Master M1 - Mido 16th January 2020
Correction Exam: Portfolio Management[1]: 2h

Notations: We consider d risky assets S^1, $S^2, \cdots, S^d$ of vector of returns R which verifies :

$$R = M + \epsilon \text{ with } R = \begin{pmatrix} R^1 \\ \vdots \\ R^d \end{pmatrix}, \ M = \begin{pmatrix} m^1 \\ \vdots \\ m^d \end{pmatrix} \text{and } \epsilon = \begin{pmatrix} \epsilon^1 \\ \vdots \\ \epsilon^d \end{pmatrix}$$

where M is a vector of $\mathbf{R}^d$, and ϵ is a Gaussian vector of expectation zero and variance-covariance matrix Σ.

We note,

$$\pi = \begin{pmatrix} \pi^1 \\ \vdots \\ \pi^d \end{pmatrix} \text{ an allocation in the risky assets } S^i,$$

$R(\pi)$ the return of a portfolio of (risky) allocation π,
m_π its expectation and σ_π its standard deviation.
We remind that,
An investment portfolio π satisfies $\pi' 1_d = 1$.
A self-financing portfolio π satisfies, $\pi' 1_d = 0$.

We note,
1_d the vector of $\mathbf{R}^d$ with all components equal to 1.
$a = 1_d' \Sigma^{-1} 1_d$, $b = 1_d' \Sigma^{-1} M$, $\pi_a = \frac{1}{a} \Sigma^{-1} 1_d$ and $\omega = \Sigma^{-1}(M - \frac{b}{a} 1_d)$.

Exercise 1: [8pts]
When there is a risk-free asset of return r_0, we remind the Security Market Line equation for all investment portfolios of returns R_P:

$$R_P - r_0 = \beta_T(P)(R_T - r_0) + \epsilon_P \qquad (1)$$

with $\beta_T(P) = \frac{\mathbf{Cov}(R_P, R_T)}{\sigma^2(R_T)}$ and ϵ_P independent from the return R_T of the Tangent Portfolio.

1. **[6pts]** Complete the table below:

Portfolio	$E(R_{P_i})$	$\beta_T(P_i)$	$\sigma(R_{P_i})$	$\sigma(\epsilon_{P_i})$
P_1	10%	1	50%	30%
P_2	?	?	0	?
P_3	12.5%	1.5	?	0%
P_4	?	3	?	40%

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Correction:
From P_1 and P_3 we get the equation of the SML and $r_0 = 5\%$ and $m_T = 10\%$. For P_1 we have $\sigma^2(R_{P_1}) = \beta_T^2(P_1)\sigma^2(R_{P_T}) + \sigma^2(\epsilon_{P_1})$
so, $(50\%)^2 = 1 \times \sigma^2(R_{P_T}) + (30\%)^2$ and therefore $\sigma(R_{P_T}) = 40\%$.
P_2 is the risk-free asset so $\beta_T(P_2) = 0$ and $\sigma(\epsilon_{P_2}) = 0$ and as obtained previously $E(R_{P_2}) = r_0 = 5\%$.
For P_3 we get $\sigma(R_{P_3}) = \beta_T(P_3)\sigma(R_{P_T}) = 60\%$.
For P_4, according to the SML $E(R_{P_4}) = 20\%$ and
$\sigma^2(R_{P_4}) = 9 \times (40\%)^2 + (40\%)^2 \Longrightarrow \sigma(R_{P_4}) = \sqrt{10} \times 40\%$.

Portfolio	$E(R_{P_i})$	$\beta_T(P_i)$	$\sigma(R_{P_i})$	$\sigma(\epsilon_{P_i})$
P_1	10%	1	50%	30%
P_2	5%	0	0	0
P_3	12.5%	1.5	60%	0%
P_4	20%	3	$\sqrt{10} \times 40\%$	40%

2. **[2pts]** Demonstrate how the SML equation (1), can be adapted to self-financing portfolios, made of risky assets only, using the fact that the SML equation can also be expressed for the d risky assets S^i as,

$$R - r_0 1_d = \beta_T(R_T - r_0) + \mathcal{E} \tag{2}$$

where β_T is the vector of the $\beta_T(R_i)$ and $\mathcal{E}$ is independent from R_T.

Correction:
Let π_P, be a self-financing portfolio.
We have, $\pi'_P(R - r_0 1_d) = \pi'_P\beta_T(R_T - r_0) + \pi'_P\mathcal{E}$ and
$\pi'_P R = R_P, \pi'_P 1_d = 0$ and if we note $\epsilon_P = \pi'_P\mathcal{E}$ then ϵ_P is independent from R_T.
Now, $\beta_T = \frac{\mathbf{Cov}(R,R_T)}{\sigma^2(R_T)}$ so, $\pi'_P\beta_T = \pi'_P\frac{\mathbf{Cov}(R,R_T)}{\sigma^2(R_T)} = \frac{\mathbf{Cov}(\pi'_P R,R_T)}{\sigma^2(R_T)} = \frac{\mathbf{Cov}(R_P,R_T)}{\sigma^2(R_T)} = \beta_T(R_P)$. Therefore, for a self-financing portfolio we can write,

$$R_P = \beta_T(P)(R_T - r_0) + \epsilon_P \tag{3}$$

with ϵ_P independent from R_T Q.E.D.

