x(t)} \sin(tx(t)) + x^9(t),\ x(7) = 0$.

Separation of variables

Background. The method of separation of variables applies to differential equations of the form $u'(t) = g(u(t))h(t)$, where u, g and h are real-valued. On an interval on which $g(u(t))$ does not not take the value 0, we may divide both sides by $g(u(t))$ and integrate. After a change of variables, this leads to:

$$\int_{u(t_0)}^{u(t)} \frac{1}{g(v)} dv = \int_{t_0}^{t} h(s) ds.$$

The method may be made rigorous (see Chapter 3), but it is often used as a heuristic way to find candidates for solutions. We write without worrying about what it means: $dx/dt = g(x)h(t)$ hence $dx/g(x) = h(t)dt$ hence $\hat{G}(x) = H(t) + K$, where $\hat{G}$ is a primitive of $1/g$, H a primitive of h, and K a constant fixed by the initial condition. This leads to the candidate solution $x(t) = \hat{G}^{-1}(H(t) + K)$. It suffices to differentiate this function to check that this is indeed a solution, and then to check that it is maximal. If Theorem 1 applies, there is a unique (maximal) solution, so we are done.

Exercise 30 (2015-I-3, done in the course's text) *(a maximal solution need not be global)* Show that the initial value problem $u'(t) = u^2(t)$ and $u(0) = 1$ has a unique maximal solution: the function $u :]-\infty, 1[\to \mathbb{R}$ defined by $u(t) = 1/(1-t)$. Prove this:

a) using Theorem 1 and the fact that we gave you the solution.

b) using the method of separation of variables.

Sketch the graph of the solution. Is this solution global?

Exercise 31 (variant of 2015-I-4) *(finding all solutions, invariance by translation in time of solutions of autonomous equations)* Let t_0 and u_0 be real numbers. Solve the initial value problem $u'(t) = u^2(t)$ and $u(0) = u_0$, discussing the cases $u_0 > 0$, $u_0 = 0$, and $u_0 < 0$, and paying attention to the definition interval of the solution. Then solve the initial value problem $u'(t) = u^2(t)$ and $u(t_0) = u_0$: a) directly; b) using the invariance by translation in time of solutions of autonomous equations. Sketch the graph of some representative solutions, explaining the link between the graph of the solutions of $u'(t) = u^2(t)$ and $u(t_0) = u_0$ and of $u'(t) = u^2(t)$ and $u(0) = u_0$.

Exercise 32 (2015-I-5) *[For $\alpha > 1$, solutions of $x' = |x|^\alpha$ "explode".]* Let $\alpha > 1$. Let $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}$. Consider the initial value problem $x'(t) = |x(t)|^\alpha$ and $x(t_0) = x_0$. After rearranging, the method of separation of variables leads to the candidate solution:

$$x(t) = \frac{x_0}{[1 - sgn(x_0)\beta(t-t_0)|x_0|^\beta]^{1/\beta}} \tag{7}$$

where $\beta = \alpha - 1$ and $sgn(x_0)$ equals 1 if $x_0 > 1$, 0 if $x_0 = 0$, and -1 if $x_0 < 0$.

a) Prove that - for a good choice of J - this is a maximal solution, and the unique one. Check that:

a1) if $x_0 = 0$, then $J = \mathbb{R}$ and $x(t) = 0$ for all t.

a2) if $x_0 > 0$, then $J =]-\infty, T[$ for some $T \in \mathbb{R}$, $x(t) > 0$ for all $t \in J$, $x(t) \to 0$ as $t \to -\infty$ and $x(t) \to +\infty$ as $t \to T$.

a3) if $x_0 < 0$, then $J =]T, +\infty[$ for some $T \in \mathbb{R}$, $x(t) < 0$ for all $t \in J$, $x(t) \to -\infty$ as $t \to T$ and $x(t) \to 0$ as $t \to +\infty$.

Sketch the graph of some representative solutions.

b) Derive (7) using the method of separation of variables.

Exercise 33 (2015-I-7) *(equations in implicit form)*

a) consider the differential equation $x'(t) = 2x(t)/t$. Find all maximal solutions defined, respectively, on a subset of $]0, +\infty[$, and on a subset of $]-\infty, 0[$. How many maximal solutions satisfy $x(1) = 2$?

b) Find all maximal solutions of the differential equation *in implicit form* $tx'(t) = 2x(t)$. How many maximal solutions satisfy $x(1) = 2$? $x(0) = 1$? Why is this consistent with Theorem 1?

Exercise 34 (2015-II-8) *(separation of variables)* Solve the following problems :

i) $x'(t) = tx(t)$ ii) $x'(t) = x^2(t),\ x(0) = 1$ iii) $x'(t) = \dfrac{1 + x^2(t)}{1 + t^2}$

Linear equations in dimension 1

Exercise 35 (2015-II-9) *(seasonal growth)* The growth of an economy may fluctuate seasonally. To study the effect of such fluctuations, solve explicitly and compare the behavior of: a) $x'(t) = rx(t)$; b) $x'(t) = rx(t) + \sin t$; and c) $x'(t) = r(1 + \sin t)x(t)$. Which of these equations are linear? with separated variables?

Hint: for b), there is a solution of the form $x(t) = \lambda \cos t + \mu \sin t$.

Exercise 36 (2015-II-10) *(linear equations)* Find all solutions of the problems : i) $x'(t) = e^t x(t)$, $x(0) = 1$; ii) $x'(t) = ax(t)+b$, $x(0) = 0$; iii) $x'(t) = x(t)+\sin(t)$, $x(0) = 1$; iv) $(1+t^2)x'(t)+tx(t) = 1$.

Exercise 37 (2015-II-11) *(Applications of first order linear equations)*

A) (game theory) In a war of attrition, two individuals, player 1 and player 2, fight for a resource without arms: they choose nonnegative waiting times t_1 and t_2, respectively, after which they leave if the other one did not leave before. The one who chose the highest waiting time gets the resource, which has a value V (if both chose the same waiting time, they both get $V/2$). Waiting is costly, so both, the winner and the loser, pay a waiting cost cT where $T = \min\{t_1, t_2\}$. Therefore, the payoff $g_1(t_1; t_2)$ for player 1 is $V - ct_2$ if $t_1 > t_2$, $V/2 - ct_1$ if $t_1 = t_2$, and $-ct_1$ if $t_1 < t_2$. If player 2 chooses his waiting time randomly according to a probability distribution with a continuous density $p(t)$, then the expected payoff of player 1 if she chooses t_1 is:

$$g_1(t_1, p(\cdot)) = \int_0^{+\infty} g_1(t_1, t_2)p(t_2)dt_2 = \int_0^{t_1} (V - ct_2)p(t_2)dt_2 - ct_1 \int_{t_1}^{+\infty} p(t_2)dt_2$$

Show that there exists a probability density $p(\cdot)$ such that the expected payoff of player 1 is independent of t_1 (hint: consider $\partial g_1(t_1, p(\cdot))/\partial t_1$). Show that if both players choose their waiting time according to this probability density, then this results in a symmetric Nash equilibrium, that is, a situation in which both players have the same (expected) strategy, and, given the behavior of the other, cannot improve their payoffs.

B) Sugar dissolves into water at a speed proportional to the difference between the current sugar concentration of the solution and the concentration at saturation (that is, the maximal sugar concentration, after which sugar ceases to dissolve into water). Consider a volume of water containing 50 kg of dissolved sugar when the solution is saturated. We introduce these 50kg in this volume of pure water at time $t = 0$. Three hours later, half of the sugar has been dissolved and half remains undissolved. How long will it take before only 10 % of the initial quantity remains undissolved?

Change of variable

Exercise 38 (2015-II-12) *(change of variable)* Solve the initial value problem $x'(t) = x(t) + 1$, $x(1) = 4$, and then the initial value problem

$$x'(t) = \frac{1}{2}\left(x(t) + \frac{1}{x(t)}\right),\ x(1) = -2.$$

Exercise 39 (2015-II-13) *(Bernoulli equations)*
They take the following form

$$x'(t) = a(t)x(t) + b(t)x^n(t),\ \text{avec}\ \ n \neq 1\,.$$

A) Make the change of variable $u(t) = x^{1-n}(t)$. Show that u satisfies a linear differential equation.

B) Use this method to solve the logistic equation $x'(t) = rx(t)(K - x(t))$, where r and K are positive constants. Which other method could you use to solve this equation?

