Master M1 - Mido: 29th October 2018

Midterm Exam: Portfolio Management [1]: 2h

Notations: We consider d risky assets $S_1,\ S_2,\ ...S_d$, whose returns over $[0,T]$ verify $R^i = m^i + \epsilon^i$. We note in vector form:

$$R = M + \epsilon \text{ with } R = \begin{pmatrix} R^1 \\ \vdots \\ R^d \end{pmatrix},\ M = \begin{pmatrix} m^1 \\ \vdots \\ m^d \end{pmatrix} \text{ and } \epsilon = \begin{pmatrix} \epsilon^1 \\ \vdots \\ \epsilon^d \end{pmatrix}$$

where M is a vector of $\mathbb{R}^d$, ϵ is a Gaussian vector of expectation zero and of matrix of variance-covariance Σ invertible. We also assume that there is a risk-free asset S_0 of return r_0.

We note:

$\Pi = \begin{pmatrix} \pi^0 \\ \pi \end{pmatrix}$ an asset allocation where π^0 is the allocation in the risk-free asset and $\pi = \begin{pmatrix} \pi^1 \\ \vdots \\ \pi^d \end{pmatrix}$ is the allocation in the risky assets S_i

R_Π the return of the portfolio Π over $[0,T]$. $E[R_\Pi]$ its expectation and $\sigma[R_\Pi]$ its standard deviation

1_d the vector of $\mathbb{R}^d$ with all components equal to 1

e_i the vector of $\mathbb{R}^d$ with all components equal to zero except for the i^{th} component which equals 1

$a = 1_d'\Sigma^{-1}1_d$ and $b = 1_d'\Sigma^{-1}M$ and we assume that $r_0 \neq \frac{b}{a}$ and $M \neq r_0 1_d$

We note Φ the cumulative distribution function for a normal law $\mathcal{N}(0,1)$ so if $Z \sim \mathcal{N}(0,1)$ then $\forall x \in \mathbb{R}, \Phi(x) = P(Z \leq x)$ and Φ^{-1} its inverse from $]0,1[$ to $\mathbb{R}$. We note ϕ the derivative of Φ, i.e the density of the distribution function of Z.

Problem : [20pts] Risk Measures and Capital Allocation

For any vector $\pi \in \mathbb{R}^d$ we define : $RM_\lambda(\pi) = -\pi'(M - r_0 1_d) + \lambda\sqrt{\pi'\Sigma\pi}$ which is called the Markowitz risk measure of parameter λ for the risk exposure π. $RM_\lambda(\pi)$ can be interpreted as the capital required for a company to hold the risky positions defined by π.

1)

[0.5pt] a) what is the relationship between π^0 and π for an investment portfolio Π of risky allocation π ?

[0.5pt] b) express as a function of r_0, M and π the expected return for an investment portfolio Π of risky allocation π

[1] Pierre Brugière University Paris 9 Dauphine

[0.5pt] c) express as a function of π and Σ the standard deviation of the returns for an investment portfolio Π of risky allocation π

[0.5pt] d) show that for any $\pi \in \mathbb{R}^d$ $RM_\lambda(\pi) = -E[R_\Pi - r_0] + \lambda\sigma(R_\Pi)$ where Π is the investment portfolio of risky allocation π

For any vector $\pi \in \mathbb{R}^d$ we define the random variable $L_\pi = -\pi'(R - r_0 1_d)$ and for $\alpha \in]0,1[$ we define $VaR_\alpha(L_\pi)$ by: $VaR_\alpha(L_\pi) = \inf\{x, P(L_\pi \leq x) \geq \alpha\}$ which is called the value at risk for the risk exposure π.

2)

[0.5pt] a) express the law of L_π as a function of $E[R_\Pi]$, r_0 and $\sigma[R_\Pi]$ where Π is the investment portfolio of risky allocation π

[1pt] b) show that $P(L_\pi \leq VaR_\alpha(L_\pi)) = \alpha$

[1pt] c) express $VAR_\alpha(L_\pi)$ as a function of $E[R_\Pi], r_0, \sigma(R_\Pi), \alpha$ and Φ

[0.5pt] d) for which value of $\lambda(\alpha)$ do we have $\forall \pi \in \mathbb{R}^d, RM_{\lambda(\alpha)}(\pi) = VaR_\alpha(L_\pi)$

For any $\pi \in \mathbb{R}^d$ we define the quantity $E_\alpha(L_\pi) = E(L_\pi | L_\pi \geq VaR_\alpha(L_\pi))$ which is called the expected shortfall for the risk exposure π.