Exercise 2: [6pts]
In this exercise we assume that Σ is invertible and that M and 1_d are not collinear, and we solve the problem $(P_{m,\sigma})$ of finding all the investment portfolios π (made of the risky assets) which satisfy $\mathbf{E}(R(\pi)) = m$ and $\sigma(R(\pi)) = \sigma$.

1. **[1pt]** Show that the problem $(P_{m,\sigma})$ is equivalent to finding $\pi \in \mathbf{R}^d$ such that
$$\begin{cases} \pi' 1_d = 1 \\ \pi' M = m \\ \pi'\Sigma\pi = \sigma^2 \end{cases}$$

Correction:
$\pi' 1_d = 1$ is the condition for π to represent an investment portfolio.
$\mathbf{E}(R(\pi)) = \pi' M$ so, the second condition is the condition on the expectation.
$\mathbf{Var}(R(\pi)) = \pi' \Sigma \pi$ so, the third condition is the condition on the variance or equivalently on the standard deviation.

2. Let $\pi_\alpha = \pi_a + \alpha\omega$

   (a) **[0.5pt]** Show that π_α is an investment portfolio.
   Correction:
   $\pi_\alpha' 1_d = \pi_a' 1_d + \alpha\omega' 1_d = 1 + \alpha(b - \frac{b}{a}a) = 1.$

   (b) **[1pt]** Calculate $m_\alpha = \mathbf{E}(R(\pi_\alpha))$ and $\sigma_\alpha = \sigma(R(\pi_\alpha))$ as a function of a, b, M, 1_d, α and $\|.\|_{\Sigma^{-1}}$.
   Correction:
   $m_\alpha = \frac{b}{a} + \alpha\|M - \frac{b}{a}1_d\|^2_{\Sigma^{-1}}$ and $\sigma_\alpha = \sqrt{\frac{1}{a} + \alpha^2\|M - \frac{b}{a}1_d\|^2_{\Sigma^{-1}}}$

   (c) **[0.5pt]** Show that $\forall m \in \mathbf{R}, \exists! \alpha \in \mathbf{R}$ such that $m_\alpha = m$.
   Correction:
   M is not colinear to 1_d so, $\|M - \frac{b}{a}1_d\|^2_{\Sigma^{-1}} \neq 0$ and so, $m_\alpha = m$ has a unique solution in α.

3. Let $\pi^* = \Sigma\pi$ and $\pi^*_\alpha = \Sigma\pi_\alpha$.

   (a) **[1pt]** Show that the problem $(P_{m_\alpha,\sigma})$ (for π) is equivalent, after a change of variable, to the problem $(P^*_{m_\alpha,\sigma})$ (for π^*) defined by,
   $$\begin{cases} (\pi^* - \pi^*_\alpha)'\Sigma^{-1}1_d = 0 \text{ (E1)} \\ (\pi^* - \pi^*_\alpha)'\Sigma^{-1}M = 0 \text{ (E2)} \\ \pi^{*\prime}\Sigma^{-1}\pi^* = \sigma^2 \end{cases}$$
   Correction:
   $\pi^* = \Sigma\pi \implies \pi = \Sigma^{-1}\pi^* \implies \pi' = \pi^{*\prime}\Sigma^{-1}$ as Σ^{-1} is symmetric, the results follows.

4. Demonstrate that

   (a) **[0.5pt]** if $\sigma < \sigma_\alpha$ then $(P^*_{m_\alpha,\sigma})$ has no solution in π^*,

   (b) **[0.5pt]** if $\sigma = \sigma_\alpha$ then $(P^*_{m_\alpha,\sigma})$ has a unique solution, which is π^*_α,

   (c) **[1pt]** if $\sigma > \sigma_\alpha$ and $d > 2$ describe the solutions of $(P^*_{m_\alpha,\sigma})$.

Correction:
We can decompose any π^* of $\mathbf{R}^d$ satisfying (E1) and (E2) as $\pi^* = \pi^*_\alpha + u$ where u belongs to the orthogonal, for the product scalar Σ^{-1}, of the vector space generated by 1_d and M (to which π^*_α belongs). We note this vector space H. With this decomposition we get $\pi^{*\prime}\Sigma^{-1}\pi^* = \sigma^2_\alpha + \|u\|^2$.
Now,

◦ $\sigma_\alpha^2 + \|u\|^2 = \sigma^2 \Longrightarrow \sigma \geq \sigma_\alpha$ which proves a).
◦ $\sigma_\alpha^2 + \|u\|^2 = \sigma_\alpha^2 \Longleftrightarrow \|u\| = 0 \Longleftrightarrow \pi^* = \pi_\alpha^*$ which proves b)
◦ if $\sigma > \sigma_\alpha$ then the set of solutions$\{\pi^*\}$ is the set $\pi_\alpha^* + \sqrt{\sigma^2 - \sigma_\alpha^2} S_H^1$ where S_H^1 is the sphere of radius 1 of H. Note that H here is not trivial reduced to $\{0\}$, as we have assumed that $d > 2$.

