Master M1 - Mido: 15th January 2019

Exam: Portfolio Management [1]: 2h

Notations: We consider d risky assets $S_1,\ S_2,\ ...S_d$, whose returns over $[0,T]$ verify $R_i = m_i + \epsilon_i$. We note in vector form:

$$R = M + \epsilon \text{ with } R = \begin{pmatrix} R_1 \\ \vdots \\ R_d \end{pmatrix},\ M = \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix} \text{ and } \epsilon = \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_d \end{pmatrix}$$

where M is a vector of $\mathbb{R}^d$, ϵ is a Gaussian vector of expectation zero and of matrix of variance-covariance Σ invertible (and therefore definite positive). We also assume that there is a risk-free asset S_0 of return r_0.

For a portfolio, we note $\pi = \begin{pmatrix} \pi_1 \\ \vdots \\ \pi_d \end{pmatrix}$ its allocation in the risky assets S_i and $\Pi = \begin{pmatrix} \pi_0 \\ \pi \end{pmatrix}$ its allocation in both the risk-free and the risky assets.

We note $R(\Pi)$ the return of this portfolio over $[0,T]$. $\mathbf{E}\big(R(\Pi)\big)$ its expectation and $\sigma\big(R(\Pi)\big)$ its standard deviation. When there is no allocation in the risk-free asset ($\pi_0 = 0$) we simply call π the portfolio and note $R(\pi)$ instead of $R(\Pi)$.

We also note:
1_d the vector of $\mathbb{R}^d$ with all components equal to 1,
$a = 1_d'\Sigma^{-1}1_d$ and $b = 1_d'\Sigma^{-1}M$ and we assume that $r_0 \neq \frac{b}{a}$.

We remind the Security Market Line equation for an investment portfolio Π_P of return R_P:

$$R_P - r_0 = \beta_T(P)(R_T - r_0) + \epsilon_P$$

with $\mathbf{E}(\epsilon_P) = 0$ and R_T and ϵ_P independent.

We note β_i the beta of the risky asset S_i with respect to the tangent portfolio i.e $\beta_i = \frac{\mathbf{Cov}(R_i,R_T)}{\mathbf{Var}(R_T)}$ and β the vector of components the β_i, i.e $\beta = (\beta_1, \cdots, \beta_d)'$.

Exercise : [6pts]

In this exercise we consider only portfolio made of the d risky assets S_i. Therefore, we represent a portfolio only by the vector π, as π_0 is always zero here. Answer to the following questions **without giving any demonstration**.

1. **[0.5pt]** Express $\mathbf{Cov}(AX, BY)$ as a function of $\mathbf{Cov}(X,Y)$, A and B.
   Correction: $A\mathbf{Cov}(X,Y)B'$

2. Let $\pi = \Sigma^{-1}(M - \frac{b}{a}1_d)$.

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(a) **[0.5pt]** Does π represent an investment or a self-financing portfolio and why ?
Correction: $\pi' 1_d = 0$, so self-financing

(b) **[0.5pt]** Express $\mathbf{E}\big(R(\pi)\big)$ as the norm of a vector.
Correction: $\|M - \frac{b}{a} 1_d\|^2_{\Sigma^{-1}}$

(c) **[0.5pt]** Express $\mathbf{Var}\big(R(\pi)\big)$ as the norm of a vector.
Correction: $\|M - \frac{b}{a} 1_d\|^2_{\Sigma^{-1}}$

3. We consider two investment portfolios (with no allocation to the risk-free asset) π_A and π_B with $\mathbf{E}(R(\pi_A)) \neq \mathbf{E}(R(\pi_B))$. Let,

$$\mathcal{F} = \left\{ \begin{pmatrix} \sigma\big(R(\pi_\lambda)\big) \\ \mathbf{E}\big(R(\pi_\lambda)\big) \end{pmatrix}, \pi_\lambda = \lambda \pi_A + (1-\lambda)\pi_B, \lambda \in \mathbb{R} \right\}.$$

Let $\rho_{A,B} = \mathbf{Correl}(R(\pi_A), R(\pi_A))$. What is the geometric nature of $\mathcal{F}$ if:

(a) **[0.25pt]** $\rho_{A,B} = -1$,
Correction: A cone.

(b) **[0.25pt]** $-1 < \rho_{A,B} < 1$
Correction: An hyperbole.