C) Which method would you use to solve the following initial value problem ? The question is not to compute the solutions, only to suggest a method.

$$x'(t) = x(t) - e^t x^3(t),\ x(0) = 1$$

Exercise 40 (2015-II-14) *(Riccati equations)* They take the following form:

$$x'(t) = a(t)x(t) + b(t)x^2(t) + c(t),\ \text{avec}\ \ n \neq 1\,.$$

A) Assume that you know a particular solution u. Shows then that $v(t) = x(t) - u(t)$ satisfies a Bernoulli equation.

B) Solve the initial value problem:

$$x'(t) = x(t) - x^2(t) - p,\ x(0) = 1\ (0 \leq p \leq 1/4)$$

Exercise 41 (2015-II-15) *(maximal solutions)* Consider the differential equation

$$y'(t) + y(t) - ty^2(t) = 0. \tag{8}$$

1/. Let $t_0 \in \mathbb{R}$. Find the maximal(s) solution(s) of this equation such that $y(t_0) = 0$.
2/. Let y be a solution of (8) that does not take the value 0. Find an explicit expression for $y(t)$. Hint: remember exercise 39.
3/. Let

$$\begin{array}{rcl} F:\mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & F(t) = (1+t)e^{-t}. \end{array}$$

(i) Study the variations of F.

(ii) Let $K \in \mathbb{R}$. Find, depending on the value of K, the number of solutions of the equation

$$F(t) = K.$$

4/. Find all maximal solutions of (8).

Part 2: systems of linear differential equations. Weeks 7, 8, 9

Week 7: exercises 42, 43, 44. Week 8: exercises 47 and 52/53. Week 9: 54 /55 (56).

Of course, time allowing, feel free to cover more exercises.

A short revision to refresh students memory on change of coordinates and diagonalization of a matrix.

Change of coordinates. Let f be a linear map from $\mathbb{R}^d$ to $\mathbb{R}^d$. Let B and $\tilde{B}$ be basis of $\mathbb{R}^d$, and A and $\tilde{A}$ be the matrices of f in the basis B and $\tilde{B}$, respectively. Then

$$A = P^{-1}\tilde{A}P$$

where P is an invertible matrix such that:

- the columns of P give the coordinates of the vectors of B in the basis $\tilde{B}$
- the columns of P^{-1} give the coordinates of the vectors of $\tilde{B}$ in the basis B.

Thus, if $d = 2$, $B = (e_1, e_2)$ is the canonical basis, and $\tilde{B} = (\eta_1, \eta_2)$ with $\eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\eta_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then $P^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

The second column of P expresses the fact that $e_2 = -1.\eta_1 + 1.\eta_2$.

A way to remember which matrix is P and which is P^{-1} is to start from the relation

$$AX = P^{-1}\tilde{A}PX$$

where X is a column vector. This relation means that if I am given a vector $\vec{v}$ with coordinates X in the basis B and want to compute the coordinates of $f(\vec{v})$ in the same basis, I can do it in two ways. First, I can multiply X by the matrix of f in the basis B, that is, by A. Alternatively, I can find the coordinates Y of $\vec{v}$ in the basis $\tilde{B}$, compute the effect of f in $\tilde{B}$ (that is, multiply by $\tilde{A}$), and then go back to B.

Thus, the effect of the matrix P must be to transform the coordinates X of $\vec{v}$ in the basis B into its coordinates $Y = PX$ in the basis $\tilde{B}$. In particular, $P\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is the first column of P, must give the coordinates of the first vector of B in the basis $\tilde{B}$. More generally, P gives the coordinates of the vectors of B in the basis $\tilde{B}$, and P^{-1} the coordinates of the vectors of $\tilde{B}$ in the basis B.

Diagonalization. To diagonalize the matrix $A \in \mathcal{M}_d(\mathbb{R})$, we first find the eigenvalues λ_1,.., λ_k by solving $det(A - \lambda I) = 0$. We then find the eigenvectors by solving $(A - \lambda_i I)X = 0$ for each $i \in \{1, .., k\}$. If A has d distinct eigenvalues, or more generally, if for each eigenvalue λ_i, the dimension of $Ker(A - \lambda_i I)$ is equal to the multiplicity of λ_i as a root of the characteristic polynomial of A, then A is diagonalizable. An eigenvector basis $\tilde{B}$ is then obtained by grouping basis of $Ker(A - \lambda_i I)$ for $i \in \{1, .., k\}$. Denoting by P^{-1} the matrix whose columns give the coordinates of the vectors of the eigenvector basis in the canonical basis, we then have $A = P^{-1}DP$ with D diagonal.

Recall that when the matrix A is triangular, then it is very easy to find its eigenvalues. Why is that so? Can you prove it?

Exercise 42 (2015-III-1) (explicit solutions and phase portrait of a saddle) Give the solutions and the phase portrait for: a) $x' = -x, y' = y$; b) $x' = x + 3y$, $y' = x - y$.

For b), give the matrix A, its eigenvalues and eigenvectors, show that A is diagonalizable in $\mathbb{R}$, give the solutions of $X' = AX$ associated to the eigenvectors, and the general solution of $X' = AX$ through the coordinates of the solutions in an eigenvector basis. Note that if we only want to draw the shape of the phase portrait, we just need to compute the eigenvalues and the associated eigenvectors.

Exercise 43 (a matrix diagonalizable in $\mathbb{C}$) Give the solutions and the phase portrait of $X' = AX$ for $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $b > 0$. Distinguish the cases $a > 0$, $a = 0$ and $a < 0$.

Exercise 44 (2015-III-2, phase portraits of a source in different basis) Let $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Give the phase portrait of $X' = P^{-1}DPX$ for :

$$a)\, P^{-1} = I; \quad b)\, P^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad c)\, P^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad d)\, P^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}; \quad e)\, P^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Exercise 45 (2015-III-3) Find the solutions and sketch the phase portraits of $X' = AX$ for:

a) $A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$; b) $A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$.

Exercise 46 (2015-III-4) *(trace-determinant plane and phase portrait)* Let $A \in \mathcal{M}_2(\mathbb{R})$. What can be said of the shape of the phase portraits of $X' = AX$ when: 1) $detA < 0$; 2) $Tr(A) > 0$; 3) $det(A) > 0$ and $Tr(A) < 0$; 4) $detA = 0$ and $Tr(A) < 0$.

Exercise 47 (2015-III-5) *(exponential of a matrix, paying attention to the condition that D and N commute)* Let t be a real number. Compute e^A and e^{tA} for the following matrices:

i) $A_1 = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$; ii) $A_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$; iii) $A_3 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$;

In each case, give the solution of $X' = A_iX$ such that $X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Exercise 48 (2015-III-7) *(comparison of three methods)* Let x_0 and y_0 be real numbers. Solve the following system of differential equations (H) and find the solution with initial condition (x_0, y_0) at $t = 0$.

$$(H) \qquad \begin{cases} x'(t) = x(t) + y(t) \\ y'(t) = 2y(t) \end{cases}$$

Same question for the system (NH):

$$(NH) \qquad \begin{cases} x'(t) = x(t) + y(t) + e^t \\ y'(t) = 2y(t) - 3e^{-t} \end{cases}$$

Solve (NH) in three ways: i) using the fact that the system is triangular; ii) finding a basis $(X_1(\cdot), X_2(\cdot))$ of the set of solutions of (H) then searching solutions of (NH) of the form $\lambda(t)X_1(t) + \mu(t)X_2(t)$; iii) computing e^{tA}, and applying Duhamel's formula.

Exercise 49 (2015-III-6) *(exponential of a 3-3 matrix)* Compute e^{tA} when $A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$ (hint: $A - 2I$ is nilpotent). Then solve the initial value problem: $x' = 2x + y + 3z, y' = 2y - z, z' = 2z, x(0) = 1, y(0) = 2, z(0) = 1$.

