Master M1 - Mido 13th January 2020

Exam: Portfolio Management [1]: 1h30

Exercise 1: [8pts]

When there is a risk-free asset of return r_0, we remind the Security Market Line equation for all investment portfolios of returns R_P:

$$R_P - r_0 = \beta_T(P)(R_T - r_0) + \epsilon_P \tag{1}$$

with $\beta_T(P) = \frac{\mathbf{Cov}(R_P, R_T)}{\sigma^2(R_T)}$ and ϵ_P independent from the return R_T of the Tangent Portfolio.

We assume that $\sigma_T = 20\%$.

Complete the table below and indicate the values of m_T and r_0.

Portfolio	$E(R_{P_i})$	$\beta_T(P_i)$	$\sigma(R_{P_i})$	$\sigma(\epsilon_{P_i})$
P_1	15%	0.5	20%	?
P_2	?	2	?	0
P_3	25%	?	20%	?
P_4	35%	1.5	?	10%

Correction:

From P_1 and P_4 we get r_0 and m_T as

$m_1 - r_0 = 0.5(m_T - r_0)$ and $m_4 - r_0 = 1.5(m_T - r_0)$, implies $r_0 = 5\%$ and $m_T = 25\%$.

Applying the SML to P_3 we get

$(25\% - 5\%) = \beta_T(P_3)(25\% - 5\%) \Longrightarrow \beta_T(P_3) = 1$

Applying the decomposition of risk for P_3 we get

$\sigma^2(P_3) = \beta_T^2(P_3)\sigma_T^2 + \sigma^2(\epsilon_{P_3}) \Longrightarrow \sigma^2(\epsilon_{P_3}) = 0$

Applying the SML to P_2 we get

$(m_2 - 5\%) = 2 \times (25\% - 5\%) \Longrightarrow m_2 = 45\%$

Applying the decomposition of risk for P_2 we get

$\sigma(P_2) = |\beta_T(P_2)|\sigma_T = 40\%$

Applying the decomposition of risk for P_4 we get

$\sigma^2(P_4) = 1.5^2 \times 0.2^2 + 0.1^2 \Longrightarrow \sigma^2(P_4) = 0.1 \Longrightarrow \sigma(P_4) = \sqrt{0.1}$

Applying the decomposition of risk for P_1 we get

$0.2^2 = 0.5^2 \times 0.2^2 + \sigma^2(\epsilon_{P_1}) \Longrightarrow \sigma(\epsilon_{P_1}) = 10\sqrt{3}\%$

[1] Pierre Brugière University Paris 9 Dauphine

Portfolio	$E(R_{P_i})$	$\beta_T(P_i)$	$\sigma(R_{P_i})$	$\sigma(\epsilon_{P_i})$
P_1	15%	0.5	20%	$10\sqrt{3}\%$
P_2	45%	2	40%	0
P_3	25%	1	20%	0
P_4	35%	1.5	$\sqrt{0.1}$	10%

Exercise 2: [8pts]
We consider the factor model $R = A + BF + \epsilon$ with

$$A = \begin{pmatrix} 1\% \\ 2\% \\ 3\% \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \quad F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \text{ and } \epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix}.$$

We assume that F and ϵ are independent, that $E[F] = 0$, $E[\epsilon] = 0$ and $\mathbf{Var}(F)$ is invertible.

1. **[1pt]** Show that the model is an APT model and calculate the values of λ_0, λ_1, λ_2. From now on we will note $\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$.

   Correction: the model is an APT model if and only it is possible to find $\lambda_0, \lambda_1, \lambda_2$ such that $E[R] = \lambda_0 1_3 + B \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$. So, the system to solve is $\begin{cases} \lambda_0 + \lambda_1 = 1\% \\ \lambda_0 + \lambda_2 = 2\% \\ \lambda_0 + \lambda_1 + \lambda_2 = 3\% \end{cases}$
   which has a unique solution $\lambda_0 = 0\%, \lambda_1 = 1\%, \lambda_2 = 2\%$. Q.E.D.