(c) **[0.25pt]** $\rho_{A,B} = 1$.
Correction: A cone.

(d) **[0.25pt]** What would be the geometric nature of $\mathcal{F}$ if $\mathbf{E}(R(\pi_A)) = \mathbf{E}(R(\pi_B))$?
Correction: A line (or potentially a single point).

4. Let π_P be an investment portfolio and π_T be the tangent portfolio and $\rho = \mathbf{Correl}(R_P, R_T)$.

(a) **[0.5pt]** Express the systematic risk of R_P as a function of $\rho, \sigma(R_P)$ and $\sigma(R_T)$.
Correction: $|\rho|\sigma(R_P)$

(b) **[0.5pt]** Express the specific risk of R_P as a function of $\rho, \sigma(R_P)$ and $\sigma(R_T)$.
Correction: $\sqrt{1-\rho^2}\sigma(R_P)$

5. Let Σ be the matrix of variance-covariance for the risky assets $S_1, S_2, \cdots, S_d$. Let π_A be the portfolio of minimum variance.

(a) **[0.5pt]** Express π_A as a function of d if $\Sigma = Id_{R^d}$.
Correction: $\pi_A = \frac{1}{d} 1_d$

(b) **[0.5pt]** Express π_A as a function of d and the σ_i if $\Sigma = diag(\sigma_i)$ with $\sigma_i > 0$.
Correction: $\pi_A = \frac{1}{\sum\limits_{i=1}^{d} \frac{1}{\sigma_i}} (\frac{1}{\sigma_1}, \cdots, \frac{1}{\sigma_d})$

6. Let Σ be a matrix of variance-covariance definite positive of $\mathbb{R}^{d\times d}$ and $(u_i)_{i\in[\![1,d]\!]}$ be an orthonormal basis of $\mathbb{R}^d$ formed by eigenvectors of Σ of eigenvalues λ_i.

   (a) **[0.25pt]** Express Σ as a function of the (u_i) and λ_i.

   Correction: $\Sigma = \sum_{i=1}^{i=d} \lambda_i u_i u_i'$

   (b) **[0.25pt]** Express Σ^{-1} as a function of the (u_i) and λ_i.

   Correction: $\Sigma = \sum_{i=1}^{i=d} \frac{1}{\lambda_i} u_i u_i'$.

7. **[0.5pt]** If X is a random variable of $\mathbb{R}^d$, express $\mathbf{E}(\|X\|^2)$ as a function of $Trace\big(\mathbf{Var}(X)\big)$ and $\mathbf{E}(X)$.
   Correction: $\mathbf{E}(\|X\|^2) = \|\mathbf{E}(X)\|^2 + Trace\big(\mathbf{Var}(X)\big)$

Exercise 2: [4pts]
We consider for the investment portfolios $P_1, P_2, \cdots, P_5$ the table below:

Portfolio	E(Return)	$\beta_T(P_i)$	$\sigma(R_{P_i})$	$\sigma(\epsilon_{P_i})$
P_1	12%	2	?%	60%
P_2	4%	?	?%	10%
P_3	?	1	?	0%
P_4	8%	1	50%	30%
P_5	10%	?	?	0%

1. **[3.5pt]** Calculate all the missing values of the table.
   Correction:
   ◦ from $P1$ and $P4$ we get $r_0 = 4\%$ and $m_T = 8\%$
   ◦ for P_2, $m_2 = r_0 \Rightarrow \beta_T(P_2) = 0$ and therefore $\sigma(R_{P_2}) = \sigma(\epsilon_{P_2}) = 10\%$
   ◦ for P_3, $\beta_T(P_3) = 1 \Rightarrow m_3 = 8\%$
   ◦ for P_4 the SML gives $0.5^2 = \sigma_T^2 + 0.3^2$ therefore $\sigma_T = 40\%$
   ◦ for P_3, $\big(\sigma_T = 40\%, \beta_T(P_3) = 1, \sigma(\epsilon_{P_3}) = 0\big) \Rightarrow \sigma(R_{P_3}) = 40\%$
   ◦ for P_5, $m_5 = 10\% \Rightarrow \beta_T(p_5) = 1.5$ and therefore $\sigma(R_{P_i}) = 60\%$.
   ◦ for P_1 $\sigma(R_{P_1}) = 4 \times 0.4^2 + 0.6^2 = 1$ so $\sigma(R_{P_1}) = 100\%$.