Exercise 50 (2015-III-8) *(system of the form $MX'(t) = AX(t) + B(t)$ with M invertible)* Solve the system:

$$\begin{cases} 2x_1'(t) + x_2'(t) - 3x_1(t) - x_2(t) = t \\ x_1'(t) + x_2'(t) - 4x_1(t) - x_2(t) = e^t, \end{cases}$$

Exercise 51 (2015-III-9) *(n^{th} order equation with polynomial right-hand-side).* Consider the n^{th} order equation $a_n x^{(n)}(t) + ... + a_1 x'(t) + a_0 x(t) = Q(t)$ where Q is a polynomial and a_0 ,...a_n are real numbers such that $a_n \neq 0$. Show that if $a_0 \neq 0$, there is a unique polynomial solution and it is of degree $\deg(Q)$. What if $a_0 = 0$ and $a_1 \neq 0$? Application: find solutions of $x''(t) + x'(t) + x(t) = t$ and $x'''(t) + x''(t) + x'(t) = t$, then solve these equations.

Exercise 52 (2015-III-10) *(resonance)* Watch movies on resonance on the web, e.g., at

http://lewebpedagogique.com/physique/quelques-videos-de-resonnances/ .

Let $\omega > 0$ and $\nu \geq 0$. Solve as a function of ν the differential equation

$$x''(t) + \omega^2 x(t) = \cos(\nu t).$$

What does this equation represent (note: students know little physics)? What happens when $\nu \to \omega$?

Exercise 53 (2015-V-1) *(phase portraits and stability)* Sketch the phase portrait of the system $X'(t) = AX$ and say if the origin is stable, asymptotically stable, or unstable for:

1) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$; 2) $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$; 3) $A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$; 4) $A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$; 5) $A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$

Exercise 54 *(damped oscillator)* Find the solutions of $x''(t) + \alpha x'(t) + \omega^2 x(t) = 0$ with $\alpha > 0$. Note: this models a damped oscillator, that is, an oscillator with friction.

Exercise 55 (2015-V-2) *(trace-determinant plane and stability)* Let $A \in \mathcal{M}_2(\mathbb{R})$. What can be said of the stability of the origin in the following cases? 1) $detA < 0$; 2) $Tr(A) > 0$; 3) $det(A) > 0$ and $Tr(A) < 0$; 4) $detA = 0$ et $Tr(A) < 0$.

Part 3: comparison, linearization, geometric approach, continuity of the flow, Lyapunov functions.

Week 10: exercises 58, 59, 62, 63 or 64. Week 11: 65, 66, 69. Weeks 12: competition between two herbivorous species. Weeks 13: 75, 76, (77).

Comparison principles

Exercise 56 (2015-II-16) *(understanding the assumptions of the comparison principle)* Let f, u, v, w be functions from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = 1-x$, $u(t) = -e^{-t}$, $v(t) = 1$, $w(t) = 2-2e^{-t}$. Show that for all real numbers t, $u'(t) < f(u(t))$, $v'(t) = f(v(t))$, $w'(t) > f(w(t))$. Is it true that for all real numbers t, $u'(t) < v'(t) < w'(t)$?

Exercise 57 (2015-II-18) *(variants of the comparison principle)* Let $f \in C^1(\mathbb{R}^2, \mathbb{R})$, u, $v \in C^1(\mathbb{R})$, and $t_0 \in \mathbb{R}$. Assume that u is a sub-solution and v a super-solution of $x'(t) = f(t, x(t))$. Why do we have: if $u(t_0) \leq v(t_0)$ then $u(t) \leq v(t)$ for all $t \geq t_0$? Show that :

1) if $u(t_0) < v(t_0)$ then $u(t) < v(t)$ for all $t \geq t_0$.

2) if $u(t_0) = v(t_0)$ then $u(t) \leq v(t)$ for all $t \geq t_0$ and $u(t) \geq v(t)$ for all $t \leq t_0$.

3) if $u(t_0) > v(t_0)$ then $u(t) > v(t)$ for all $t \leq t_0$. What can be said for $t \geq t_0$?

Exercise 58 (2015-II-17) *(basic sufficient conditions for nonexplosion)* Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function. Show that any maximal solution of $x'(t) = f(t, x(t))$ is global:

i) if f is bounded over $I \times \mathbb{R}^n$;

ii) if there exists constants A et B such that: $\forall (t, x) \in \mathbb{R} \times \mathbb{R}^n$, $||f(t, x)|| \leq A||x|| + B$.

Exercise 59 (2015-II-19) *(don't be scared)* Show that all maximal solutions of the following differential equation are global: $x'(t) = e^t x(t) + e^{-t} \sin(t^2 x(t)) - \arctan(1 + t^2 + x^2(t))$.

Exercise 60 *(explosion in dimension 1: theory)* Let $f : \mathbb{R}^2 \to R$ be C^1. Let $(J, x(\cdot))$ be a maximal solution of $x'(t) = f(t, x(t))$ such that $x(t) \to +\infty$ as $t \to \sup J$. Prove that:

i) if there exist K and $\bar{x}$ in $\mathbb{R}$ such that for all (t, x) in $\mathbb{R}^2$: $x \geq \bar{x} \Rightarrow f(t, x) \leq Kx$, then $\sup J = +\infty$.

ii) if there exist (strictly) positive real numbers K, $\bar{x}$ and ε such that, for all (t, x) in $\mathbb{R}^2$, $x \geq \bar{x} \Rightarrow f(t, x) \geq Kx^{1+\varepsilon}$, then $\sup J < +\infty$.

Exercise 61 *(explosion in dimension 1: examples)* In the following cases, does the solution of the initial value problem $x'(t) = f(t, x)$, $x(0) = x_0$ explode in forward time?

1) $f(t, x) = t + x$, x_0 arbitrary ; 2) $f(t, x) = tx$, $x_0 = 5$; 3) $f(t, x) = x^2$, $x_0 > 0$;
4) $f(t, x) = x^2$, $x_0 = 0$; 5) $f(t, x) = x^2 + t$, $x_0 = 0$; 6) $f(t, x) = x^2 - t$, $x_0 < 0$;
7) $f(t, x) = x^2 - t$, $x_0 > 0$ such that $x_0^2 - t > 1/(2x_0)$; 8) $f(t, x) = x^{4/3}/\sqrt{1 + x^2}$, $x_0 > 0$;
9) $f(t, x) = x^3/\sqrt{1 + x^2}$, $x_0 > 0$; 10) $f(t, x) = x^{4/3}/\sqrt{1 + \sin^2(x)}$, $x_0 > 0$;
11) $f(t, x) = x^{4/3}/\sqrt{1 + \sin^2(x)}$, $x_0 \leq 0$; 12) $f(t, x) = \ln x$, $x_0 = 2$.

Exercise 62 (2015-II-20) *(explosion by comparison)* Consider the logistic equation $x'(t) = rx(t)(K - x(t))$. Show that if a maximal solution $(J =]a, b[, x(\cdot))$ satisfies $x(0) > K$, then $a > -\infty$. Hint: show that $y(t) = x(-t) \to +\infty$ as $t \to -a$ and that for t large enough, $y'(t) \geq y^\alpha(t)$ for some $\alpha > 1$. What can be said of the interval of definition of the solutions of the equations of Exercise 23?

Exercise 63 (2015-II-21) *(logistic growth with varying carrying capacity)* Let $(J, \bar{x}(\cdot))$ be a maximal solution of

$$x'(t) = x(t)(K + \sin t - x(t)) \tag{9}$$

such that $x(0) > 0$. Using the fact that $|\sin t| \leq 1$ and comparison principles, show that $\sup J = +\infty$, and that as $t \to +\infty$, $\liminf \bar{x}(t) \geq K - 1$ and $\limsup \bar{x}(t) \leq K + 1$.

Exercise 64 (2015-II-22) *(adapted from 2013 mid-semester exam)*

1) Let $K > 0$. Sketch the phase line of the differential equation $x'(t) = x^2(t)(K - x(t))$.

2) Consider the differential equation $x'(t) = x^2(t)(1 - x(t) - e^{-t})$. Let $(J, x(\cdot))$ be a maximal solution defined in $t = 0$ and such that $x(0) > 0$.

a) Show that $x(t) > 0$ for all $t \in J$.

b) Show that for all $t \in J \cap [0, +\infty[$, $x'(t) \leq x^2(t)(1 - x(t))$. Conclude that $\sup J = +\infty$ and that $\limsup_{t \to +\infty} x(t) \leq 1$.

c) Let $\varepsilon > 0$. Show that there exists $t_0 \in \mathbb{R}$ such that for all $t \geq t_0$, $x'(t) \geq x^2(t)(1 - \varepsilon - x(t))$.

d) Show that $x(t) \to 1$ when $t \to +\infty$.