3)

[0.5pt] a) show that if $a \in \mathbb{R}$ and $Z \sim \mathcal{N}(0,1)$ then $E(Z|Z \geq a) = \frac{\phi(a)}{1-\Phi(a)}$

[0.5pt] b) show that if $a \in \mathbb{R}$ and $X \sim \mathcal{N}(m, \sigma^2)$ then $E(X|X \geq a) = m + \frac{\phi(\frac{a-m}{\sigma})}{1-\Phi(\frac{a-m}{\sigma})}\sigma$

[1.5pt] c) express $E_\alpha(L_\pi)$ as a function of $E[R_\Pi], r_0, \sigma[R_\Pi], \alpha, \phi$ and Φ

[0.5pt] d) for which value of $\lambda(\alpha)$ do we have $\forall \pi \in \mathbb{R}^d, RM_{\lambda(\alpha)}(\pi) = E_\alpha(L_\pi)$

In this section we consider the derivative $\frac{\partial RM_\lambda}{\partial e_i}(\pi)$ as a row vector representing the derivative, calculated at point π, of $RM_\lambda(\pi)$ in the direction of vector e_i.

4)

[1pt] a) show that $\forall \pi \in \mathbb{R}^d, RM_\lambda(\pi) \leq \sum_{i=1}^{d} RM_\lambda(\pi^i e_i)$

[1pt] b) show that $\forall \pi \in \mathbb{R}^d, RM_\lambda(\pi) = \sum_{i=1}^{d} \pi^i \frac{\partial RM_\lambda}{\partial e_i}(\pi)$

[1pt] c) how can you interpret the results $4a$) and $4b$) in terms of capital allocation and diversification effect ?

5)

[2pt] a) show that $\exists \lambda_0 \in \mathbb{R}$ such that

$$\begin{cases} \inf_{\pi \in \mathbb{R}^d} RM_\lambda(\pi) = -\infty \text{ if } \lambda < \lambda_0 \text{ and} \\ \inf_{\pi \in \mathbb{R}^d} RM_\lambda(\pi) = 0 \text{ if } \lambda \geq \lambda_0 \end{cases}$$

and express λ_0 as a function of $M, r_0, 1_d$ and Σ

[0.5pt] b) show that if $\lambda > \lambda_0$ then $\forall \pi \in \mathbb{R}^d \setminus \{0\}, RM_\lambda(\pi) > 0$

[0.5pt] c) show that $\exists \pi \in \mathbb{R}^d$ such that $E(R_\Pi - r_0) > 0$

From now on we consider that $\lambda > \lambda_0$ (as defined in 5a)) and for any $\pi \in \mathbb{R}^d \setminus \{0\}$ we define the Return on Risk-Adjusted Capital as the quantity $RORAC_\lambda(\pi) = \frac{-E(L_\pi)}{RM_\lambda(\pi)}$ and we call $\mathcal{D} = \{\pi^* \in \mathbb{R}^d \setminus \{0\}, RORAC_\lambda(\pi^*) = \sup_{\pi \in \mathbb{R}^d} RORAC_\lambda(\pi)\}$

6)

[1.5pt] a) show that $\pi \in \mathbb{R}^d \setminus \{0\}$ maximizes $RORAC_\lambda(\pi)$ if and only if the investment portfolio Π of risky allocation π maximizes the Sharpe Ratio $\frac{E(R_\Pi - r_0)}{\sigma(R_\Pi)}$

[1.5pt] b) using a) determine $\mathcal{D}$ and $\frac{E(R_{\Pi^*} - r_0)}{\sigma(R_{\Pi^*})}$ for $\pi^* \in \mathcal{D}$

[1pt] c) calculate $RORAC_\lambda(\pi^*)$ for $\pi^* \in \mathcal{D}$ as a function of λ and λ_0

[2pt] d) for $\pi^* \in \mathcal{D}$ calculate $\frac{-E[L_{\pi^{*i} e_i}]}{\pi^{*i} \frac{\partial RM_\lambda}{\partial e_i}(\pi^*)}$ when $m^i \neq r_0$ and $\pi^{*i} \neq 0$ as a function of λ and λ_0