Portfolio	E(Return)	$\beta_T(P_i)$	$\sigma(R_{P_i})$	$\sigma(\epsilon_{P_i})$
P_1	12%	2	100%	60%
P_2	4%	0	10%	10%
P_3	8%	1	40%	0%
P_4	8%	1	50%	30%
P_5	10%	1.5	60%	0%

2. **[0.5pt]** What is r_0 and m_T ?
   Correction: as seen above $r_0 = 4\%$ and $m_T = 8\%$

Exercise 3 : [6pts]
Let $(u_i)_{i\in[\![1,d]\!]}$ be an orthonormal basis of eigenvectors of Σ associated with the eigenvalues λ_i.

1. **[0.5pt]** Prove that $\forall i$, $\lambda_i > 0$.
   Correction: Let u_i be an eigenvector for the eigenvalue λ_i.
   $\Sigma u_i = \lambda_i u_i \Rightarrow u_i'\Sigma u_i = u_i'\lambda_i u_i$. As Σ is definite positive $u_i'\Sigma u_i > 0$ and therefore $\lambda_i > 0$.

   Let,
   $f_i = \langle R, \frac{u_i}{\sqrt{\lambda_i}}\rangle = \frac{1}{\sqrt{\lambda_i}} u_i' R$
   $F = (f_1, \cdots, f_d)'$ be the vector of components the f_i,
   $F[k] = (f_1, \cdots, f_k)'$,
   $R[k] = \sum\limits_{i=1}^{k} f_i \sqrt{\lambda_i} u_i$ and
   $\epsilon[k] = R - R[k]$.

2. **[0.5pt]** Prove that $R = R[d]$.

   Correction: The u_i form an orthonormal basis of $\mathbb{R}^d$ so $R = \sum\limits_{i=1}^{d} \langle R, u_i\rangle u_i$

   $= \sum\limits_{i=1}^{d} \langle R, \frac{u_i}{\sqrt{\lambda_i}}\rangle \sqrt{\lambda_i} u_i = R[d]$. Q.E.D.

3. **[1pt]** Prove that $R[k]$ is Gaussian and independent from $\epsilon[k]$.
   Correction: $(R, R-R[k])$ is a linear transformation of the Gaussian vector R and therefore is Gaussian. So, if the covariance is zero the two components are independent. $\mathbf{Cov}(R[k], R-R[k]) = \mathbf{Cov}(\sum\limits_{i=1}^{k} u_i' R u_i, \sum\limits_{j=k+1}^{d} u_j' R u_j)$

   $= \sum\limits_{i=1}^{k} \sum\limits_{j=k+1}^{d} \mathbf{Cov}(u_i u_i' R, u_j u_j' R) = \sum\limits_{i=1}^{k} \sum\limits_{j=k+1}^{d} u_i u_i' \mathbf{Cov}(R, R) u_j u_j'$
   but $\forall i \neq j$, $u_i' \mathbf{Cov}(R, R) u_j = 0$ so, $\mathbf{Cov}(R[k], R - R[k]) = 0$ Q.E.D.

4. **[0.5pt]** Prove that F is Gaussian and that $\mathbf{Var}(F) = Id_{\mathbb{R}^d}$.
   Correction: F is obtained from R by a linear transformation so is Gaussian. $\mathbf{Cov}(f_i, f_j) = \frac{1}{\sqrt{\lambda_i}\sqrt{\lambda_j}} u_i' \Sigma u_j$ so if $i = j$ this is $\frac{1}{\sqrt{\lambda_i}\sqrt{\lambda_i}} \lambda_i = 1$ and if $i \neq j$ it is zero. Q.E.D.

5. **[1pt]** Calculate $Trace(\mathbf{Var}(R[k]))$ as a function of the λ_i.

