Master M1 - Mido: 29th October 2018

Midterm Exam: Portfolio Management [1]: 2h

Notations: We consider d risky assets S_1, S_2, $...S_d$, whose returns over $[0, T]$ verify $R^i = m^i + \epsilon^i$. We note in vector form:

$$R = M + \epsilon \text{ with } R = \begin{pmatrix} R^1 \\ \vdots \\ R^d \end{pmatrix}, \ M = \begin{pmatrix} m^1 \\ \vdots \\ m^d \end{pmatrix} \text{ and } \epsilon = \begin{pmatrix} \epsilon^1 \\ \vdots \\ \epsilon^d \end{pmatrix}$$

where M is a vector of $\mathbb{R}^d$, ϵ is a Gaussian vector of expectation zero and of matrix of variance-covariance Σ invertible. We also assume that there is a risk-free asset S_0 of return r_0.

We note:

$\Pi = \begin{pmatrix} \pi^0 \\ \pi \end{pmatrix}$ an asset allocation where π^0 is the allocation in the risk-free asset and $\pi = \begin{pmatrix} \pi^1 \\ \vdots \\ \pi^d \end{pmatrix}$ is the allocation in the risky assets S_i

R_Π the return of the portfolio Π over $[0, T]$. $E[R_\Pi]$ its expectation and $\sigma[R_\Pi]$ its standard deviation

1_d the vector of $\mathbb{R}^d$ with all components equal to 1

e_i the vector of $\mathbb{R}^d$ with all components equal to zero except for the i^{th} component which equals 1

$a = 1_d' \Sigma^{-1} 1_d$ and $b = 1_d' \Sigma^{-1} M$ and we assume that $r_0 \neq \frac{b}{a}$ and $M \neq r_0 1_d$

We note Φ the cumulative distribution function for a normal law $\mathcal{N}(0, 1)$ so if $Z \sim \mathcal{N}(0, 1)$ then $\forall x \in \mathbb{R}, \Phi(x) = P(Z \leq x)$ and Φ^{-1} its inverse from $]0, 1[$ to $\mathbb{R}$. We note ϕ the derivative of Φ, i.e the density of the distribution function of Z.

Problem : [20pts] Risk Measures and Capital Allocation

For any vector $\pi \in \mathbb{R}^d$ we define : $RM_\lambda(\pi) = -\pi'(M - r_0 1_d) + \lambda\sqrt{\pi'\Sigma\pi}$ which is called the Markowitz risk measure of parameter λ for the risk exposure π. $RM_\lambda(\pi)$ can be interpreted as the capital required for a company to hold the risky positions defined by π.

1)

[0.5pt] a) what is the relationship between π^0 and π for an investment portfolio Π of risky allocation π ?

Correction: for an investment portfolio the sum of the weight is 1, so $\pi^0 = 1 - \pi' 1_d$

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[0.5pt] b) express as a function of r_0, M and π the expected return for an investment portfolio Π of risky allocation π

Correction: $E[R_\Pi] = \pi^0 r_0 + \pi' M = (1 - \pi' 1_d) r_0 + \pi' M = r_0 + \pi'(M - r_0 1_d)$

[0.5pt] c) express as a function of π and Σ the standard deviation of the returns for an investment portfolio Π of risky allocation π

Correction: $Var[R_\Pi] = \pi' \Sigma \pi$ so, $\sigma(R_\Pi) = \sqrt{\pi' \Sigma \pi}$

[0.5pt] d) show that for any $\pi \in \mathbb{R}^d$ $RM_\lambda(\pi) = -E[R_\Pi - r_0] + \lambda \sigma(R_\Pi)$ where Π is the investment portfolio of risky allocation π

Correction: Trivial from the previous questions

For any vector $\pi \in \mathbb{R}^d$ we define the random variable $L_\pi = -\pi'(R - r_0 1_d)$ and for $\alpha \in]0,1[$ we define $VaR_\alpha(L_\pi)$ by: $VaR_\alpha(L_\pi) = \inf\{x, P(L_\pi \leq x) \geq \alpha\}$ which is called the value at risk for the risk exposure π.