A first look at stability and linearization

Exercise 65 *(linearization in dimension 1)*

What is the stability of the origin for the equation $x' = \lambda x$, with $\lambda \in \mathbb{R}$? Let x^* be an equilibrium of the equation $x' = f(x)$, with $f : \mathbb{R} \to \mathbb{R}$ of class C^1. Show that x^* is asymptotically stable if $f'(x^*) < 0$ and unstable if $f'(x^*) > 0$: a) directly; b) using the linearization theorem.

Exercise 66 (2015-V-3) Discuss the stability of the equilibria (or the equilibrium) of the system.

$$\begin{cases} x'(t) = y(t) \\ y'(t) = (x^3(t) - 1)e^{-x^2(t)} \end{cases}$$

Geometric approach

Exercise 67 *(vector fields, graphs and phase line)* (no justification needed) Sketch the velocity field, the graphs of some solutions and the phase line of the differential equation $x' = x(K - x)$, $K > 0$.

Exercise 68 *(understanding Euler's method)*

Let (E) denote the differential equation $x'(t) = x(t) + t$.

a) Show that there is a solution whose trajectory is a straight line then solve (E).

b) Sketch the vector field at points with integer coordinates (n, m) with $0 \leq n \leq 2$ and $-1 \leq m \leq 1$.

c) Consider the initial value problem $x'(t) = x(t) + t$ and $x(0) = 0$. Let $y_\lambda(\cdot)$ denote the approximate solution obtained by applying Euler's method with a step size of λ. Compute $y_\lambda(t)$ for $t \in [0, 2]$ in the following cases: $\lambda = 2$; $\lambda = 1$; $\lambda = 1/2$. How does the gap between the true solution and the approximate solution evolve as λ diminishes?

d) Show that for the initial value problem $x'(t) = x(t) + t$ and $x(0) = -1$, Euler's method gives the exact solution, for any step size. Why is it so?

Exercise 69 *(reading vector fields)* Do Exercise 4 of the page with vector fields (Figure 1.29 has nothing to do with exercice 4; it just helps students for Exercise 5).

Problem: Competition between two herbivorous species.

Consider the initial value problem

$$\begin{cases} x'(t) = (2 - x(t) - y(t))x(t), & x(0) = x_0 \geq 0, \\ y'(t) = (1 - x(t) - y(t))y(t), & y(0) = y_0 \geq 0. \end{cases} \tag{10}$$

This system of differential equations models competition between two herbivorous species, with respective population densities $x(t)$ and $y(t)$. When these densities are low, there is a profusion of grass, and the growth-rates are respectively of 2 for the first species and of 1 for the second one. When the population densities increase, grass becomes scarcer, and the growth rates of both species are reduced by the amount $x(t)+y(t)$. They are thus of $2-x(t)-y(t)$ and $1-x(t)-y(t)$, respectively. Note that the growth-rate of the first species is always higher than the growth-rate of the second species. Thus, we expect that the first species will win the competition, and that the second will go extinct. This is what we will show, in two different ways.

Part I (generalities and comparison principle).

1/. Show that (10) has a unique maximal solution $(J, (x(\cdot), y(\cdot)))$.

2/. Using the first equation, show that if $x_0 = 0$ then for all $t \in J$, $x(t) = 0$, and that if $x_0 > 0$, then for all $t \in J$, $x(t) > 0$. Show similarly that $y(t) \geq 0$ for all $t \in J$, with a strict inequality if $y_0 > 0$.

3/. Show that, for all $t \in J \cap \mathbb{R}_+$, $x(t) \leq x_0 e^{2t}$ and $y(t) \leq y_0 e^t$. Conclude that $\mathbb{R}_+ \subset J$.

4/. Show that for all $t \in \mathbb{R}_+$, $x'(t) \leq (2-x(t))x(t)$. Show that $\limsup_{t\to+\infty} x(t) \leq 2$. Show similarly that $\limsup_{t\to+\infty} y(t) \leq 1$.

5/. In the following three cases, show that $(x(t), y(t))$ converges as $t \to +\infty$ and find its limit.

(i) $x_0 = y_0 = 0$; (ii) $x_0 > 0$, $y_0 = 0$; (iii) $x_0 = 0$, $y_0 > 0$.

In what follows, we assume $x_0 > 0$ and $y_0 > 0$, so that $x(t) > 0$ and $y(t) > 0$ for all t in J.

Part II (stable and unstable equilibria)

1/. Compute the three equilibria.

2/. For each equilibrium (x^*, y^*), find the linearized system, say whether the equilibrium is hyperbolic, and give the nature (source, sink, saddle,...) and the stability (stable, asymptotically stable, unstable) of the origin for the linearized system. What can we deduce on the stability of (x^*, y^*) for the initial, nonlinear system?

Part III (nullclines and directions of movement).

This section shows that $(x(t), y(t)) \to (2, 0)$ as $t \to +\infty$. Let us divide $(\mathbb{R}_+^*)^2$ in three regions, corresponding to different signs of x' and y', so that the fact that $(x(t), y(t))$ belongs to one of this zone determines the general direction of movement. Let

$$V(x,y) = x + y; \quad v(t) = V(x(t), y(t)); \quad A = \{(x,y) \in (\mathbb{R}_+^*)^2 | V(x,y) \leq 1\};$$

$$B = \{(x,y) \in (\mathbb{R}_+^*)^2 | 1 < V(x,y) < 2\}; \text{ and } C = \{(x,y) \in (\mathbb{R}_+^*)^2 | V(x,y) \geq 2\}.$$

0/. Draw these regions and the general direction of movement in the interior of these zones and on their boundaries.

1/. Case 1: if there exists $t_b \in \mathbb{R}_+$ such that $(x(t_b), y(t_b)) \in B$.

a) We want to show that $(x(t), y(t)) \in B$ for all $t \geq t_b$.

a1) Why is is enough to show that for all $t \geq t_b$, $1 < v(t) < 2$.

a2) Assume by contradiction that there is a time $s > t_b$ such that $v(s) < 1$. Show this implies that there exists $T > t_b$ such that $v(t) > 1$ for all $t \in [t_b, T[$ and $v(T) \geq 1$. Show that $v(T) = 1$ and $v'(T) > 0$. Deduce that this implies that there exists $\bar{t} \in]t_b, T[$ such that $v(\bar{t}) < 1$. Find a contradiction and conclude that $v(t) > 1$ for all $t \geq t_b$.

a3) Show similarly that $v(t) < 2$ for all $t \geq t_b$.

b) Show that on $[t_b, +\infty[$, $x(\cdot)$ is increasing, and $y(\cdot)$ is decreasing. Conclude that there exists $(x^*, y^*) \in \mathbb{R}^2$ such that $x^* \geq x(t_b)$, $y^* \leq y(t_b)$ and $(x(t), y(t)) \to (x^*, y^*)$ as $t \to +\infty$.

c) Show that (x^*, y^*) is an equilibrium and that $(x(t), y(t)) \to (2, 0)$ as $t \to +\infty$.

2/. Case 2 : if there exists $t_a \in \mathbb{R}_+$ such that $(x(t_a), y(t_a)) \in A$.

a) We want to show that there exists $T > t_a$ such that $v(T) > 1$. Assume by contradiction that $v(t) \leq 1$ for all $t \geq t_a$. Show that on $[t_a, +\infty[$, $x(\cdot)$ and $y(\cdot)$ are increasing. Deduce from this that there exists $(x^*, y^*) \in \mathbb{R}^2$ such that $x^* \geq x(t_a)$, $y^* \geq y(t_a)$, and $(x(t), y(t)) \to (x^*, y^*)$ as $t \to +\infty$. Find a contradiction and conclude.

b) Show that exists $t_b \geq t_a$ such that $(x(t_b), y(t_b)) \in B$, hence $(x(t), y(t)) \to (2, 0)$ as $t \to +\infty$.

3/. Case 3 : if there exists $t_c \in \mathbb{R}_+$ such that $(x(t_c), y(t_c)) \in C$. Show that if $v(t) \geq 2$ for all $t \geq t_c$, then $(x(t), y(t)) \to (2, 0)$ as $t \to +\infty$, and that otherwise, there exists $t_b \geq t_c$ such that $(x(t_b), y(t_b)) \in B$. Conclude that $(x(t), y(t)) \to (2, 0)$ as $t \to +\infty$.