2. **[1pt]** In the reduced model $R = A + BF$ find the expressions of the allocations of the investment portfolios which are without risk and calculate the returns of these portfolios.

   Correction: π is the allocation of an investment portfolio which is risk free in the reduced model if an only if π satisfies $\pi' 1_3 = 1$ and $\mathbf{Var}(R(\pi)) = 0$
   now, $\mathbf{Var}(R(\pi)) = \mathbf{Var}(\pi' BF) = \pi' B \mathbf{Var}(F)$
   so, $\mathbf{Var}(R(\pi)) = 0 \Longleftrightarrow \pi' B = 0$
   so, π is a solution if an only if $\begin{cases} \pi_1 + \pi_3 = 0 \\ \pi_2 + \pi_3 = 0 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$

which has a unique solution $\pi = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. The return of this portfolio has to be $\lambda_0 = 1\%$ which is verified easily as in the reduced economy $\mathbf{E}(R(\pi)) = 1 \times 2\% + 1 \times 2\% - 1 \times 3\% = 0\%$.

3. **[1pt]** In the reduced model $R = A + BF$ show that there is no self-financing portfolio which is without risk.

   Correction: π is the allocation of a self-financing portfolio without risk in the reduced model if an only if $\begin{cases} \pi_1 + \pi_3 = 0 \\ \pi_2 + \pi_3 = 0 \\ \pi_1 + \pi_2 + \pi_3 = 0 \end{cases}$ and the only solution is $\pi = 0$ which is a non-invested portfolio.

4. In the factor model, let $R(\pi)$ be the return of the portfolio π and β^* be the solution of

$$\arg\min_{\beta \in \mathbb{R}^2} \mathbf{Var}(R(\pi) - \beta' F)$$

   (a) **[1pt]** express β^* as a function of $\mathbf{Cov}(F, R(\pi))$ and Σ_F
   Correction: $\mathbf{Var}(R(\pi) - \beta' F) = \mathbf{Var}(R(\pi)) - 2\mathbf{Cov}(R(\pi), \beta' F) + \mathbf{Var}(\beta' F) = \mathbf{Var}(R(\pi)) - 2\mathbf{Cov}(R(\pi), F)\beta + \beta' \mathbf{Var}(F)\beta$
   So, $\frac{\partial \mathbf{Var}(R(\pi) - \beta' F)}{\partial \beta} = 2\mathbf{Cov}(R(\pi), F) - 2\beta' \mathbf{Var}(F)$
   which cancels for $\beta^* = \Sigma_F^{-1} \mathbf{Cov}(F, R(\pi))$

   (b) **[1pt]** express β^* as a function of B and π and give the expression of $\mathbf{Var}(R(\pi) - \beta^{*\prime} F)$ as a function of π and $\mathbf{Var}(\epsilon)$
   Correction: $\Sigma_F^{-1} \mathbf{Cov}(F, R(\pi)) = \Sigma_F^{-1} \mathbf{Cov}(F, R)\pi = \Sigma_F^{-1} \mathbf{Cov}(F, F) B' \pi$
   so $\beta^* = B' \pi$ and
   $\mathbf{Var}(R(\pi) - \beta^{*\prime} F) = \mathbf{Var}(\pi' BF + \pi' \epsilon - \beta^{*\prime} F) = \pi' \mathbf{Var}(\epsilon) \pi$

   (c) **[0.5pt]** show that $\mathbf{E}(R(\pi)) = \lambda_0 + \beta^{*,\prime} \lambda$ if π is an investment portfolio
   Correction: $E(R(\pi)) = E(\pi' R) = \pi'(\lambda_0 1_3 + B\lambda) = \lambda_0 \pi' 1_3 + \pi' B \lambda$ and $\pi' B = \beta^{*\prime}$ and $\pi' 1_3 = 1$ for an investment portfolio. Q.E.D

   (d) **[0.5pt]** show that $\mathbf{E}(R(\pi)) = \beta^{*,\prime} \lambda$ if π is a self-financing portfolio
   Correction: consequence of the previous question as this time $\pi' 1_3 = 0$.