2)

[0.5pt] a) express the law of L_π as a function of $E[R_\Pi]$, r_0 and $\sigma[R_\Pi]$ where Π is the investment portfolio of risky allocation π

Correction: $L_\pi \sim \mathcal{N}(-E[R_\Pi - r_0], \sigma^2[R_\Pi])$

[1pt] b) show that $P(L_\pi \leq VaR_\alpha(L_\pi)) = \alpha$

Correction: for a normal distribution the repartition function is a bijection from $\mathbb{R}$ to $]0,1[$ so $\exists! \beta$ such that $P(L_\pi \leq \beta) = \alpha$ and it is easy to check that $VaR_\alpha(L_\pi) = \beta$

[1pt] c) express $VAR_\alpha(L_\pi)$ as a function of $E[R_\Pi], r_0, \sigma(R_\Pi), \alpha$ and Φ

Correction: $P(L_\pi \leq VaR_\alpha(L_\pi)) = \alpha$

$\iff P(-E[R_\Pi - r_0] + \sigma(R_\Pi) Z \leq VaR_\alpha(L_\pi)) = \alpha$

$\iff P(Z \leq \frac{1}{\sigma(R_\Pi)}(E[R_\Pi - r_0] + VaR_\alpha(L_\pi))) = \alpha$

$\iff \Phi(\frac{1}{\sigma(R_\Pi)}(E[R_\Pi - r_0] + VaR_\alpha(L_\pi))) = \alpha$

$\iff VaR_\alpha(L_\pi) = -E[R_\Pi - r_0] + \Phi^{-1}(\alpha)\sigma(R_\Pi)$

[0.5pt] d) for which value of $\lambda(\alpha)$ do we have $\forall \pi \in \mathbb{R}^d, RM_\lambda(\pi) = VaR_\alpha(L_\pi)$

Correction: for $\lambda(\alpha) = \Phi^{-1}(\alpha)$

For any $\pi \in \mathbb{R}^d$ we define the quantity $E_\alpha(L_\pi) = E(L_\pi | L_\pi \geq VaR_\alpha(L_\pi))$ which is called the expected shortfall for the risk exposure π.

3)

[0.5pt] a) show that if $a \in \mathbb{R}$ and $Z \sim \mathcal{N}(0,1)$ then $E(Z | Z \geq a) = \frac{\phi(a)}{1-\Phi(a)}$

Correction: $E(Z|Z \geq a) = \int_a^{+\infty} z \frac{1}{\sqrt{2\pi}} exp(-\frac{z^2}{2}) \frac{1}{1-\Phi(a)} dz = \frac{1}{1-\Phi(a)}[-\frac{1}{\sqrt{2\pi}} exp(-\frac{z^2}{2})]_a^{+\infty} = \frac{\phi(a)}{1-\Phi(a)}$

[0.5pt] b) show that if $a \in \mathbb{R}$ and $X \sim \mathcal{N}(m, \sigma^2)$ then $E(X|X \geq a) = m + \frac{\phi(\frac{a-m}{\sigma})}{1-\Phi(\frac{a-m}{\sigma})}\sigma$

Correction: $E(X|X \geq a) = E(m + \sigma Z | m + \sigma Z \geq a) = m + \sigma E(Z | Z \geq \frac{a-m}{\sigma}) = m + \frac{\phi(\frac{a-m}{\sigma})}{1-\Phi(\frac{a-m}{\sigma})}\sigma$ from the previous question

[1.5pt] c) express $E_\alpha(L_\pi)$ as a function of $E[R_\Pi], r_0, \sigma[R_\Pi], \alpha, \phi$ and Φ

Correction: $L_\pi \sim \mathcal{N}(-E[R_\Pi - r_0], \sigma^2[R_\Pi])$
so if we define $m = -E[R_\Pi - r_0]$ and $\sigma = \sigma[R_\Pi]$ we get $E_\alpha(L_\pi) = m + \frac{\phi(\frac{a-m}{\sigma})}{1-\Phi(\frac{a-m}{\sigma})}\sigma$ with $a = m + \Phi^{-1}(\alpha)\sigma$ so, $E_\alpha(L_\pi) = m + \frac{\phi(\Phi^{-1}(\alpha))}{1-\Phi(\Phi^{-1}(\alpha))}\sigma = m + \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}\sigma$ $= -E[R_\Pi - r_0] + \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}\sigma[R_\Pi]$