4/. Show that if $x_0 > 0$ and $y_0 > 0$, then $(x(t), y(t)) \to (2, 0)$ as $t \to +\infty$.

Part IV (Lyapunov method) In this part, we give another proof of the fact that $(x(t), y(t)) \to (2, 0)$ as $t \to +\infty$. Thus, we are not allowed to use Part III.

1/. Let $w(t) = \ln(x(t)/y(t))$. Show that for all $t \in J$, $w'(t) = 1$. Deduce from this that $x(t)/y(t) \to +\infty$ as $t \to +\infty$. Using Part 1, conclude that $y(t) \to 0$ as $t \to +\infty$.

2/. Let $\varepsilon > 0$. Show that there exists $T_\varepsilon \in \mathbb{R}_+$ such that for all $t \geq T_\varepsilon$, $x'(t) \geq x(t)(2 - \varepsilon - x(t))$. Conclude that $\liminf x(t) \geq 2 - \varepsilon$ and then, using Part 1, that $x(t) \to 2$ as $t \to +\infty$.

Periodic solutions and continuity of the flow [for students preparing competitive exams]

Exercise 70 (2015-V-7) *(openness of the basin of attraction of asymptotically stable equilibria)*

Consider an autonomous differential equation $X'(t) = F(X(t))$ with $F : \mathbb{R}^d \to \mathbb{R}^d$ of class C^1. Let X^* be an asymptotically stable equilibrium and let $B(X^*)$ denote its basin of attraction; that is, $B(X^*)$ is the set of initial positions X_0 such that the solution with initial condition $X(0) = X_0$ is defined for all times $t \geq 0$ and converges towards X^* as $t \to +\infty$. Show that $B(X^*)$ is open.

Exercise 71 *(periodic solutions)*

Let $T > 0$. Let $F : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be C^1 and T-periodic: $F(t + T, X) = F(t, X)$, $\forall (t, X) \in \mathbb{R}^{d+1}$. Let $X(\cdot)$ be a global solution of $X'(t) = F(t, X(t))$, which we denote by $(*)$.

1. Let $k \in \mathbb{Z}$. Define $Y : \mathbb{R} \to \mathbb{R}$ by $Y(t) = X(kT + t)$. Show that $Y(\cdot)$ is solution of $(*)$.
2. Show that if $X(T) = X(0)$, then $X(\cdot)$ is T-periodic. Prove the same result without assuming $X(\cdot)$ global.
3. Show that if a solution of an autonomous equation takes the same value twice, then it is global and periodic.

Exercise 72 *(periodic solutions in dimension 1)*

Let $T > 0$. Denote by $(*)$ the differential equation $x'(t) = f(t, x(t))$ with $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of class C^1 and T-periodic.

1) Let $x : J \to \mathbb{R}$ be a solution such that $x(T) = x(0)$. Show that x is T-periodic and global.

2) Let a and b be real numbers, with $a < b$. Assume that solutions with initial conditions $x(0) = a$ and $x(0) = b$ are defined at time T (that is, T is included in their interval of definition).

2a) Show that for any $x_0 \in [a, b]$, the solution with initial condition $x(0) = x_0$ is defined at time T. Let $\phi_T(x_0)$ denote the value of this solution at time T and let $g(x_0) = \phi_T(x_0) - x_0$.

2b) Graphically, what is the interpretation of $g(x_0)$? Why is the fonction $g : [a, b] \to \mathbb{R}$ continuous?

2c) Show that if $g(a)g(b) < 0$, then there exists a periodic solution.

3) Assume that there exists two global solutions such that one goes to $+\infty$ and the other one to $-\infty$ as $t \to +\infty$. Show that there exists a periodic solution.

4) Assume that the function f is strictly increasing with respect to its second variable x. Assume for simplicity that all solutions are global so that the function g may be defined on the whole of $\mathbb{R}$.

4a) Let $T' \in \mathbb{R}^*$. Without using that f is T-periodic, show that there is at most one periodic solution of period T' (hint: $x(T) - x(0) = \int_0^T x'(t)\, dt$).

4b) Show that g is strictly increasing.

4c) Let $x_0 \in \mathbb{R}$, let $x(\cdot)$ denote the solution such that $x(0) = x_0$. For all k in $\mathbb{N}$, let $u_k = x(kT)$ and $v_k = u_{k+1} - u_k$.

4d) Show that $x((k+1)T) = \phi_T(x(kT))$ (hint: $y(t) = x(kT + t)$ is solution of $(*)$).

4e) Assume $x(T) > x(0)$. Show that $v_1 > 0$ and then that (u_k) and (v_k) are increasing. Conclude that $u_k \to +\infty$ as $k \to +\infty$, that $x(t) \to +\infty$ as $t \to +\infty$, and that $x(\cdot)$ is not periodic.

4f) Similarly, show that if $x(T) < x(0)$, then $x(t) \to -\infty$ as $t \to +\infty$.

4g) Show that there is at most one periodic solution, that if it exists, it is T-periodic, and that as $t \to +\infty$, all solutions above the periodic solution go to $+\infty$ and all solutions below it go to $-\infty$.

4h) (difficult) Show that even if f is not periodic, there exists at most one periodic solution. Hint in this note.[5]

Exercise 73 *(periodic solutions in dim 1: an example)* Consider the equation $x' = x + \cos t$. Using questions 3) and 4) of Exercise 72, show that there exists exactly one periodic solution and that all other solutions go to infinity as $t \to +\infty$. Solve explicitly the equation and check these results.

Exercise 74 *(trap and periodic solutions in dim 1)* Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be C^1. Let a and b be real numbers, with $a < b$. Assume that f is T-periodic and that:

$$\forall t \in \mathbb{R}^+, f(t, a) > 0, \text{ and } f(t, b) < 0.$$

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(0) = y_0 \end{cases} \qquad (C_{y_0})$$

[5] Assume that there are two periodic solutions $x(\cdot)$ and $y(\cdot)$ with periods T_x and T_y, respectively, and that $x(0) < y(0)$. Using that f is strictly increasing in its second variable, show that for any positive integer p, $y(pT_x) \geq y(T_x) > y(0)$ and then that there exists $\alpha > 0$, independent of p, such that for any t in $\mathbb{R}$, if $|t - pT_x| < \alpha$, then $y(t) > y(0)$. Finally, use that there exists positive integers p and q such that $|qT_y - pT_x| < \alpha$ and conclude.

1. Study the existence of solutions to the problem (C_{y_0}). Show that for any $y_0 \in [a,b]$, any (maximal) solution y of (C_{y_0}) takes its values in $[a,b]$ for positive times. Conclude that it is defined on the whole of $\mathbb{R}_+$.

2. Using question 2) of Exercise 72, show that there exists a periodic solution.

Linearization, Hamiltonian systems, Lyapunov functions

Exercise 75 (2015-V-4) *(linearization, constant of movement)* Consider the following system:

$$\begin{cases} x'(t) = 2\varepsilon x(t) + y(t) \\ y'(t) = -5x(t) - x^3(t). \end{cases} \tag{E}$$

1. Show that this system has a unique equilibrium:

2. Give the nature of the linearized system in this equilibrium. For which values of ε can we deduce an information on the stability of the equilibrium of the system (E) ?

3. We now consider the case $\varepsilon = 0$. Let

$$H(x,y) = \frac{1}{2}y^2 + \frac{5}{2}x^2 + \frac{1}{4}x^4.$$

   a. Show that H is constant along trajectories of (E).

   b. Show that all trajectories are bounded. Conclude that all solutions of (E) are global.

   c. Is the equilibrium stable? Asymptotically stable?

   d. Show that all solutions are periodic.

Exercise 76 (2015-V-5) *(Hamiltonian systems)* Consider a system of differential equations $\dot{X}(t) = F(X(t))$ with $F : \mathbb{R}^2 \to \mathbb{R}^2$. This is a *Hamiltonian system* if there exists a differentiable function $H : \mathbb{R}^2 \to \mathbb{R}$ such that $F_1 = \partial H/\partial y$ and $F_2 = -\partial H/\partial x$; that is, if the system may be written as:

$$\begin{cases} \dot{x}(t) &= &\frac{\partial H}{\partial y}(x(t), y(t)) \\ \dot{y}(t) &= &-\frac{\partial H}{\partial x}(x(t), y(t)). \end{cases}$$

1. Show that H is then a constant of movement.

2. Let $H(x,y) = -xy$, for x, y in $\mathbb{R}$. Give the Hamiltonian system associated to H and his phase portrait.

3. Show that the following system is Hamiltonian and find a Hamiltonian function:

$$\begin{cases} \dot{x} = -x \\ \dot{y} = y - 3x^2. \end{cases}$$

   Give the nature of the equilibria and sketch the trajectories.

