Master M1 - Mido: 2 November 2017

Partiel: Portfolio Management [1]: time 2h

Notations: We consider d risky assets S^1, S^2, $...S^d$, with returns on $[0,T]$ satisfying $R^i = m^i + \epsilon^i$. We note:

$$R = M + \epsilon \text{ avec } R = \begin{pmatrix} R^1 \\ \vdots \\ R^d \end{pmatrix},\ M = \begin{pmatrix} m^1 \\ \vdots \\ m^d \end{pmatrix} \text{and } \epsilon = \begin{pmatrix} \epsilon^1 \\ \vdots \\ \epsilon^d \end{pmatrix}$$

where M is a vector of $\mathbf{R}^d$, ϵ is a random vector of expectation zero and variance-covariance matrix Σ invertible. We note :

$$\pi = \begin{pmatrix} \pi^1 \\ \vdots \\ \pi^d \end{pmatrix} \text{ the vector of the allocations at time zero in the assets } S^i,$$

R^π the return of the portfolio π on $[0,T]$, m^π the expectation of its return and σ^π the standard deviation of its return.

1_d the vector of $\mathbf{R}^d$ with all its components equal to 1.
For a function $L(\pi)$ we note $\frac{\partial L}{\partial \pi}$ the raw vector $[\frac{\partial L}{\partial \pi^1}, \cdots, \frac{\partial L}{\partial \pi^d}]$
We note $a = 1_d'\Sigma^{-1}1_d$ and $b = 1_d'\Sigma^{-1}M$

Exercise 1 : [4pts]
Answer without any justification to the following questions :
1)**[0.25pt]** Express $E[R^\pi]$ as a function of π and M
2)**[0.25pt]** Express $Var[R^\pi]$ as a function of π and Σ
3)**[0.50pt]** Express $Cov(R^{\pi_1}, R^{\pi_2})$ as a function of π_1, π_2 and Σ

If X and Y are two random vectors of $\mathbf{R}^k$ and $\mathbf{R}^l$,and A is a matrix of $\mathbf{R}^{m\times k}$ and B a matrix of $\mathbf{R}^{n\times l}$
4)**[0.50pt]** Express $Cov(X,Y)$ as a function of $E[XY']$, $E[X'Y]$, $E[X]$ and $E[Y]$
5)**[0.50pt]** Express $Cov(AX,BY)$ as a function of A, B and $Cov(X,Y)$
6)**[0.50pt]** Express $Cov(X,Y)$ as a function of $Cov(Y,X)$

7)**[0.50pt]** Express $\frac{\partial f}{\partial \pi}$ when $f(\pi) = \pi'\Sigma\pi + \alpha_1\pi' M + \alpha_2$

We consider a strategy (x_0, π)
8)**[0.25pt]** what is the value of $1_d'\pi$ for an investment strategy
9)**[0.25pt]** what is the value of $1_d'\pi$ for a self-financing strategy
10)**[0.25pt]** what does x_0 represent for an investment strategy

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11)**[0.25pt]** what does x_0 represent for a self financing strategy

Exercise 2 : [6pts]

We assume that $M \neq 0$ and we note $\pi_M = \frac{1}{b}\Sigma^{-1}M$ and R_M the return of the portfolio (x_0, π_M)

1)**[0.25pt]** Proove that π_M is the allocation of an investment portfolio

We consider (x_0, π_P) an investment portfolio whose return is noted R_P

2)**[0.25pt]** Show that : $Cov(R_P, R_M) = 0 \Longrightarrow E[R_P] = 0$

We consider the problem $\inf_{\alpha \in \mathbf{R}} Var(R_P - \alpha R_M)$

3)**[0.25pt]** Show that the inf is reached for a unique value of α noted $\alpha_M(P)$

4)**[0.25pt]** Express $\alpha_M(P)$ as a function of $Cov(R_M, R_P)$, $Var(R_M)$ and $Var(R_P)$

