Master 280 - Mido : 26 June 2020
Pierre Brugière [1]: Credit Risk 1h30

[12pts]Exercise 1:

1. which of these models are the more likely to be used by a financial analyst
   (a) Altman's model
   (b) Ohlson's model
   (c) Intensity models
2. which of these models is the more likely to be used by a derivatives trader
   (a) Altman's model
   (b) Structural models
   (c) Intensity models
3. In a structural model of default a stochastic process is used to model the evolution of
   (a) The asset side of the balance sheet
   (b) The liability side of the balance sheet
   (c) Both the asset side and the liability side of the balance sheet
4. For a company with EUR 2 billion of assets and EUR 1.6 billion of liabilities and with a volatility for the assets of 20% what is the distance to default at a horizon time of one year.
   (a) 0.8926
   (b) 0.9234
   (c) 1.1157
5. Which equation(s) correspond(s) to a Cox Process for τ
   (a) $P(\tau > t) = E(exp(-\int_0^t \lambda(s, X_s)ds))$
   (b) $P(\tau < t) = E(exp(-\int_0^t \lambda(s, X_s)ds))$
   (c) $P(t < \tau < t + dt | \tau > t, \{X_s, s \in [0, t]\}) = \lambda(t, X_t)dt$
6. What are the possibilities offered by Cox processes

[1] Pierre Brugière Université Paris 9 Dauphine

(a) to model an intensity of default depending on time
(b) to model an intensity of default depending on some macro economic variables
(c) to model a correlation of defaults between bonds

7. In an intensity model, if the intensity of default of a 1 year zero coupon bond is 1% per annum and if the bond has a recovery rate of 20% then the spread of the bond should be:

   (a) 0.2%
   (b) 0.8%
   (c) 1%

8. In a CDO, priced with a constant correlation ρ and probability of default p for the constituents, then when ρ decreases:

   (a) the price of the senior tranche is always increasing
   (b) the price of the equity tranche is always increasing
   (c) the price of the junior tranche is always increasing
   (d) none of the above

9. In a CDO when the correlation decreases

   (a) the "Diversity Score" increases
   (b) the "Diversity Score" decreases
   (c) the "Diversity Score" stays the same

10. In the default model $X_i = 1 \Longleftrightarrow Z_i < \tilde{p}$ what can be said about $\frac{1}{n}\sum_{i=1}^{n} X_i$ when n tends to $+\infty$

   (a) it converges in law towards $\mathcal{L}(\tilde{p})$
   (b) it converges almost surely towards $E(\tilde{p})$
   (c) it converges towards a normal law

11. Which of the following assertions are true. The copula of a Gaussian vector is determined in a unique way by

   (a) the matrix of correlation of its components
   (b) the matrix of variance-covariance of its components
   (c) the means and standard deviations of its components.

12. Let (U_1, U_2) be a random vector, which of the following random vector has the same copula as (U_1, U_2)

   (a) $(exp(U_1), exp(U_2))$

(b) $(2U_1+1, 3U_2-4)$

(c) $(1-U1, U2)$.

[4pts] **Exercise 2 :**

Let $\mathcal{E}(\mu)$ be the exponential law of parameter $\mu > 0$. Such a law is defined by $\forall t \in [0,+\infty[, P(\mathcal{E}(\mu) > t) = exp(-\mu t)$.

Let $(X_s)_{s\in[0,+\infty[}$ be a random process and $(\mathcal{E}_i)_{i\in[\![1,d]\!]}$ be a random variables of exponential law of parameter 1 independent of the process $(X_s)_{s\in[0,+\infty[}$ and independent between them.

Let c be a strictly positive number and λ be a continuous function such that $\forall x \in \mathbf{R}, \forall s \in \mathbf{R}^+, \lambda(s,x) > c$.