Exercise 77 (2015-V-6) *(a system with four equilibria, Lyapunov functions)* Consider the following system:

$$\begin{cases} x'(t) = -x(t)^2 - y(t)^2 + 1 \\ y'(t) = -2x(t)y(t). \end{cases} \tag{S}$$

1. Find the equilibria of this system.

2. For each of these equilibria, give the linearized system, and discuss its nature and stability.

3. Deduce whenever possible information on the stability of the corresponding equilibrium of (S).

4. Let
$$V(x,y) = y^2x + \frac{x^3}{3} - x.$$

   a. Show that (x_0, y_0) is an equilibrium point of (S) if and only if it is a critical point of V.

   b. Show that V decreases along trajectories. When do we have $\frac{d}{dt}V(x(t), y(t)) = 0$?

5. We study the behavior of the system close to the equilibrium $(1,0)$. We want to show that there exists a neighborhood $\mathcal{V}$ of this point such that, for every initial position (x_0, y_0) in $\mathcal{V}$, $\forall t > 0$,
$$\|(x(t), y(t)) - (1,0)\| \leq e^{-t}\|(x_0, y_0) - (1,0)\|, \tag{11}$$
where $(x(t), y(t))$ is the unique solution of (S) with initial position (x_0, y_0) (and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^2$).

   a. Show that
$$\frac{\partial V}{\partial x}(x,y) = 2(x-1) + \|(x,y) - (1,0)\|\varepsilon_1((x,y) - (1,0)),$$
   and
$$\frac{\partial V}{\partial y}(x,y) = 2y + \|(x,y) - (1,0)\|\varepsilon_2((x,y) - (1,0)),$$
   where $\varepsilon_i((x,y) - (1,0))$, $i = 1,2$ goes to 0 when (x,y) goes to $(1,0)$.

   b. Let $\mathcal{V}$ be a neighborhood of $(1,0)$ such that $||\varepsilon((x,y) - (1,0))|| < 1$, where ε is the vector with components $(\varepsilon_1, \varepsilon_2)$. Let $(x_0, y_0) \in \mathcal{V}$. Show that for t sufficiently small:
$$\frac{d}{dt}\|(x(t), y(t)) - (1,0)\|^2 \leq -2\|(x(t), y(t)) - (1,0)\|^2.$$
   (recall Cauchy-Schwartz inequality: $|(X, X')| \leq \|X\|\|X'\|$).
   Conclude that (11) holds. Explain why this inequality actually holds for all times $t > 0$.

Exercise 78 *(prey-predator)* Consider a simple version of the standard Lotka-Volterra prey-predator model:
$$\begin{cases} \dot{x} = x(y-b) \\ \dot{y} = y(a-x), \end{cases}$$
where a and b are positive constants.

1. Show that if x_0 and y_0 are strictly positive, then for all times t, $x(t) > 0$ and $y(t) > 0$.

2. Find the equilibria.

3. Study the linearized system in each of these equilibria and deduce their stability.

4. Is this a Hamiltonian system? Let $G(x,y) = x + y - a\ln x - b\ln y$. Show that $G(x,y)$ is a constant of movement (for solutions such that $x_0 > 0$, $y_0 > 0$).

5. Show that trajectories are periodic and that, denoting by T their period:
$$\frac{1}{T}\int_0^T x(t)dt = a, \text{ et } \frac{1}{T}\int_0^T y(t)dt = b.$$

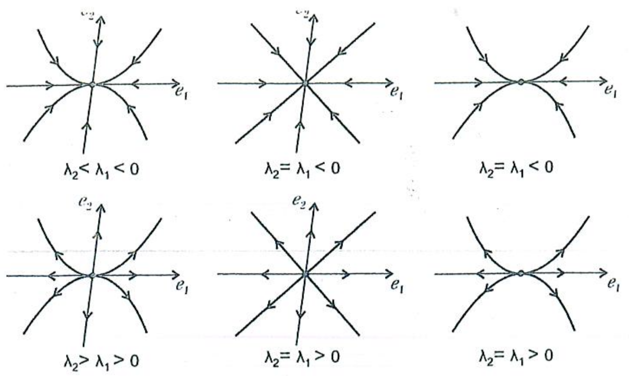


FIG. 7.3 – Nœuds (attractifs en haut, répulsifs en bas)

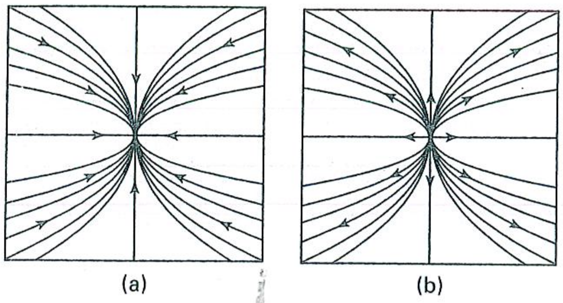


Figure 3.3 Phase portraits for (a) a sink and (b) a source.

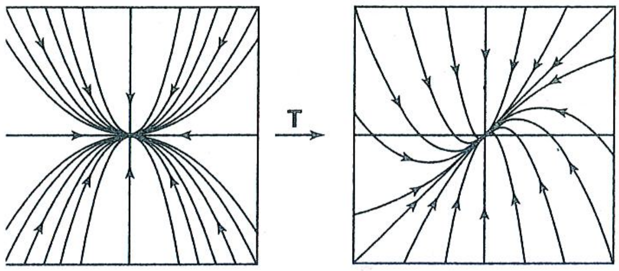
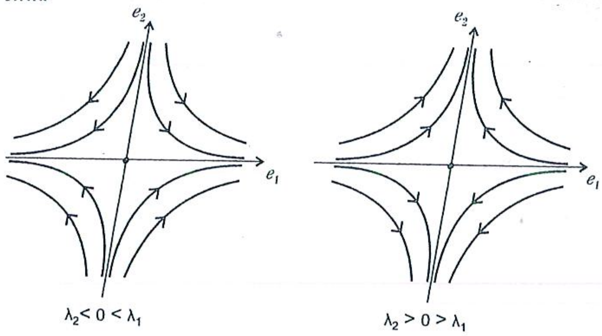


Figure 3.7 The change of variables T in the case of a (real) sink.



– Cols

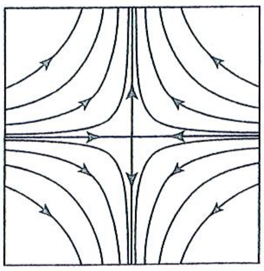


Figure 3.1 Saddle phase portrait for $x' = -x$, $y' = y$.

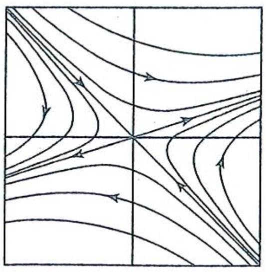


Figure 3.2 Saddle phase portrait for $x' = x + 3y$, $y' = x - y$.

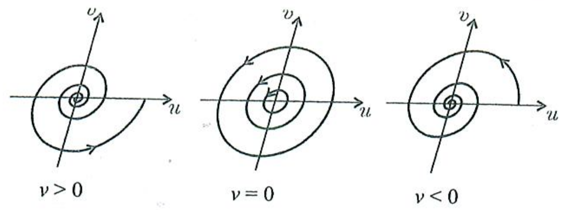


FIG. 7.5 – Foyers (répulsif à gauche, attractif à droite) et centre

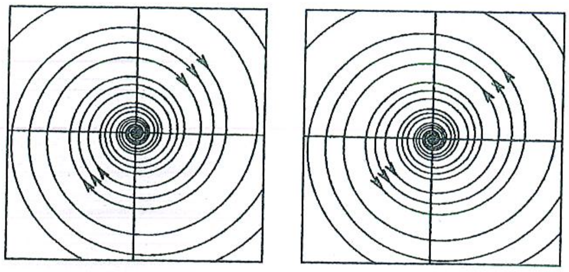


Figure 3.5 Phase portraits for a spiral sink and a spiral source.