[0.5pt] d) for which value of $\lambda(\alpha)$ do we have $\forall \pi \in \mathbb{R}^d, RM_{\lambda(\alpha)}(\pi) = E_\alpha(L_\pi)$

Correction: for $\lambda(\alpha) = \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$

In this section we consider the derivative $\frac{\partial RM_\lambda}{\partial e_i}(\pi)$ as a row vector representing the derivative, calculated at point π, of $RM_\lambda(\pi)$ in the direction of vector e_i.

4)

[1pt] a) show that $\forall \pi \in \mathbb{R}^d, RM_\lambda(\pi) \leq \sum_{i=1}^{d} RM_\lambda(\pi^i e_i)$

Correction: $\sum_{i=1}^{d} RM_\lambda(\pi_i e_i)$ is the sum of two terms :
the first term $\sum_{i=1}^{d} (\pi^i e_i)'(M - r_0 1_d) = \pi'(M - r_0 1_d)$ by linearity of the scalar product
the second term $\sum_{i=1}^{d} \lambda\sqrt{(\pi^i e_i)'\Sigma(\pi^i e_i)} = \lambda \sum_{i=1}^{d} \|\pi^i e_i\|_\Sigma \geq \lambda \|\sum_{i=1}^{d} \pi^i e_i\|_\Sigma$ by triangular inequality applied to the norm $\|.\|_\Sigma$ which proves the result.

[1pt] b) show that $\forall \pi \in \mathbb{R}^d, RM_\lambda(\pi) = \sum_{i=1}^{d} \pi^i \frac{\partial RM_\lambda}{\partial e_i}(\pi)$

Correction: $RM_\lambda(\pi) = -\pi'(M - r_0 1_d) + \lambda\|\pi\|_\Sigma$.
$RM_\lambda(\pi)$ is positive homogeneous of degree 1 as $\forall t > 0, RM_\lambda(t\pi) = tRM_\lambda(\pi)$ so Euler's formula proves the result

[1pt] c) how can you interpret the results $4a$) and $4b$) in terms of capital allocation and diversification effect ?

Correction: $RM_\lambda(\pi)$ is the capital required to hold the position π. Equation 4a) shows the diversification effect and that less capital is required to hold the positions on an aggregated basis than to hold them on a separate basis. Equation 4b) shows a way to allocate the capital between all the positions after taking into account the diversification effect

5)

[2pt] a) show that

$$\begin{cases} \inf_{\pi \in \mathbb{R}^d} RM_\lambda(\pi) = -\infty \text{ if } \lambda < \lambda_0 \text{ and} \\ \inf_{\pi \in \mathbb{R}^d} RM_\lambda(\pi) = 0 \text{ if } \lambda \geq \lambda_0 \end{cases}$$

and express λ_0 as a function of $M, r_0, 1_d$ and Σ

Correction: $RM_\lambda(\pi) = -\langle \Sigma\pi, M - r_0 1_d\rangle_{\Sigma^{-1}} + \lambda\|\Sigma\pi\|_{\Sigma^{-1}}$
For $\pi \in \mathbb{R}^d$ we use the decomposition $\Sigma\pi = \beta(M - r_0 1_d) + v$ where v orthogonal to $M - r_0 1_d$ for $\langle . \rangle_{\Sigma^{-1}}$ and $\beta \in \mathbb{R}$
then $RM_\lambda(\pi) = -\beta\|M - r_0 1_d\|^2_{\Sigma^{-1}} + \lambda\sqrt{\beta^2\|M - r_0 1_d\|^2_{\Sigma^{-1}} + \|v\|^2_{\Sigma^{-1}}}$

then if $\lambda < \|M - r_0 1_d\|_{\Sigma^{-1}}$ by taking $v = 0$ we get $\lim_{\beta \longrightarrow +\infty} RM_\lambda(\pi) = -\infty$
and if $\lambda \geq \|M - r_0 1_d\|_{\Sigma^{-1}}$ then $RM_\lambda(\pi) \geq 0$ and reaches its minimum value 0 for $\pi = 0$.
So the critical λ is $\lambda_0 = \|M - r_0 1_d\|_{\Sigma^{-1}}$