5. **[2pt]** find a numerical example of a matrix C and vector A such that

$$R = A + CF + \epsilon$$

is an APT model for which in the reduced model $R = A + CF$ there is no risk-free investment portfolio but some risk-free self-investing portfolios.

Correction: Let $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and $A = \begin{pmatrix} 2\% \\ 4\% \\ 3\% \end{pmatrix}$ then the system
$\begin{cases} \lambda_0 + \lambda_1 = 2\% \\ \lambda_0 + \lambda_2 = 4\% \\ \lambda_0 + \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2 = 3\% \end{cases}$
has some solutions (not unique). For example $\lambda_0 = 0, \lambda_1 = 2\%$ and $\lambda_2 = 4\%$, so the model is an APT model and a portfolio π has no risk in the reduced economy iff $\begin{cases} \pi_1 + \frac{1}{2}\pi_3 = 0 \\ \pi_2 + \frac{1}{2}\pi_3 = 0 \end{cases}$ which implies automatically that $\pi_1 + \pi_2 + \pi_3 = 0$ which means that the portfolio is self-financing. Indeed, here the risk-free self-financing portfolios in the reduced economy are the portfolios of the form $\pi = \begin{pmatrix} \frac{\alpha}{2} \\ \frac{\alpha}{2} \\ -\alpha \end{pmatrix}$ where $\alpha \neq 0$. Q.E.D.

Exercise 3.[6pts]

We consider en economy with d risky assets of vector of returns R and a risk-free asset r_0. We assume that $\mathbf{Var}(R)$ is invertible.

1. **[2pts]** (most diversified portfolio) Show that it is possible to build some portfolios R_D which have the same correlation of returns with all the d risky assets, i.e such that, $\exists c \in \mathbb{R}, \forall i \in [\![1, d]\!],\ \rho(R_i, R_D) = c$.

   Correction: the correlation is constant across all risky assets iff
   $\exists \lambda \in \mathbb{R}, \forall i \in [\![1, d]\!]\ \mathbf{Cov}(R_i, R_D) = \lambda\sigma_i$.
   which is equivalent to $\mathbf{Cov}(R_, R_D) = \lambda\sigma$ where σ is the vector of components the σ_i.
   Now, $\mathbf{Cov}(R, R)\pi_D = \lambda\sigma \iff \pi_D = \lambda\Sigma^{-1}\sigma$ which gives the form of the risky allocation of these portfolios. Q.E.D.

We assume that $R \sim \mathcal{N}(M, \Sigma)$ and remind that the allocation of the tangent portfolio is

$$\frac{1}{b - r_0 a}\Sigma^{-1}(M - r_0 1_d)$$

2. **[2pts]** (Market Portfolio, Tangent Portfolio) show that once Σ is fixed there is a unique value of M for which the Tangent Portfolio coincides with the Market Portfolio.
   Correction: let $Cap = (c_1, c_2, \cdots, c_d)$ be the allocation of the market portfolio (which represents the relative market capitalisations of all the risky assets). The Tangent Portfolio is equal to the Market portfolio iff $\frac{1}{b-r_0 a}\Sigma^{-1}(M - r_0 1_d) = Cap \iff M = r_0 1_d + (b - r_0 a)\Sigma Cap$. Q.E.D.

3. **[2pts]** (most Diversified Portfolio, Tangent Portfolio) show that the Tangent Portfolio has the same correlation with all the d risky assets iff all the d risky assets have the same Sharpe Ratio.
   Correction: here the condition is equivalent to $\frac{1}{b-r_0 a}\Sigma^{-1}(M - r_0 1_d) \propto \Sigma^{-1}\sigma \iff \exists \lambda \in \mathbb{R}$ such that $M = r_0 1_d + \lambda\sigma$ which is equivalent to saying that all the d risky assets have the same Sharpe Ratio Q.E.D.