Exercise 3 : [6pts]

In this exercise we assume that the d risky assets satisfy, $R = A + BF$ with A constant vector of $\mathbf{R}^d$, B constant matrix of $\mathbf{R}^{d\times k}$ and F random variable of $\mathbf{R}^k$ such that $\mathbf{Var}(F)$ is invertible.

1. We consider a self-financing portfolio $\pi \in \mathbf{R}^d$ and note $R(\pi)$ its return.

   (a) **[1pt]** Show that: $R(\pi)$ risk-free $\Longleftrightarrow \pi' B = 0$.
   Correction:
   $R(\pi)$ risk-free $\Longleftrightarrow \mathbf{Var}(R(\pi)) = 0 \Longleftrightarrow \pi'\mathbf{Var}(R)\pi = 0$
   but $\mathbf{Var}(R) = B\mathbf{Var}(F)B'$ so,
   $R(\pi)$ risk-free $\Longleftrightarrow (B'\pi)'\mathbf{Var}(F)B'\pi = 0$ and as $\mathbf{Var}(F)$ is invertible it is equivalent to $B'\pi = 0$, which is equivalent to $\pi' B = 0$.

   (b) **[1pt]** Show that the following two propositions are equivalent:
   (P1) All risk-free self-financing portfolios have a return of zero.
   (P2) $(\pi' 1_d = 0$ and $B'\pi = 0) \Longrightarrow \pi' A = 0$.
   Correction:
   A risk-free self-financing portfolio is an allocation π which satisfies $\pi' 1_d = 0$ and $B'\pi = 0$ and for such a portfolio the return is $\pi' R = \pi' A + \pi' BF = \pi' A$ so the result follows.

   (c) **[1pt]** Using the previous results show that if all risk-free self-financing portfolios have a return of zero then,
   $\exists \lambda_0 \in \mathbf{R}$ and $\exists \lambda \in \mathbf{R}^k$ such that $A = \lambda_0 1_d + B\lambda$.
   Correction:
   Calling $b_1, b_2, \cdots, b_k$ the column vectors of B we have,
   $\Big(\pi' 1_d = 0 \text{ and } B'\pi = 0\Big) \Longleftrightarrow \pi \in Vec\{1_d, b_1, b_2, \cdots, b_k\}^\perp$.
   So, if (P_1) is true we have,
   $\pi \in Vec\{1_d, b_1, b_2, \cdots, b_k\}^\perp \Longrightarrow \pi \in Vec\{A\}^\perp$ and as a consequence $Vec\{1_d, b_1, b_2, \cdots, b_k\}^\perp \subset Vec\{A\}^\perp$ and consequently,
   $Vec\{A\} \subset Vec\{1_d, b_1, b_2, \cdots, b_k\}$. Which proves the result.

   (d) **[1pt]** Using the previous results show that if all risk-free self-financing portfolios have a return of zero then,
   $\exists \lambda_0 \in \mathbf{R}$ and $\exists \mu \in \mathbf{R}^k$ such that $E(R) = \lambda_0 1_d + B\mu$.
   Correction:
   The condition implies that $A = \lambda_0 1_d + B\lambda$ and as consequence, $E(R) = A + BE(F) = \lambda_0 1_d + B(E(F) + \lambda)$ and the result follows by taking $\mu = E(F) + \lambda$.

2. **[1pt]** Demonstrate that $(P3)$ implies $(P4)$ where,
(P3) $\exists \lambda_0 \in \mathbf{R}$ and $\exists \mu \in \mathbf{R}^k$ such that, $E(R) = \lambda_0 1_d + B\mu$.
(P4) All self-financing portfolios without risk have a return of zero.
Correction:
Let π be a self-financing portfolio without risk. In this case, the return is constant and therefore equal to the expected returns which is,
$\pi' E(R) = \lambda_0 \pi' 1_d + \pi' B\mu$.
As there is no risk, $\pi' B = 0$ and as the portfolio is self-financing, $\pi' 1_d = 0$.
Therefore, $\pi' E(R) = 0$ and the return, which is equal to the expected return, is zero Q.E.D.

3. **[1pt]** Demonstrate that $(P5)$ implies $(P6)$ where,
(P5) All risk-free self-financing portfolios have a return of zero.
(P6) All risk-free investment portfolios have the same return.
Correction:
Let π_P, π_Q be two investment portfolios without risk (and therefore with constant returns). Let $\pi_S = \pi_P - \pi_Q$.
π_S is a self-financing portfolio as $\pi'_S 1_d = \pi'_P 1_d - \pi'_Q 1_d = 1 - 1 = 0$.
The return of π_S is $R(\pi_S) = \pi'_S R = \pi'_P R - \pi'_Q R = R(\pi_P) - R(\pi_Q)$ and therefore is constant as the difference between two constants.
So, $(P5)$ implies that $R(\pi_S) = 0$ and therefore that $R(\pi_P) = R(\pi_Q)$. Q.E.D.