5)**[1pt]** Calculate the numerical value of $E(R_P - \alpha_M(P)R_M)$

6)**[1pt]** Express $Var(R_P - \alpha_M(P)R_M)$ as a function of $Var(R_P)$ and of the correlation between R_P et R_M noted $\rho_{P,M}$

7)**[0.50pt]** Show, based on the previous results, that for any investment portfolio P of return R_P we have :

$R_P = \alpha_{P,M}R_M + \epsilon_{P,M}$ with $\alpha_{P,M} \in \mathbf{R}$ and

i) $E(\epsilon_{P,M}) = 0$

ii) $Var(\epsilon_{P,M}) \leq Var(R_P)$

8)**[1.5pt]** Show that if an investment portfolio (x_0, π_Q) verifies:

$Cov(R_P, R_Q) = 0 \Longrightarrow E[R_P] = 0,$

then $\pi_Q = \pi_M$

9)**[1pt]** what do 7) and 8) imply in terms of "risk remuneration" for an investment portfolio.

Exercise 3 : [5pts]

We note R_Z the return of an asset belonging or not to the universe of portfolios that can be built with the risky assets $S^1, S^2, \cdots S^d$. We want to solve

$$\begin{cases} \inf_{\pi \in \mathbf{R}^d} \mathbf{Var}(R_Z - \pi' R) \\ \quad \pi' 1_d = 1 \end{cases}$$

1)**[1.00pt]** write the expression of the Lagrangian $L(\pi, \lambda)$

2)**[0.50pt]** Calculate $\frac{\partial L}{\partial \pi}$

3)**[0.50pt]** Justify that the inf is reached for π solution of $\frac{\partial L}{\partial \pi} = 0$

4)**[2.00pt]** Show that the solution of the problem can be expressed as :

$\pi = \pi_a + \Sigma^{-1}[U_Z - \pi_a' U_Z 1_d]$, what are the expressions of U_Z and π_a ?

5)**[1.00pt]** To which situation corresponds the case $U_Z = 0$ and how the solution of the optimization problem can be found directly in this case

Exercise 4 : [5pts]

We recall that if $\mathcal{P}$ is the set of all investment portfolios that can be built then $\{(\sigma_\pi, m_\pi)', \pi \in \mathcal{P}\}$ is delimited by the set $\{(\sigma_\pi, m_\pi)', \pi \in \mathcal{F}\}$ where $\mathcal{F}$ is the set of investment portfolios of the form $\frac{1}{a}\Sigma^{-1}1_d + \lambda\Sigma^{-1}(M - \frac{b}{a}1_d)$ where $\lambda \in \mathbf{R}$

1)**[1pt]** Show that : $(\alpha \in \mathbf{R}, \pi_1 \in \mathcal{F}$ et $\pi_2 \in \mathcal{F}) \Longrightarrow \alpha\pi_1 + (1-\alpha)\pi_2 \in \mathcal{F}$
2)**[1pt]** Show that : $(\alpha \in \mathbf{R}\setminus\{0\}, \pi_1 \in \mathcal{P}\setminus\mathcal{F}$ and $\pi_2 \in \mathcal{F}) \Longrightarrow \alpha\pi_1+(1-\alpha)\pi_2 \in \mathcal{P}\setminus\mathcal{F}$

We assume in all what follows that $M \neq \frac{b}{a}1_d$, π_P, $\pi_Q \in \mathcal{F}$ with $m_P \neq m_Q$ and that $\pi_R \in \mathcal{P}\setminus\mathcal{F}$ with $m_R \neq m_P$.
3)**[1pt]** Show that : $\forall\pi \in \mathcal{F}\ \exists\alpha \in \mathbf{R}$, $\pi = \alpha\pi_P + (1-\alpha)\pi_Q$
4)**[2pt]** Show that : $\forall(\sigma, m)'$ to the right of $\mathcal{F}$ we can find α_1, $\alpha_2 \in \mathbf{R}$, such that the investment portfolio of allocation $\alpha_1\pi_P + \alpha_2\pi_Q + (1-\alpha_1-\alpha_2)\pi_R$ as an expected return of m and a standard deviation of the return of σ.