

Portfolio Management

TD7. Factor Models and APT

We consider N risky assets S_1, \dots, S_N and a risk-free asset S_0 of return r^0 . We note R^i the return of the risky asset i over $[0, T]$ and $\pi_N \in \mathbb{R}^N$ a vector whose component π^i corresponds to the % of the initial wealth invested in the risky asset S^i at time 0. We consider a K -factors model : $R^i = a^i + b_1^i f^1 + \dots + b_K^i f^K + \varepsilon^i$ with $\Sigma_F = [cov(f^i, f^j)]$ invertible $\mathbb{E}[f^i] = 0$, $\mathbb{E}[\varepsilon^i] = 0$, $cov(\varepsilon^i, f^j) = 0$ and $\Sigma_\varepsilon = [cov(\varepsilon^i, \varepsilon^j)]$ invertible diagonal. We note matricially : $R = A + BF + E$

Exercise 1 *Considering all sources of risks as factors*

1. Determine B^* and F^* in order to write the model as a factor model : $R = A + B^*F^*$ with $E[F^*] = 0$ and Σ_{F^*} invertible.
2. In a factor model $R = A + BF + E$ show that B satisfies $B = Cov(R, F)\Sigma_F^{-1}$

Exercise 2 *Regression on K given factors*

1. Show that $E[\|R\|^2] = \|E(R)\|^2 + E[\|R - E(R)\|^2]$
2. Show that $E[\|R - E(R)\|^2] = Trace(Var(R))$.

Let R be the vector of returns of d assets and F be a random vector of dimension K such that Σ_F is invertible and $E[F] = 0$. We search the vector A^* and the matrix B^* which minimizes $E[\|R - A - BF\|^2]$.

3. a) how do you compare this result to an optimization "component by component" ("asset by asset") where you solve for all $i \in [1, d]$ $\min_{a^i, b_1^i, \dots, b_K^i} E[(R^i - a^i - b_1^i f^1 - \dots - b_K^i f^K)^2]$
 - b) show that $A^* = E(R)$ and $B^* = Cov(R, F)\Sigma_F^{-1}$
 - c) (subsidiary question) show that if we define $\langle, \rangle : \mathbb{R}^{N \times K} \times \mathbb{R}^{N \times K} \rightarrow \mathbb{R}$ by $\langle M, N \rangle = Trace(M^T N)$ then \langle, \rangle defines a scalar product
 - d) show that $Cov(R - A^* - B^*F, F) = 0$
 - e) show that we can write $R = A^* + B^*F + \mathcal{E}$ with $E[\mathcal{E}] = 0$ and $Cov(F, \mathcal{E}) = 0$
 - f) what happens if we do not make the assumption that $E(F) = 0$?

Exercise 3 *Principal Component Analysis*

Let $(u_i)_{i \in [1, N]}$ be a set of eigenvectors of $Var(R)$ which form an orthonormal basis of \mathbb{R}^N

1. justify why we can write $R = \sum_{i=1}^N \langle R, u_i \rangle u_i$
2. Calculate $Cov(\langle R, u_i \rangle, \langle R, u_j \rangle)$

3. Deduct a decomposition $R = A + BF$ with A constant vector, B constant matrix and F random vector of \mathbb{R}^N with Σ_F diagonale invertible and $E(F) = 0$.