FINDING THE PORTFOLIO OF MINIMUM VARIANCE

We search for π_m solution of $\begin{cases} \underset{\pi}{Min} \ \ \pi'\Sigma\pi \\ \pi' 1_d = 1 \end{cases}$

We define $a = \|1_d\|^2_{\Sigma^{-1}}$ and $b = \langle M, 1_d \rangle_{\Sigma^{-1}}$.

$$L(\pi,\lambda) = \pi'\Sigma\pi - \lambda(\pi' 1_d - 1) \ \ \text{(strictly convexe)}$$

$$\frac{\partial L}{\partial \pi} = 0 \Rightarrow 2\pi'\Sigma - \lambda 1_d' = 0 \Rightarrow 2\Sigma\pi = \lambda 1_d \Rightarrow \pi = \frac{\lambda}{2}\Sigma^{-1} 1_d$$

$$\pi' 1_d = 1 \Rightarrow \frac{\lambda}{2} 1_d' \Sigma^{-1} 1_d = 1 \Rightarrow \frac{\lambda}{2} = \frac{1}{\|1_d\|^2_{\Sigma^{-1}}} = \frac{1}{a}. \text{So}$$

- $$\pi_m = \frac{1}{a}\Sigma^{-1} 1_d$$
- $$m_m = \pi_m' M = M'\pi_m = M'\frac{1}{a}\Sigma^{-1} 1_d = \frac{\langle M, 1_d \rangle_{\Sigma^{-1}}}{a} = \frac{b}{a}$$
- $$\sigma_m^2 = \left(\frac{1}{a}\Sigma^{-1} 1_d\right)' \Sigma \left(\frac{1}{a}\Sigma^{-1} 1_d\right) = \frac{1}{a^2} 1_d' \Sigma^{-1} 1_d = \frac{a}{a^2} = \frac{1}{a}$$

FINDING ALL THE EFFICIENT PORTFOLIOS (1)

$$\pi_E \text{ solution of } \begin{cases} \underset{\pi}{Min}\ \pi'\Sigma\pi \\ \pi' M = r \\ \pi' 1_d = 1 \end{cases} \Leftrightarrow \begin{cases} \underset{\pi}{Min}\ \pi'\Sigma\pi \\ M'\pi = r \\ 1_d'\pi = 1 \end{cases}$$

$$L(\pi,\lambda,\eta) = \pi'\Sigma\pi - \lambda(M'\pi - r) - \eta(1_d'\pi - 1) \quad \text{(strictly convexe)}$$

$$\frac{\partial L}{\partial \pi} = 0 \Rightarrow \pi = \Sigma^{-1}\left(\frac{\lambda}{2}M + \frac{\eta}{2}1_d\right)$$

$$1_d'\pi = 1 \Rightarrow \frac{\lambda}{2}1_d'\Sigma^{-1}M + \frac{\eta}{2}1_d'\Sigma^{-1}1_d = 1 \Rightarrow \frac{\lambda}{2}b + \frac{\eta}{2}a = 1 \Rightarrow \frac{\eta}{2} = \frac{1}{a}\left(1 - \frac{\lambda}{2}b\right)$$

$$M'\pi = r \Rightarrow \frac{\lambda}{2}M'\Sigma^{-1}M + \frac{\eta}{2}M'\Sigma^{-1}1_d = m \Rightarrow \frac{\lambda}{2}\|M\|^2_{\Sigma^{-1}} + \frac{1}{a}\left(1 - \frac{\lambda}{2}b\right)b = m \Rightarrow \frac{\lambda}{2}\left(\|M\|^2_{\Sigma^{-1}} - \frac{b^2}{a}\right) = m - \frac{b}{a}$$

$$\text{but } \|M\|^2_{\Sigma^{-1}} - \frac{b^2}{a} = \left\langle M, M - \frac{b1_d}{a}\right\rangle_{\Sigma^{-1}} = \left\langle M - \frac{b1_d}{a}, M - \frac{b1_d}{a}\right\rangle_{\Sigma^{-1}} \text{ because } \left\langle 1_d, M - \frac{b1_d}{a}\right\rangle_{\Sigma^{-1}} = b - b\frac{a}{a} = 0$$

$$\left(\frac{b}{a}1_d \text{ projection of } M \text{ on Vect}\{1_d\} \text{ for } <.,.>_{\Sigma^{-1}} \text{ so } M - \frac{b}{a}1_d \text{ and } 1_d \text{ are orthogonal for } <.,.>_{\Sigma^{-1}}\right).$$

$$\text{we assume } M - \frac{b}{a}1_d \neq 0 \left(\text{otherwise the only return that can be expected for all portfolios is } \frac{b}{a}\right)$$

$$\text{so } \frac{\lambda}{2} = \left(m - \frac{b}{a}\right)\frac{1}{\left\|M - \frac{b}{a}1_d\right\|^2_{\Sigma^{-1}}}$$

FINDING ALL THE EFFICIENT PORTFOLIOS (2)

$$\pi = \frac{\lambda}{2}\Sigma^{-1}M + \frac{\eta}{2}\Sigma^{-1}1_d = \frac{\lambda}{2}\Sigma^{-1}M + \frac{1}{a}\left(1 - \frac{\lambda}{2}b\right)\Sigma^{-1}1_d = \frac{\lambda}{2}\Sigma^{-1}\left(M - \frac{b}{a}1_d\right) + \frac{1}{a}\Sigma^{-1}1_d. \text{ So}$$

- $$\pi_E = \frac{\Sigma^{-1}1_d}{a} + \left(m_E - \frac{b}{a}\right)\frac{\Sigma^{-1}\left(M - \frac{b}{a}1_d\right)}{\left\|M - \frac{b}{a}1_d\right\|^2_{\Sigma^{-1}}}$$