   Correction: $\mathbf{Var}(R[k]) = \mathbf{Var}(\sum\limits_{i=1}^{k} f_i \sqrt{\lambda_i} u_i)$ as the f_i are independent

   $\mathbf{Var}(R[k]) = \sum\limits_{i=1}^{k} \lambda_i \mathbf{Var}(f_i u_i)$ and $\mathbf{Var}(f_i u_i) = u_i \mathbf{Var}(f_i) u_i' = u_i u_i'$

   Now, $Trace(u_i u_i') = Trace(u_i' u_i) = 1$ so finally $\mathbf{Var}(R[k]) = \sum\limits_{i=1}^{k} \lambda_i$

6. **[0.5pt]** Rewrite $R = \sum_{i=1}^{k} f_i u_i \sqrt{\lambda_i} + \epsilon[k]$ matricially under the form:
$R = BF[k] + \epsilon[k]$ where B is a $\mathbb{R}^{n \times k}$ matrix and find the expression of the i-th vector column b_i of B as a function of the u_i and λ_i.
Correction: $B = [b_1 | \cdots | b_k]$ with $b_i = u_i \sqrt{\lambda_i}$

7. **[0.5pt]** Show that $\mathbf{Cov}(F[k], \epsilon[k]) = 0$.
Correction: for $i \in [\![1, k]\!]$ and $j \in [\![k+1, d]\!]$, $\mathbf{Cov}(f_i, f_j) = 0$
so, $\mathbf{Cov}(f_i, \epsilon[k]) = \mathbf{Cov}(f_i, \sum_{j=k+1}^{d} f_j u_j \sqrt{\lambda_j}) = 0$ which proves the result.

8. **[1pt]** Show that $(R = BF[k] + \epsilon[k]$ and $R = CF[k] + \epsilon[k]) \Longrightarrow C = B$.
Correction: $\mathbf{Cov}(R, F[k]) = \mathbf{Cov}(BF[k], F[k]) + \mathbf{Cov}(\epsilon[k], F[k])$
$= B\mathbf{Cov}(F[k], F[k]) = B$ so, B is determined in a unique way as $\mathbf{Cov}(R, F[k])$.
Q.E.D.

9. **[0.5pt]** From what precedes find a factor decomposition of R in the format $R = A + HG[k] + \eta[k]$ where: $Var(G[k])$ is an invertible $\mathbb{R}^{k \times k}$ matrix, $\mathbf{Cov}(G[k], \eta[k]) = 0$ and $E(\eta[k]) = 0$.
Correction: $\eta[k] = \epsilon[k] - E(\epsilon[k])$, $H = B$, $G[k] = F[k]$ and
$A = E(\epsilon[k]) = E[R] - BE[F]$ satisfy the conditions.

Exercise 4: [4pts]

Let Q be a self-financing portfolio, with a vector of allocation π_Q to the risky assets. We note $\beta_T(Q) = \frac{\mathbf{Cov}(R_Q, R_T)}{\mathbf{Var}(R_T)}$ and $\beta_i = \frac{\mathbf{Cov}(R_i, R_T)}{\mathbf{Var}(R_T)}$ for the risky assets, S_1, S_2, $...S_d$. We note β the vector of components the β_i, i.e $\beta = (\beta_1, \cdots, \beta_d)'$.

1. (a) **[1.5pt]** Express $\beta_T(Q)$ as a function of π_Q and β.
Correction: $\mathbf{Cov}(R_Q, R_T)\frac{1}{\mathbf{Var}(R_T)} = \mathbf{Cov}(\pi'_Q R, R_T)\frac{1}{\mathbf{Var}(R_T)}$
$= \pi'_Q \mathbf{Cov}(R, R_T)\frac{1}{\mathbf{Var}(R_T)} = \pi'_Q \beta$.

   (b) **[0.5pt]** Express $R(Q)$ as a function of π_Q, R, and r_0.
Correction: $R(Q) = \pi'_Q R + \pi^0_Q r_0 = \pi'_Q R - \pi'_Q 1_d r_0 = \pi'_Q (R - r_0 1_d)$.

2. **[2pt]** Is the SML equation satisfied for R_Q or how does it have to be modified? (do not forget to check the properties of independence and zero expectation in the decomposition obtained).
Correction: if R is the vector of returns of the risky assets and ϵ the vector of the ϵ_i, then according to the SML: $R - r_0 1_d = (R_T - r_0)\beta + \epsilon$.
From this we get: $\pi'_Q R - r_0 \pi'_Q 1_d = (R_T - r_0)\pi'_Q \beta + \pi'_Q \epsilon$
so, $R_Q = \beta_T(Q)(R_T - r_0) + \epsilon_Q$ with $\epsilon_Q = \pi'_Q \epsilon$ and still R_T and ϵ_Q are independent and $E(\epsilon_Q) = 0$ as $E(\pi'_Q \epsilon) = \pi'_Q E(\epsilon)$.