

## Portfolio management

### TD3. Markowitz portfolio theory

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We consider  $d$  risky assets  $S^1, \dots, S^d$  and a risk-free asset  $S^0$ . At  $t = 0$ , the agents invest their wealth in  $S^0, S^1, \dots, S^d$  and keep their investment unchanged until the terminal date  $T > 0$ .

We note  $R^i$  the return of asset  $i$  over  $[0, T]$ , and  $R$  the vector  $R := (R^1, \dots, R^d)'$ . The risk-free asset  $S^0$  has a deterministic return  $R^0$ . We note  $M = (m^1, \dots, m^d)'$  the expectation of  $R$  and  $\Sigma$  the variance-covariance matrix of  $R$ .

We assume that  $M \neq R^0 \mathbf{1}_d$  and that  $\Sigma$  is invertible.

#### Exercise 1

1. We consider a first investor who invests according to the following criterion

$$\max_{\pi \in \mathbb{R}^d} \mathbb{E}[R^\pi] - \frac{a}{2} \text{var}(R^\pi),$$

where  $a$  is a positive constant.

- determine the allocation  $\pi_1$  chosen by this investor.
- compute  $m_1 := \mathbb{E}[R^{\pi_1}]$  and  $\sigma_1^2 := \text{var}(R^{\pi_1})$ .

2. A second investor invests according to the following criteria

$$\max_{\pi \in \mathbb{R}^d} \mathbb{E}[u(R^\pi)]$$

where  $u$  is the quadratic utility function defined by  $u(x) = x - \frac{\lambda}{2}x^2$ .

- give the allocation  $\pi_2$  chosen by this investor.

**Exercise 2** here we take  $d = 2$ , no risk free asset and we note

$$\text{var}(R^i) = (\sigma^i)^2, \quad \text{and} \quad \rho = \frac{\text{cov}(R^1, R^2)}{\sigma^1 \sigma^2}.$$

We assume that  $\sigma^1 < \sigma^2$  and  $m^1 < m^2$ . We note  $R^\alpha$  the return of the portfolio obtained by allocating the proportion  $\alpha$  of the initial wealth into the asset  $S^1$ .

1. Express  $m^\alpha := \mathbb{E}[R^\alpha]$  and  $(\sigma^\alpha)^2 := \text{var}(R^\alpha)$  as a function of  $\alpha$ .
2. Give the relationship between  $m^\alpha$  et  $(\sigma^\alpha)^2$ .
3. Determine the efficient frontier and draw it in the case  $\rho = 0$ .
4. Determine the efficient frontier when the investor is not allowed to short-sell the assets.

**Exercise 3** Here we assume  $d = 2$ ,  $R^0 = 0$  and :

$$\Sigma = \begin{pmatrix} 0.2 & 0.2 * 0.99 \\ 0.2 * 0.99 & 0.2 \end{pmatrix} \quad \text{and} \quad M - R^0 \mathbf{1} = M = \begin{pmatrix} 0.08 \\ 0.08 \end{pmatrix}$$

1. Compute explicitly the efficient portfolio of variance  $\sigma^2 = 1$ .
2. We assume that an investor does not know the exact expected value of  $R^1$  and estimates this value at 0.09. Give for such an investor the efficient portfolio with variance  $\sigma^2 = 1$ . Comment the result.

**Exercise 4** We consider the following mean-variance problem

$$(P) : \begin{cases} \sup_{\pi} \mathbb{E}[R^\pi] \\ \text{var}(R^\pi) = \sigma^2 \\ \mathbf{1}'\pi = 1 \end{cases}$$

1. Give the Lagrangien associated to the problem  $(P)$  and the first order optimality conditions.
2. Prove that for any efficient portfolio  $\pi(\sigma)$ , where  $\sigma$  is not the minimal variance level ( $\sigma^2 > \frac{1}{a}$ ), there exists a portfolio  $\pi^-(\sigma)$  on the frontier of the set of admissible portfolios such that

$$\text{cov}(R^{\pi(\sigma)}, R^{\pi^-(\sigma)}) = 0$$

3. Prove that  $\pi^-(\sigma)$  is in the inefficient part of the frontier.