Master M1 - Mido: 15th January 2018
Exam: Portfolio Management [1]: 2h

Notations: We consider d risky assets S^1, S^2, $...S^d$, whose returns over $[0,T]$ verify $R^i = m^i + \epsilon^i$. We note in vector form:

$$R = M + \epsilon \text{ with } R = \begin{pmatrix} R^1 \\ \vdots \\ R^d \end{pmatrix}, \ M = \begin{pmatrix} m^1 \\ \vdots \\ m^d \end{pmatrix} \text{ and } \epsilon = \begin{pmatrix} \epsilon^1 \\ \vdots \\ \epsilon^d \end{pmatrix}$$

where M is a vector of $\mathbb{R}^d$, ϵ is a random vector of expectation zero and of matrix of variance-covariance Σ invertible.

We note:

$$\pi = \begin{pmatrix} \pi^1 \\ \vdots \\ \pi^d \end{pmatrix} \text{ an allocation, at time 0, in the risky assets } S^i,$$

∘ R_π the return of the portfolio π over $[0,T]$
∘ 1_d the vector of $\mathbb{R}^d$ with all components equal to 1
∘ $a = 1_d'\Sigma^{-1}1_d$ and $b = 1_d'\Sigma^{-1}M$
∘ r^0 the risk free rate of the risk free asset S^0 and we assume that $r^0 \neq \frac{b}{a}$

Exercice 1: [10pts]
Give without any justification the answers to the following questions
[0.25pt] 1) Express $Cov(R_{\pi_1}, R_{\pi_2})$ as a function of π_1, π_2 and Σ
Correction: $\pi_1'\Sigma\pi_2$
[0.25pt] 2)if P is an investment portfolio and π_P its vector of allocation in the risky assets, express its allocation in the risk free asset π_P^0 as a function of π_P and 1_d
Correction: $\pi_P^0 = 1 - \pi_P'1_d$
[0.25pt] 3)if Q is a self financing portfolio and π_Q its vector of allocation in the risky assets, express its allocation in the risk free asset π_Q^0 as a function of π_Q and 1_d
Correction: $\pi_Q^0 = -\pi_Q'1_d$
[0.25pt] 4) what represents the portfolio $\pi_a = \frac{1}{a}\Sigma^{-1}1_d$?
Correction: the portfolio made of the risky assets of minimum variance
[0.25pt] a) express $E(R_{\pi_a})$ as a function on a and b
Correction: $\frac{b}{a}$
[0.25pt] b) express $Var(R_{\pi_a})$ as a function on a and b
Correction: $\frac{1}{a}$

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[0.25pt] 5) If an investment portfolio C is of the form $\pi_= \frac{1}{\alpha}\Sigma^{-1}(M - r^0 1_d)$
a) what is the value of α ?
Correction: $\alpha = b - r^o a$
[0.25pt] a) what is the name of this investment portfolio ?
Correction: the Tangent Portfolio
[1pt] 6) From the Security Market Line theorem which says that for any investment portfolio P: $R_P - r^o = Cov(R_P, R_T)(R_T - r_0) + \epsilon_P$ deduct a similar equation satisfied by a self financing portfolio Q of return R_Q
Correction: $R_Q = Cov(R_Q, R_T)(R_T - r_0) + \epsilon_Q$ with R_T and ϵ_Q Gaussian and independent
[0.5pt] 7) In a factor model $R = A + BF + \mathcal{E}$ with $Cov(F, \mathcal{E}) = 0$ what is the expression of $Var(R)$ as a function of B, $Var(F)$ and $Var(\mathcal{E})$
Correction: $Var(R) = BVar(F)B' + Var(\mathcal{E})$
[0.5pt] 8) if Σ is a matrix of $\mathbb{R}^d \times \mathbb{R}^d$ which is symmetric definite positive with eigenvalues $(\lambda^i)_{i\in[\![1,d]\!]}$ and a corresponding basis of orthonormal eigenvectors $(u_i)_{i\in[\![1,d]\!]}$ express Σ with the λ^i and u_i
Correction: $\Sigma = \sum_{i=1}^{i=d} \lambda^i u_i u_i'$

Exercice 2: [3pts]
1) We consider the following table :

Portefeuille	E(Return)	$\beta_T(P_i)$	$\sigma(R_{P_i})$	$\sigma(\epsilon_{P_i})$
P_1	7%	0.5	5%	0%
P_2	?	1	10%	10%
P_3	16%	2	20%	0%
P_4	6%	?	?	10%