[0.5pt] b) show that if $\lambda > \lambda_0$ then $\forall \pi \in \mathbb{R}^d \setminus \{0\}, RM_\lambda(\pi) > 0$
Correction: The result is clear from the expression
$RM_\lambda(\pi) = -\beta\|M - r_0 1_d\|^2_{\Sigma^{-1}} + \lambda\sqrt{\beta^2\|M - r_0 1_d\|^2_{\Sigma^{-1}} + \|v\|^2_{\Sigma^{-1}}}$

[0.5pt] c) show that if $\pi \in \mathbb{R}^d$ such that $E(R_\Pi - r_0) > 0$
Correction: $M - r_0 1_d \neq 0 \implies \exists \pi \in \mathbb{R}^d$ such that $\pi'(M - r_0 1_d) \neq 0$ so either $\pi'(M - r_0 1_d) > 0$ or $-\pi'(M - r_0 1_d) > 0$ Q.E.D

From now on we consider that $\lambda > \lambda_0$ and for any $\pi \in \mathbb{R}^d \setminus \{0\}$ we define the Return on Risk-Adjusted Capital as the quantity $RORAC_\lambda(\pi) = \frac{-E(L_\pi)}{RM_\lambda(\pi)}$ and we call $\mathcal{D} = \{\pi^* \in \mathbb{R}^d \setminus \{0\}, RORAC_\lambda(\pi^*) = \sup_{\pi \in \mathbb{R}^d} RORAC_\lambda(\pi)\}$

[1.5pt] a) show that $\pi \in \mathbb{R}^d \setminus \{0\}$ maximizes $RORAC_\lambda(\pi)$ if and only if the investment portfolio Π of risky allocation π maximizes the Sharpe Ratio $\frac{E(R_\Pi - r_0)}{\sigma(R_\Pi)}$
Correction: $\arg\max_{\pi \in \mathbb{R}^d \setminus \{0\}} RORAC_\lambda(\pi) = \arg\max_{\pi \in \mathbb{R}^d \setminus \{0\}} \frac{E(R_\Pi - r_0)}{-E(R_\Pi - r_0) + \lambda\sigma(R_\Pi)}$
$= \arg\max_{\pi \in \mathbb{R}^d \setminus \{0\}} \frac{\frac{E(R_\Pi - r_0)}{\lambda\sigma(R_\Pi)}}{-\frac{E(R_\Pi - r_0)}{\lambda\sigma(R_\Pi)} + 1}$ so we search $= \arg\max_{x(\pi) = \frac{E(R_\Pi - r_0)}{\lambda\sigma(R_\Pi)} < 1} \frac{x(\pi)}{1 - x(\pi)}$ as $\frac{E(R_\Pi - r_0)}{\lambda\sigma(R_\Pi)} <$ 1. As we can find π such that $\frac{E(R_\Pi - r_0)}{\lambda\sigma(R_\Pi)} > 0$ the maximum will be positive and will be obtained for the positive values of $x(\pi)$ as close as possible to 1 so $\arg\max_{\pi \in \mathbb{R}^d \setminus \{0\}} RORAC_\lambda(\pi) = \arg\max_{\pi \in \mathbb{R}^d \setminus \{0\}} \frac{E(R_\Pi - r_0)}{\lambda\sigma(R_\Pi)}$