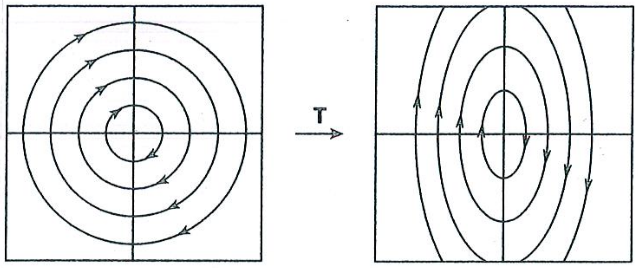


Figure 3.8 The change of variables T in the case of a center.

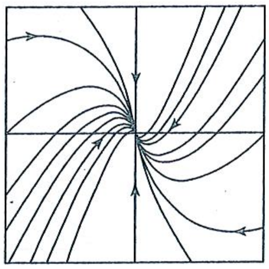


Figure 3.6 Phase portrait for a system with repeated negative eigenvalues.

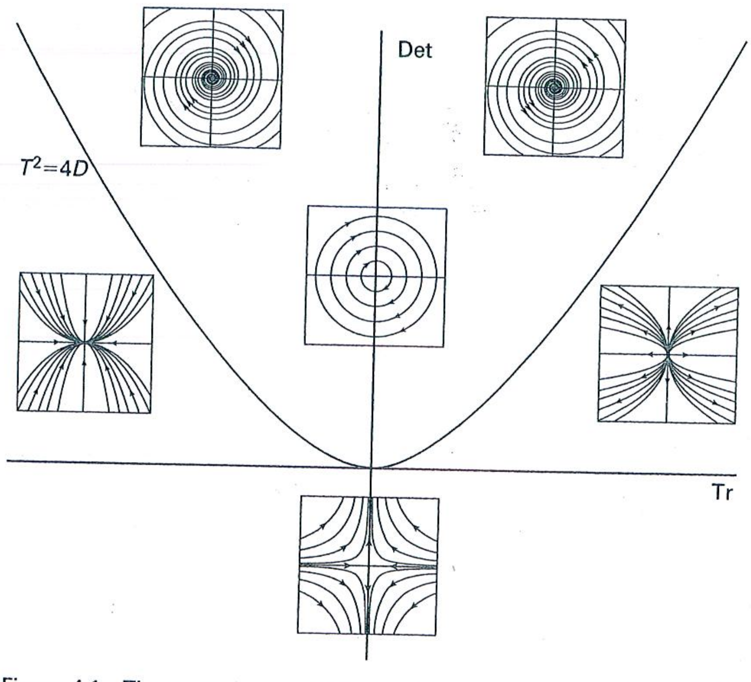


Figure 4.1 The trace-determinant plane. Any resemblance to any of the authors' faces is purely coincidental.

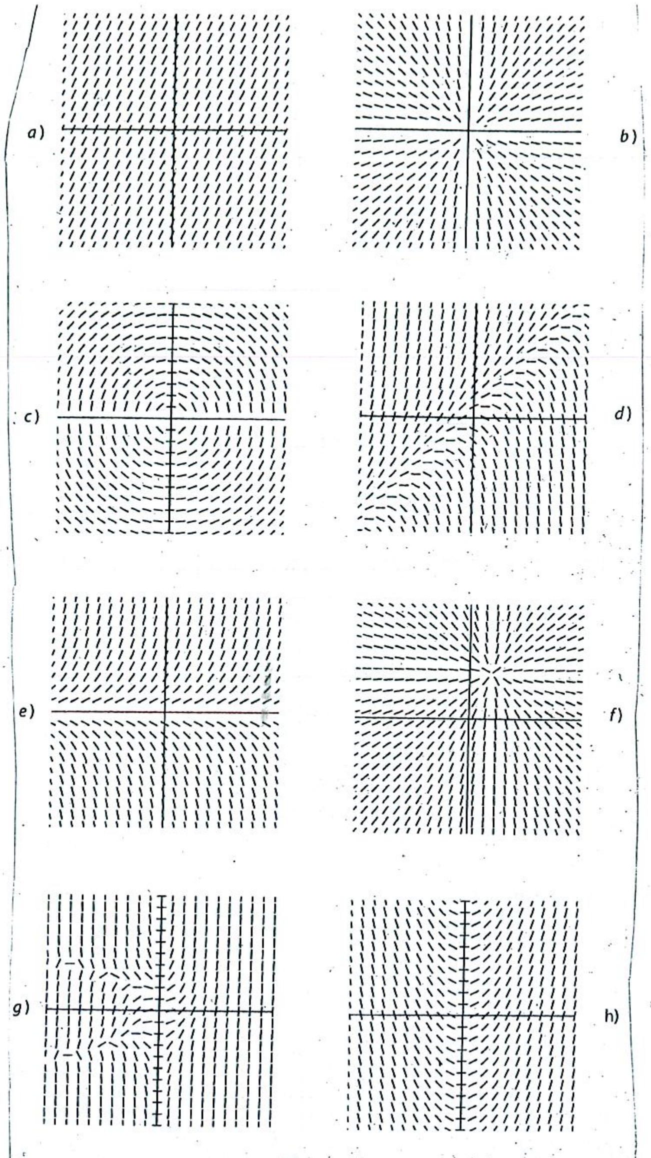


Fig. 1.39.

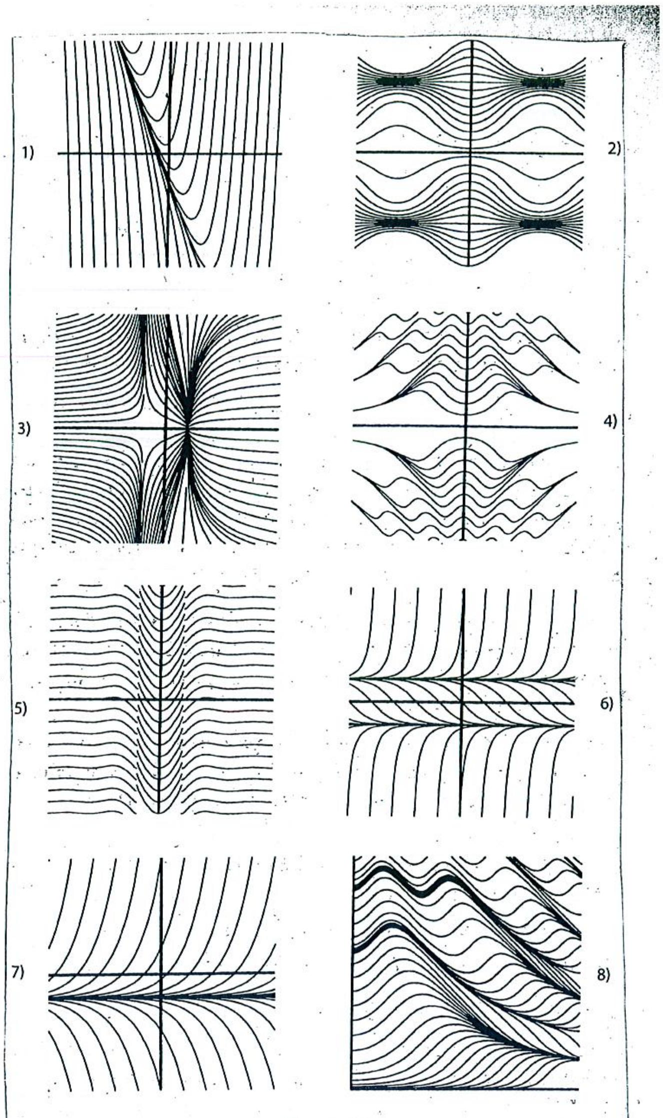


Fig. 1.40.

4. Sur chacun des huit champs de directions de la figure 1.39 :

(a) Trouver ~~l'isocline I_0~~. l'ensemble des points où la pente est nulle

(b) Trouver les points où la pente n'est pas définie.

(c) Dessiner quelques solutions.

(d) Marier chaque dessin avec l'une des équations ci-dessous.

1.	$x' = 2$	5.	$x' = x/t$
2.	$x' = t$	6.	$x' = -t/x$
3.	$x' = x - t$	7.	$x' = (x-2)/(t-1)$
4.	$x' = x$	8.	$x' = tx^2 + t^2$.

