Portfolio Management

TD4. CAPM

We consider an economy with d risky assets $S^1, \ldots, S^d$, a risk free asset S^0 and one period $[0, T]$. We note R^i the return of asset i, $R := (R^1, R^2, \cdots, R^d)$, $M := \mathbb{E}[R]$, $\Sigma := \left(\sigma_{ij} = cov(R^i, R^j)\right)_{1 \leq i,j \leq d}$, $a = 1_d' \Sigma^{-1} 1_d$ and $b = M' \Sigma^{-1} 1_d$
As in the course a portfolio of risky assets π_λ on the frontier for the risky investment portfolios can be written as
$\pi_\lambda = \frac{1}{a} \Sigma^{-1} 1_d + \lambda \Sigma^{-1} (M - \frac{b}{a} 1_d)$

Exercise 1 *Tangent Portfolio*
1. Express $m_\lambda := E[R^{\pi_\lambda}]$ as a function of a, b, λ and $\|M - \frac{b}{a} 1_d\|_{\Sigma^{-1}}$.
2. Express $\sigma_\lambda^2 := Var[R^{\pi_\lambda}]$ as a function of a, b, λ and $||M - \frac{b}{a} 1_d||_{\Sigma^{-1}}$.
3. Show that $(\frac{\partial \sigma_\lambda}{\partial \lambda}, \frac{\partial m_\lambda}{\partial \lambda})'$ is colinear to $(\lambda, \sigma_\lambda)'$
4. deduct that the "tangent portfolio" is obtained for $\lambda = \frac{1}{b - r_0 a}$
5. deduct that the "tangent portfolio" is $\pi_T = \frac{1}{b - r_0 a} \Sigma^{-1} (M - r_0 1_d)$

Exercise 2 *Security Market Line*
1. Show that $i = 1, \ldots, d$

$$R^i - r_0 \; = \; \beta^i (R^{\pi_T} - r_0) + \varepsilon^i \quad \text{with} \quad \beta^i := \frac{cov(R^i, R^{\pi_T})}{var(R^{\pi_T})} \text{ and } (\varepsilon^1, \cdots \varepsilon^d) \text{independant from } R^{\pi_T}$$

where π_T is the tangent portfolio
2. Show that
a) $E[R - r_0 1_d] = M - r_0 1_d$
b) $\beta = (b - r_0 a) \frac{(M - r_0 1_d)}{\|M - r_0 1_d\|^2_{\Sigma^{-1}}}$
c) $E[R^{\pi_T} - r_0] = \frac{1}{b - r_0 a} \|M - r_0 1_d\|^2_{\Sigma^{-1}}$
d) conclude that $E(\varepsilon^i) = 0$
3. We assume $R^0 = 2\%$, complete the following table :

Asset	expected return	beta	systematic risk	specific risk
Tangent portfolio	10 %		20%	–
asset 1		0.8		10%
asset 2	-5%			30%