[1.5pt] b) using a) determine $\mathcal{D}$ and $\frac{E(R_{\Pi^*} - r_0)}{\sigma(R_{\Pi^*})}$ for $\pi^* \in \mathcal{D}$
Correction: we want to maximize $\frac{E(R_\Pi - r_0)}{\sigma(R_\Pi)} = \frac{\pi'(M - r_0 1_d)}{\sigma(R_\Pi)} = \frac{\langle \Sigma\pi, M - r_0 1_d\rangle_{\Sigma^{-1}}}{\|\Sigma\pi\|_{\Sigma^{-1}}}$.
If we consider $\pi \in \mathbb{R}^d$ and decompose $\Sigma\pi = \beta(M - r_0 1_d) + v$ where v orthogonal to $M - r_0 1_d$ for $\langle . \rangle_{\Sigma^{-1}}$ then $\frac{\langle \Sigma\pi, M - r_0 1_d\rangle_{\Sigma^{-1}}}{\|\Sigma\pi\|_{\Sigma^{-1}}} = \frac{\beta\|M - r_0 1_d\|_{\Sigma^{-1}}}{\sqrt{\beta^2\|M - r_0 1_d\|^2_{\Sigma^{-1}} + \|v\|_{\Sigma^{-1}}{}^2}}$ and the maximum is attained for $v = 0$ and $\beta > 0$ and has for value $\|M - r_0 1_d\|_{\Sigma^{-1}}$ so $\mathcal{D} = \{\beta\Sigma^{-1}(M - r_0 1_d), \beta > 0\}$ and $\forall \pi^* \in \mathcal{D}$, $\frac{E(R_{\Pi^*} - r_0)}{\sigma(R_{\Pi^*})} = \|M - r_0 1_d\|_{\Sigma^{-1}}$

[1pt] c) calculate $RORAC_\lambda(\pi^*)$ for $\pi^* \in \mathcal{D}$ as a function of λ and λ_0
$RORAC_\lambda(\pi) = \frac{-E(L_\pi)}{RM_\lambda(\pi)} = \frac{\pi'(M - r_0 1_d)}{-\pi'(M - r_0 1_d) + \lambda\sqrt{\pi'\Sigma\pi}}$
and $\pi^{*\prime}(M - r_0 1_d) = \beta\|M - r_0 1_d\|^2_{\Sigma^{-1}}$ and $\pi^{*\prime}\Sigma\pi^* = \beta^2\|M - r_0 1_d\|^2_{\Sigma^{-1}}$
so $RORAC_\lambda(\pi^*) = \frac{1}{\frac{\lambda}{\lambda_0} - 1}$

[2pt] d) for $\pi^* \in \mathcal{D}$ calculate $\frac{-E[L_{\pi^{*i} e_i}]}{\pi^{*i}\frac{\partial RM_\lambda}{\partial e_i}(\pi^*)}$ when $m^i \neq r_0$ and $\pi^{*i} \neq 0$
$-E[L_{\pi^{*i} e_i}] = (\pi^{*i} e_i)'(M - r_0 1_d) = \pi^{*i}(m^i - r_0)$ and
$\frac{\partial RM_\lambda}{\partial \pi}(\pi) = -(M - r_0 1_d)' + \lambda\frac{(\Sigma\pi)'}{\sqrt{\pi'\Sigma\pi}}$
as $\pi^* = \beta\Sigma^{-1}(M - r_0 1_d)$ with $\beta > 0$ and $\pi^{*\prime}\Sigma\pi^* = \beta^2\lambda_0^2$ we get

$\frac{\partial RM_\lambda}{\partial \pi}(\pi^*) = -(M - r_0 1_d)' + \lambda\beta\frac{(M-r_0 1_d)'}{\lambda_0\beta}$ and
$\pi^{*i}\frac{\partial RM_\lambda}{\partial e_i}(\pi^*) = -\pi^{*i}(m^i - r_0) + \frac{\lambda}{\lambda_0}\pi^{*i}(m^i - r_0)$
so, $\frac{-E[L_{\pi^{*i}e_i}]}{\pi^{*i}\frac{\partial RM_\lambda}{\partial e_i}(\pi^*)} = \frac{\pi^{*i}(m^i-r_0)}{\pi^{*i}(m^i-r_0)(-1+\frac{\lambda}{\lambda_0})} = \frac{1}{-1+\frac{\lambda}{\lambda_0}}$