We remind the Security Market Line equation: $R_{P_i} - r_0 = \beta_T(P_i)(R_T - r_0) + \epsilon_{P_i}$

[1.50pt] a) from the table deduct r_0, m_T and σ_T
Correction:
From P_1 and P_3 we get $7\% - r_0 = 0.5(m_T - r^0)$ and $16\% - r_0 = 2(m_T - r^0)$ so, $\frac{16\% - r^0}{7\% - r^0} = 4$ and thus $r^0 = 4\%$ and $m_T = 10\%$.
For portfolio 1: $\sigma(\epsilon_{P_1}) = 0 \Longrightarrow \sigma(R_{P_1}) = \beta_T(P_1)\sigma_T \Longrightarrow \sigma_T = 10\%$
[1.50pt] b) complete the table
$E(R_{P_2}) - 4\% = 1 \times (10\% - 4\%) \Longrightarrow E(R_{P_2}) = 10\%$
$E(R_{P_4}) - 4\% = \beta_T(P_4)(10\% - 4\%) \Longrightarrow \beta_T(P_4) = \frac{1}{3}$
$\sigma(R_{P_4})^2 = \beta_T(P_4)^2\sigma_T^2 + \sigma(\epsilon_{P_4})^2 \Longrightarrow \sigma(R_{P_4})^2 = \frac{1}{9}(10\%)^2 + (10\%)^2$
$\Longrightarrow \sigma(R_{P_4}) = \frac{\sqrt{10}}{3}10\%$

Exercice 2: [3pts]
We consider the factor model $R = \lambda^0 1_3 + BF + \mathcal{E}$ with,

$$R = \begin{pmatrix} R^1 \\ R^2 \\ R^3 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 2 & 0 \end{pmatrix}, F = \begin{pmatrix} f^1 \\ f^2 \end{pmatrix} \text{ and } \mathcal{E} = \begin{pmatrix} \epsilon^1 \\ \epsilon^2 \\ \epsilon^3 \end{pmatrix}$$

with the assumptions: $\mathbf{Var}(F)$ invertible, $\mathbf{E}(\mathcal{E}) = 0$ and $\mathbf{Cov}(F, \mathcal{E}) = 0$

1)

[1pt] a) exhibit a self financing portfolio π_S such that $\mathbf{Cov}(R_{\pi_S}, F) = 0$ and calculate for this self financing portfolio $E(R_{\pi_S})$

Correction:

$\pi_S = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ works and $E(R_{\pi_S}) = 0$ (it will be the case for all self financing portfolio in this model)

[1pt] b) Is it possible to find in this model an investment portfolio π such that $\mathbf{Cov}(R_\pi, F) = 0$?

Correction:

$\mathbf{Cov}(R_\pi, F) = \pi' BVar(F)$ so $\mathbf{Cov}(R_\pi, F) = 0 \Longleftrightarrow \pi' B = 0$ but

$\pi' B = 0 \Longrightarrow \pi' \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 0$ and $\pi' \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 0 \Longrightarrow \pi' 1_3 = 0$ so it is not possible to find and investment portfolio such that $\mathbf{Cov}(R_\pi, F) = 0$

[1pt] c) we assume that $\mathbf{Var}(F) = 0.01 \times Id_{\mathbb{R}^2}$ and that $\mathbf{Var}(\mathcal{E}) = 0.01 \times Id_{\mathbb{R}^3}$ calculate the correlation between R^1 and R^2

Correction:

$Var(R^1) = 3 \times 0.01$, $Var(R^2) = 3 \times 0.01$

$Cov(R^1, R^2) = Cov(f^1 + f^2 + \epsilon^1, 2f^2 + \epsilon^2) = 2 \times 0.01$ so $\rho = \frac{2}{3}$

Problem: [10pts]

We assume that $E(R)$ is not colinear to 1_d and for $\lambda \in \mathbb{R}$ we call (P_λ) the problem to determine (B, f) solutions of

$$(P_\lambda) \begin{cases} E(R - \lambda 1_d - B(f - \lambda)) = 0 \\ Cov(R - \lambda 1_d - B(f - \lambda), f) = 0 \end{cases}$$

where B is a vector of $\mathbb{R}^d$ and f a random variable not constant taking values in $\mathbb{R}$

[2pt]1) Show that: (B, f) solution of $(P_\lambda) \Longrightarrow M - \lambda 1_d = \frac{Cov(R,f)}{Var(f)}(E(f) - \lambda)$

Correction:

$Cov(R - \lambda 1_d - B(f - \lambda), f) = 0 \Longleftrightarrow Cov(R - Bf, f) = 0$

