Master M1 - Mido 16th January 2020
Exam: Portfolio Management[1]: 2h

Notations: We consider d risky assets S^1, $S^2, \cdots, S^d$ of vector of returns R which verifies :

$$R = M + \epsilon \text{ with } R = \begin{pmatrix} R^1 \\ \vdots \\ R^d \end{pmatrix}, \ M = \begin{pmatrix} m^1 \\ \vdots \\ m^d \end{pmatrix} \text{ and } \epsilon = \begin{pmatrix} \epsilon^1 \\ \vdots \\ \epsilon^d \end{pmatrix}$$

where M is a vector of $\mathbf{R}^d$, and ϵ is a Gaussian vector of expectation zero and variance-covariance matrix Σ.

We note,

$$\pi = \begin{pmatrix} \pi^1 \\ \vdots \\ \pi^d \end{pmatrix} \text{ an allocation in the risky assets } S^i,$$

$R(\pi)$ the return of a portfolio of (risky) allocation π,
m_π its expectation and σ_π its standard deviation.
We remind that,
An investment portfolio π satisfies $\pi' 1_d = 1$.
A self-financing portfolio π satisfies, $\pi' 1_d = 0$.

We note,
1_d the vector of $\mathbf{R}^d$ with all components equal to 1.
$a = 1_d' \Sigma^{-1} 1_d$, $b = 1_d' \Sigma^{-1} M$, $\pi_a = \frac{1}{a} \Sigma^{-1} 1_d$ and $\omega = \Sigma^{-1}(M - \frac{b}{a} 1_d)$.

Exercise 1: [8pts]
When there is a risk-free asset of return r_0, we remind the Security Market Line equation for all investment portfolios of returns R_P:

$$R_P - r_0 = \beta_T(P)(R_T - r_0) + \epsilon_P \tag{1}$$

with $\beta_T(P) = \frac{\mathbf{Cov}(R_P, R_T)}{\sigma^2(R_T)}$ and ϵ_P independent from the return R_T of the Tangent Portfolio.

1. **[6pts]** Complete the table below:

Portfolio	$E(R_{P_i})$	$\beta_T(P_i)$	$\sigma(R_{P_i})$	$\sigma(\epsilon_{P_i})$
P_1	10%	1	50%	30%
P_2	?	?	0	?
P_3	12.5%	1.5	?	0%
P_4	?	3	?	40%

[1]Pierre Brugière University Paris 9 Dauphine

2. **[2pts]** Demonstrate how the SML equation (1), can be adapted to self-financing portfolios, made of risky assets only, using the fact that the SML equation can also be expressed for the d risky assets S^i as,

$$R - r_0 1_d = \beta_T (R_T - r_0) + \mathcal{E} \tag{2}$$

where β_T is the vector of the $\beta_T(R_i)$ and $\mathcal{E}$ is independent from R_T.

Exercise 2: [6pts]

In this exercise we assume that Σ is invertible and that M and 1_d are not collinear, and we solve the problem $(P_{m,\sigma})$ of finding all the investment portfolios π (made of the risky assets) which satisfy $\mathbf{E}(R(\pi)) = m$ and $\sigma(R(\pi)) = \sigma$.

1. **[1pt]** Show that the problem $(P_{m,\sigma})$ is equivalent to finding $\pi \in \mathbf{R}^d$ such that
$$\begin{cases} \pi' 1_d = 1 \\ \pi' M = m \\ \pi' \Sigma \pi = \sigma^2 \end{cases}$$

2. Let $\pi_\alpha = \pi_a + \alpha\omega$
   (a) **[0.5pt]** Show that π_α is an investment portfolio.
   (b) **[1pt]** Calculate $m_\alpha = \mathbf{E}(R(\pi_\alpha))$ and $\sigma_\alpha = \sigma(R(\pi_\alpha))$ as a function of a, b, M, 1_d, α and $\|.\|_{\Sigma^{-1}}$.
   (c) **[0.5pt]** Show that $\forall m \in \mathbf{R}, \exists! \alpha \in \mathbf{R}$ such that $m_\alpha = m$.

3. Let $\pi^* = \Sigma\pi$ and $\pi^*_\alpha = \Sigma\pi_\alpha$.
   (a) **[1pt]** Show that the problem $(P_{m_\alpha,\sigma})$ (for π) is equivalent, after a change of variable, to the problem $(P^*_{\alpha,\sigma})$ (for π^*) defined by,
$$\begin{cases} (\pi^* - \pi^*_\alpha)' \Sigma^{-1} 1_d = 0 \text{ (E1)} \\ (\pi^* - \pi^*_\alpha)' \Sigma^{-1} M = 0 \text{ (E2)} \\ \pi^{*\prime} \Sigma^{-1} \pi^* = \sigma^2 \end{cases}$$

4. Demonstrate that
   (a) **[0.5pt]** if $\sigma < \sigma_\alpha$ then $(P^*_{\alpha,\sigma})$ has no solution in π^*,
   (b) **[0.5pt]** if $\sigma = \sigma_\alpha$ then $(P^*_{\alpha,\sigma})$ has a unique solution, which is π^*_α,
   (c) **[1pt]** if $\sigma > \sigma_\alpha$ and $d > 2$ describe the solutions of $(P^*_{\alpha,\sigma})$.

Exercise 3 : [6pts]

In this exercise we assume that the d risky assets satisfy, $R = A + BF$ with A constant vector of $\mathbf{R}^d$, B constant matrix of $\mathbf{R}^{d\times k}$ and F random variable of $\mathbf{R}^k$ such that $\mathbf{Var}(F)$ is invertible.

1. We consider a self-financing portfolio $\pi \in \mathbf{R}^d$ and note $R(\pi)$ its return.

   (a) **[1pt]** Show that: $R(\pi)$ risk-free $\Longleftrightarrow \pi' B = 0$.

   (b) **[1pt]** Show that the following two propositions are equivalent:
   (P1) All risk-free self-financing portfolios have a return of zero.
   (P2) $(\pi' 1_d = 0$ and $B'\pi = 0) \Longrightarrow \pi' A = 0$.

   (c) **[1pt]** Using the previous results show that if all risk-free self-financing portfolios have a return of zero then,
   $\exists \lambda_0 \in \mathbf{R}$ and $\exists \lambda \in \mathbf{R}^k$ such that $A = \lambda_0 1_d + B\lambda$.

   (d) **[1pt]** Using the previous results show that if all risk-free self-financing portfolios have a return of zero then,
   $\exists \lambda_0 \in \mathbf{R}$ and $\exists \mu \in \mathbf{R}^k$ such that $E(R) = \lambda_0 1_d + B\mu$.

2. **[1pt]** Demonstrate that $(P3)$ implies $(P4)$ where,
   (P3) $\exists \lambda_0 \in \mathbf{R}$ and $\exists \mu \in \mathbf{R}^k$ such that, $E(R) = \lambda_0 1_d + B\mu$.
   (P4) All self-financing portfolios without risk have a return of zero.

3. **[1pt]** Demonstrate that $(P5)$ implies $(P6)$ where,
   (P5) All risk-free self-financing portfolios have a return of zero.
   (P6) All risk-free investment portfolios have the same return.