

Portfolio management

TD2. Optimization and portfolio selection

We consider $d \geq 1$ risky assets S^1, \dots, S^d and a risk-free asset S^0 . We note $R_T = (R_T^1, \dots, R_T^d)'$ the random vector for the returns of S^1, \dots, S^d , over the period $[0, T]$, M the expectation of R_T and R^0 the deterministic return of S^0 . We assume that the variance-covariance matrix Σ of R_T is invertible and that $\mathbf{E}(R_T) \neq R^0 \mathbf{1}_d$. We note $\Pi = (\pi^0, \pi)'$ the vector of allocations for an investment portfolio where $\pi \in \mathbf{R}^d$ defines the allocation in the risky assets and π^0 in the risk-free asset.

Exercise 1.

Assume that $d = 2$ and that investments are made only in the risky assets, i.e. $\pi^0 = 0$.

- 1) Does the expected return of a risky portfolio determines in a unique way the variance of its return ?
- 2) Express π for the portfolio of minimum variance in function of σ_1, σ_2 and the covariance of the returns of the two risky assets $\rho_{1,2}$.

Exercise 2.

Consider a portfolio $\Pi = (\pi^0, \pi)' \in \mathbf{R}^{d+1}$, $d \geq 1$

- 1) Give the expression of the terminal wealth $W_T^\pi(x_0)$ at time T for the buy and hold strategy Π and the initial capital x_0 .

We consider the optimisation problem

$$(P_\sigma) \begin{cases} \sup_{\pi \in \mathbf{R}^d} \mathbf{E}(W_T^\pi(x_0)) \\ \text{var}(W_T^\pi(x_0)) = x_0^2 \sigma^2 \end{cases} \quad \text{where } \sigma > 0 \text{ is fixed}$$

- 2) Show that (P_σ) is equivalent to

$$\begin{cases} \sup_{\pi \in \mathbf{R}^d} \mathbf{E}(W_T^\pi(x_0)) \\ \text{var}(W_T^\pi(x_0)) \leq x_0^2 \sigma^2 \end{cases}$$

For $\lambda > 0$, we define the Lagrangian $L^\lambda(\pi) := \mathbf{E}R^\pi - \lambda(\text{var}R^\pi - \sigma^2)$.

- 3) Show that $\sup_{\pi \in \mathbf{R}^d} L^\lambda(\pi) < +\infty$ and that the sup is attained for a particular value of π that we note π_λ
- 4) Can you choose $\lambda(\sigma) > 0$ such that $\text{var}R^{\pi_{\lambda(\sigma)}} = \sigma^2$?
- 5) Explain why the efficient frontier is $\{(\sigma(\pi_{\lambda(\sigma)}), E(R^{\pi_{\lambda(\sigma)}}), \sigma > 0)\}$.

Exercise 3.

Let L be a random variable. The quantile of L is defined by:

$$q_L(\alpha) = \inf\{x, P(L \leq x) \geq \alpha\}.$$

In finance L usually represents the distributions of possible losses ($L > 0$ means a loss) and $q_L(\alpha)$ is called the α -value at risk and is noted $\text{VAR}_\alpha(Z)$

- 1) Show that $q_L(\alpha) = \sup\{x, P(L \leq x) < \alpha\}$ and $P(L \leq q_L(\alpha)) \geq \alpha$.
- 2) Show that $P(L \leq \text{VAR}_{95\%}(L)) \geq 95\%$ and $P(L > \text{VAR}_{95\%}(L)) \leq 5\%$.
Let X_1, X_2, ν_1, ν_2 be independent variables with $X_i \sim N(0, 1)$. We assume that $P(\nu_i = 10) = 0.02$ and $P(\nu_i = 0) = 0.98$. We note $L_i = X_i + \nu_i$.
- 3) Show that $P(L_i > 4) < 3\%$ and $P(\frac{1}{2}L_1 + \frac{1}{2}L_2 > 4) > 3\%$
- 4) Deduct from 3) that $\text{VAR}_{97\%}(\frac{1}{2}L_1 + \frac{1}{2}L_2) > \frac{1}{2}\text{VAR}_{97\%}(L_1) + \frac{1}{2}\text{VAR}_{97\%}(L_2)$
- 5) How do you interpret the previous result and what problem does it show about the utilization of the VAR as a risk measure?