$\Longleftrightarrow Cov(R, f) - BCov(f, f) = 0 \Longleftrightarrow B = \frac{Cov(R,f)}{Var(f)}$

$E(R - \lambda 1_d - B(f - \lambda)) = 0 \Longleftrightarrow E(R) - \lambda 1_d - B(E(f) - \lambda) = 0$

$\Longleftrightarrow B = \frac{M - \lambda 1_d}{E(f) - \lambda}$ (necessarily we must search f such that $E(f) - \lambda \neq 0$ as $E(R) - \lambda 1_d \neq 0$)

So, if (B, f) is a solution, necessarily $E(f) \neq \lambda$ and $\frac{Cov(R,f)}{Var(f)} = \frac{M - \lambda 1_d}{E(f) - \lambda}$ which implies $M - \lambda 1_d = \frac{Cov(R,f)}{Var(f)}(E(f) - \lambda)$ Q.E.D

The result of question 1) can be admitted for the rest of the problem.

[0.5pt]2) Show that: (B, λ) solution of $(P_\lambda) \Longrightarrow E(f) \neq \lambda$

Correction:

by absurd: $E(f) = \lambda$ and (B, f) solution $\Longrightarrow M - \lambda 1_d = 0$ which is in contradiction with the hypothesis. Q.E.D

For $x \in \mathbb{R}^d$ we note $f_x = x'R$ (if π is a portfolio allocation we have then $f_\pi = \pi' R = R_\pi$ and we use the notations indifferently)

[1pt] 3) Show that:
$M - \lambda 1_d = \frac{Cov(R,f_x)}{Var(f)}(E(f_x) - \lambda) \Longrightarrow x$ and $\Sigma_R^{-1}(M - \lambda 1_d)$ are colinear
Correction:
$M - \lambda 1_d = \frac{Cov(R,f_x)}{Var(f)}(E(f_x) - \lambda) \Longleftrightarrow M - \lambda 1_d = \frac{Cov(R,x'R)}{Var(f)}(E(f_x) - \lambda)$
$\Longleftrightarrow M - \lambda 1_d = \frac{\Sigma x}{Var(f)}(E(f_x) - \lambda) \Longleftrightarrow x = \Sigma^{-1}(M - \lambda 1_d)\frac{Var(f)}{E(f_x) - \lambda}$ Q.E.D

From now on until the end we assume that $\lambda \neq \frac{b}{a}$

4)
[0.5pt] a) show that $\exists \pi_\lambda$ investment portfolio colinear to $\Sigma_R^{-1}(M - \lambda 1_d)$
Correction:
we search π_λ as $\alpha \Sigma^{-1}(M - \lambda 1_d)$
$1_d' \alpha \Sigma^{-1}(M - \lambda 1_d) = 1 \Longleftrightarrow \alpha(B - \lambda a) = 1$ so $\alpha = \frac{1}{B - \lambda a}$ and $\pi_\lambda = \frac{\Sigma^{-1}(M - \lambda 1_d)}{B - \lambda a}$

[1pt] b) express $E(\pi_\lambda' R) - \lambda$ as a function of Σ_R^{-1}, M, λ, b and a.
Correction:
$E(\pi_\lambda' R) - \lambda = \pi_\lambda' M - \lambda \pi_\lambda' 1_d = \pi_\lambda'(M - \lambda 1_d) = \frac{(M - \lambda 1_d)' \Sigma_R^{-1}(M - \lambda 1_d)}{b - \lambda a} = \frac{\|M - \lambda 1_d\|^2_{\Sigma_R^{-1}}}{b - \lambda a}$

[0.5pt] c) express $Var(\pi_\lambda' R)$ as a function of Σ_R^{-1}, M, λ, b and a.
Correction:
$Var(\pi_\lambda' R) = \frac{1}{(b - \lambda a)^2}(M - \lambda 1_d)' \Sigma_R^{-1} \Sigma_R \Sigma_R^{-1}(M - \lambda 1_d) = \frac{\|M - \lambda 1_d\|^2_{\Sigma_R^{-1}}}{(b - \lambda a)^2}$

[1pt] d) show that $M - \lambda 1_d = \frac{Cov(R, f_{\pi_\lambda})}{\sigma^2_{f_{\pi_\lambda}}}(E(f_{\pi_\lambda}) - \lambda)$