5. Déterminer quel(s) dessin(s) de la figure 1.40 représente(nt) quelques solutions de chacune des équations ci-dessous :

(a)	$x' = \sin tx$	(d)	$x' = (\sin t)(\sin x)$
(b)	$x' = x^2 - 1$	(e)	$x' = x/(t^2 - 1)$
(c)	$x' = 2t + x$	(f)	$x' = (\sin 3t)/(1 - t^2)$.

Attention : plusieurs dessins peuvent être associés à la même équation, et un dessin peut n'être associé à aucune.

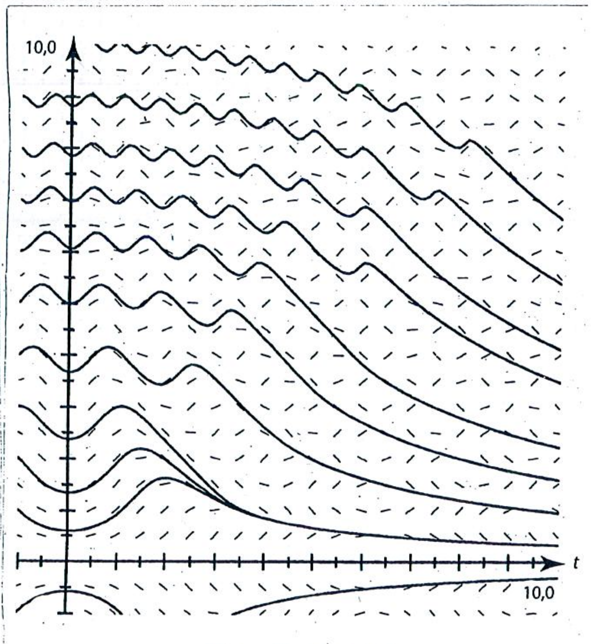


Fig. 1.29. $x' = \sin tx$.

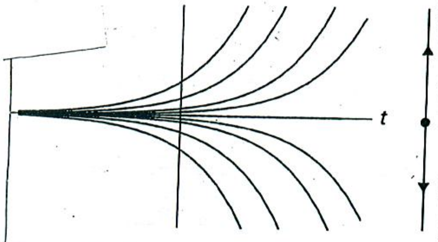


Figure 1.1 The solution graphs and phase line for $x' = ax$ for $a > 0$. Each graph represents a particular solution.

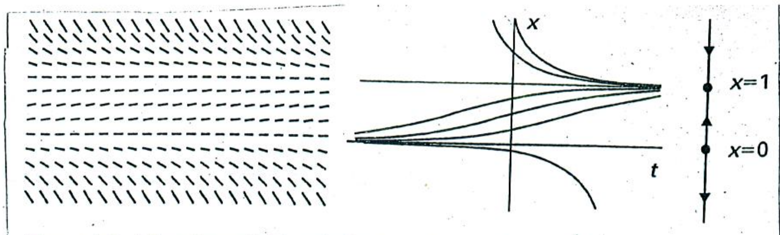


Figure 1.3 The slope field, solution graphs, and phase line for $x' = ax(1 - x)$.

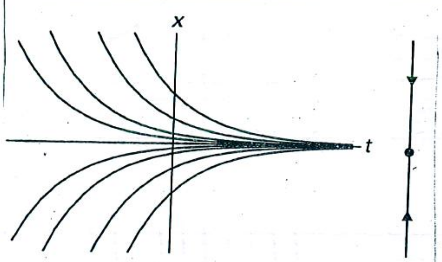


Figure 1.2 The solution graphs and phase line for $x' = ax$ for $a < 0$.

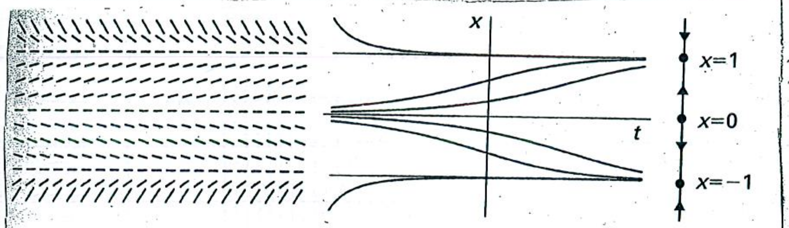


Figure 1.5 The slope field, solution graphs, and phase line for $x' = x - x^3$.

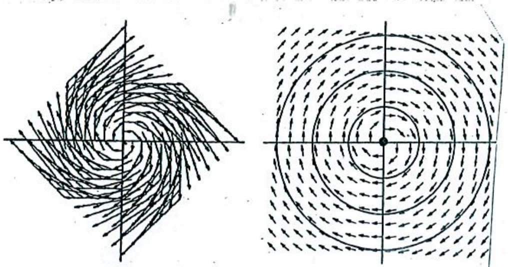


Figure 2.1 The vector field, direction field, and several solutions for the system $x' = y$, $y' = -x$.

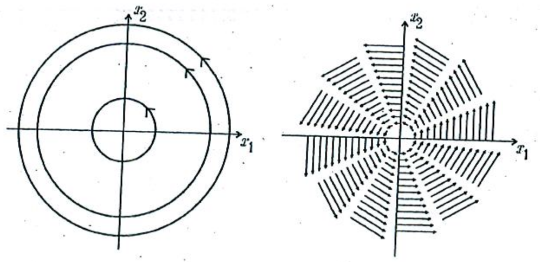


FIG. 1.3 – Portrait de phase (à gauche) et champ de vecteurs (à droite)

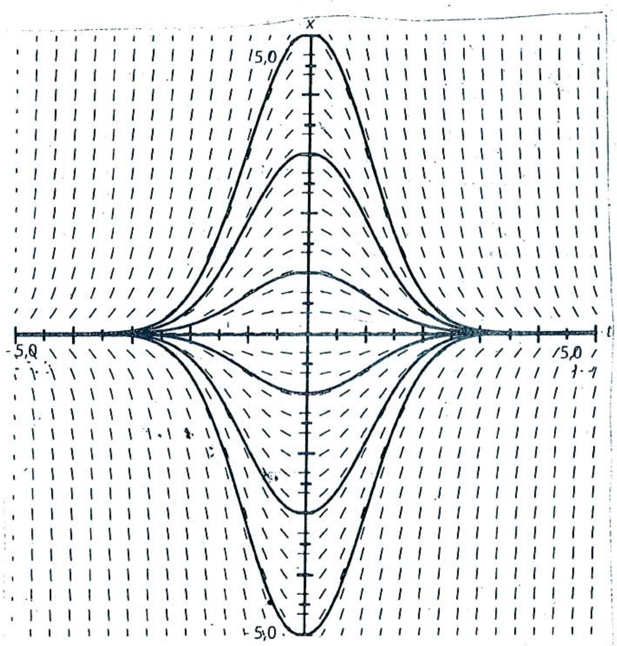
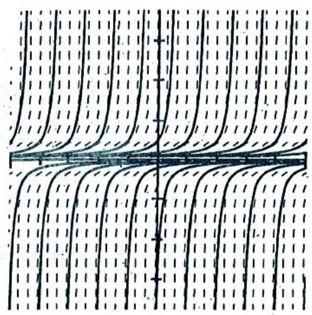
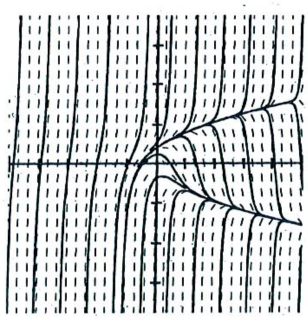


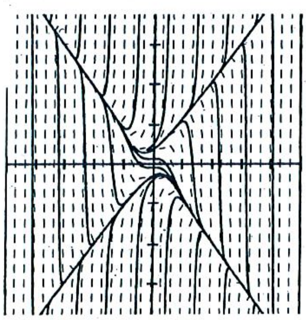
Fig. 1.4. $x' = -tx$.



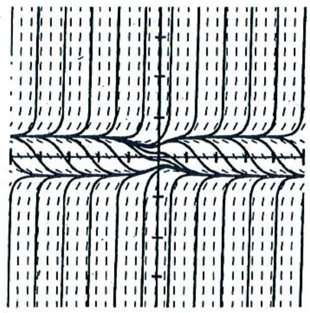
$x' = x^2$



$x' = x^2 - t$



$x' = x^2 - t^2$



$x' = x^2 - \frac{t^2}{1 + t^2}$