Let $(X_i)_{i\in[\![1,d]\!]}$ be the Bernouilli random variables defined by

$$X_i = 1 \Longleftrightarrow \mathcal{E}_i < \int_0^T \lambda(s, X_s)ds$$

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1. [1*pt*] Calculate the expression of $P(X_i = 1)$

   solution:

   $P(X_i=1) = E\Big(1_{\mathcal{E}_i<\int_0^T \lambda(s,X_s)ds}\Big) = E\Big(E\big[1_{\mathcal{E}_1<\int_0^T \lambda(X_s)ds} | \int_0^T \lambda(s,X_s)ds\big]\Big)$
   $= 1 - E\Big(exp(-\int_0^T \lambda(s,X_s)ds)\Big)$

2. [2*pt*] Show that for $i \neq j$ $Cov(X_i,X_j) = Var\Big[exp\Big(-\int_0^T \lambda(s,X_s)ds\Big)\Big]$

   solution:

   $Cov(X_i,X_j) = Cov(X_1,X_2) = Cov(1-X_1, 1-X_2)$
   $E((1-X_1)(1-X_2)) = E\Big[E\Big(1_{\mathcal{E}_1>\int_0^T \lambda(s,X_s)ds}1_{\mathcal{E}_2>\int_0^T \lambda(s,X_s)ds} | \int_0^T \lambda(s,X_s)ds\Big)\Big]$
   $= E\Big[E\Big(1_{\mathcal{E}_1>\int_0^T \lambda(s,X_s)ds} | \int_0^T \lambda(s,X_s)ds\Big)E\Big(1_{\tau_2>\int_0^T \lambda(s,X_s)ds} | \int_0^T \lambda(s,X_s)ds\Big)\Big]$
   $= E\Big[exp\Big(-\int_0^t \lambda(s,X_s)ds\Big)exp\Big(-\int_0^T \lambda(s,X_s)ds\Big)\Big] = E\Big[\big(exp(-\int_0^T \lambda(s,X_s)ds)\big)^2\Big]$
   and $E(1-X_1) = P(X_1=0) = E\Big(exp(-\int_0^T \lambda(s,X_s)ds)\Big)$.
   Therefore, $Cov(X_i,X_j) = Var\Big[exp\Big(-\int_0^T \lambda(s,X_s)ds\Big)\Big]$Q.E.D.

3. [1*pt*] What is sufficient and necessary condition on $\lambda(.,.)$ for the X_i, X_j to have a correlation of zero when $i \neq j$. Is the correlation zero when λ is purely deterministic ?

   solution:

   The correlation is zero if an only if the covariance is zero and this is true if an only if $\int_0^T \lambda(s,X_s)ds$ is deterministic. Yes

[4pts] **Exercise 3 :**

Let $(X_i,Y_i)_{i\in[\![1,n]\!]}$ be a sample of i.i.d random variables of the same law as (X,Y) which has a density.

Let $K_n = \frac{2}{n(n-1)} \sum_{i<j} sgn(X_i - X_j) sgn(Y_i - Y_j)$ where $sgn(x) = 1$ if $x > 0$ and -1 otherwise.

1. $[2pts]$ Show that $E[K_n] = P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0)$

   solution:
   We remark that $sgn(X_i - X_j) sgn(Y_i - Y_j)) = 1_{(X_i - X_j)(Y_i - Y_j) > 0} - 1_{(X_i - X_j)(Y_i - Y_j) < 0}$ and there are $\frac{n(n_1)}{2}$ terms in the sum and all the expectations have the same value so,
   $E[K_n] = P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0)$

2. $[2pts]$ Show that $Var[K_n]$ tends to zero when n tends to ∞.
   solution:
   Let $Z_{(i,j)} = sgn(X_i - X_j) sgn(Y_i - Y_j)$ for $i \neq j$. If $\{i, j\}$ and $\{k, l\}$ have no indice in common then $Z_{(i,j)}$ and $Z_{(k,l)}$ are independent and therefore $cov(Z_{(i,j)}, Z_{(k,l)}) = 0$. In the calculation of $Var(K_n)$ the generic terms is $cov(Z_{(i,j)}, Z_{(k,l)})$ and the number of terms for which $\{i, j\}$ and $\{k, l\}$ have at least an indice in common are of order $O(n^3)$ and therefore will be negligeable when divided by n^4.