Correction:
$Cov(R, f_{\pi_\lambda}) = Cov(R, \pi_\lambda' R) = Cov(R, R)\pi_\lambda = \Sigma_R \Sigma_R^{-1} \frac{M - \lambda 1_d}{b - \lambda a} = \frac{M - \lambda 1_d}{b - \lambda a}$ so,
$\frac{Cov(R, f_{\pi_\lambda})}{\sigma^2_{f_{\pi_\lambda}}}(E(f_{\pi_\lambda}) - \lambda) = \frac{M - \lambda 1_d}{b - \lambda a} \frac{(b - \lambda a)^2}{\|M - \lambda 1_d\|^2_{\Sigma_R^{-1}}} \frac{\|M - \lambda 1_d\|^2_{\Sigma_R^{-1}}}{b - \lambda a} = M - \lambda 1_d$ Q.E.D

[0.5pt] e) calculate B_λ such that $(B_\lambda, f_{\pi_\lambda})$ is a solution of (P_λ)
Correction:
$\pi_\lambda = \Sigma_R^{-1} \frac{M - \lambda 1_d}{b - \lambda a}$ and $B_\lambda = \frac{Cov(R, f_{\pi_\lambda})}{Var(f_{\pi_\lambda})} \Longrightarrow B_\lambda = \frac{M - \lambda 1_d}{b - \lambda a} \frac{(b - \lambda a)^2}{\|M - \lambda 1_d\|^2_{\Sigma_R^{-1}}}$
$= (b - \lambda a) \frac{M - \lambda 1_d}{\|M - \lambda 1_d\|^2_{\Sigma_R^{-1}}}$ Q.E.D

[1pt] f) deduct from a) that for (B, f) solutions of (P) we can write:
$R = \lambda 1_d + B(f - \lambda) + \mathcal{E}$ with
$E(\mathcal{E}) = 0$, $Cov(f, \mathcal{E}) = 0$ and $Trace(Var(\mathcal{E})) < Trace(Var(R))$
Correction:
$\mathcal{E} = R - \lambda 1_d - B(f - \lambda)$ so by definition of (P_λ) id (B, f) is a solution $E(\mathcal{E}) = 0$

and $Cov(f, \mathcal{E}) = 0$
in such a situation $Var(R) = Var(\lambda_0 1_d + B(f - \lambda_0) + \mathcal{E}) = Var(Bf + \mathcal{E}) = Var(Bf) + Var(\mathcal{E}) = BVar(f)B' + Var(\mathcal{E})$
but by hypothesis $Var(f) > 0$ and $B \neq 0$ so $Trace(B'Var(f)B') > 0$ and so $Trace(Var(\mathcal{E})) < Trace(Var(R))$ Q.E.D

[1pt] g) show (using what has been demonstrated so far) that for any investment portfolio of return R_Q we have
$E(R_Q) - \lambda = \frac{Cov(R_Q, f_{\pi_\lambda})}{Var(f_{\pi_\lambda})}(E(f_{\pi_\lambda}) - \lambda)$

Correction:
if we call π_Q the allocation of the portfolio Q we can multiply by π'_Q each term of the equation $M - \lambda 1_d = \frac{Cov(R, f_{\pi_\lambda})}{\sigma^2_{f_{\pi_\lambda}}}(E(f_{\pi_\lambda}) - \lambda)$ and using the fact that $\pi'_Q 1_d = 1$, $\pi'_Q R = R_Q$ and $\pi'_Q M = E(R_Q)$ we get
$\pi'_Q M - \lambda \pi'_Q 1_d = \pi'_Q \frac{Cov(R, f_{\pi_\lambda})}{\sigma^2_{f_{\pi_\lambda}}}(E(f_{\pi_\lambda}) - \lambda)$
$\iff E(R_Q) - \lambda = \frac{Cov(\pi'_Q R, f_{\pi_\lambda})}{\sigma^2_{f_{\pi_\lambda}}}(E(f_{\pi_\lambda}) - \lambda)$
$\iff E(R_Q) - \lambda = \frac{Cov(R_Q, f_{\pi_\lambda})}{\sigma^2_{f_{\pi_\lambda}}}(E(f_{\pi_\lambda}) - \lambda)$

[0.5pt] h) show (using what has been demonstrated so far) that for any self financing portfolio of return R_Q we have
$E(R_Q) = \frac{Cov(R_Q, f_{\pi_\lambda})}{Var(f_{\pi_\lambda})}(E(f_{\pi_\lambda}) - \lambda)$

Correction:
if we call π_Q the allocation of the portfolio Q the calculation is the same as previously but this time $\pi'_Q 1_d = 0$ which leads to the new equation

[0.5pt] i) tell (without demonstration) what is the geometric interpretation of the portfolio π_λ?

Correction:
it is the tangent portfolio obtained by taking $r^0 = \lambda$