Master M1 - Mido 28th October 2019

Correction Exam: Portfolio Management [1]: 2h

Notations: We consider a risk-free asset S^0 of risk-free rate r_0 and d risky assets S^1, $S^2, \cdots, S^d$ of vector of returns R which verifies :

$$R = M + \epsilon \text{ with } R = \begin{pmatrix} R^1 \\ \vdots \\ R^d \end{pmatrix}, \; M = \begin{pmatrix} m^1 \\ \vdots \\ m^d \end{pmatrix} \text{ et } \epsilon = \begin{pmatrix} \epsilon^1 \\ \vdots \\ \epsilon^d \end{pmatrix}$$

where M is a vector of $\mathbf{R}^d$, and ϵ is a Gaussian vector of expectation zero and of variance-covariance matrix Σ invertible.
We note,

$$\Pi = \begin{pmatrix} \pi_0 \\ \pi \end{pmatrix} \text{ a portfolio allocation for which,}$$

π_0 is the allocation in the risk-free asset S^0 and

$$\pi = \begin{pmatrix} \pi^1 \\ \vdots \\ \pi^d \end{pmatrix} \text{ is the allocation in the risky assets } S^i.$$

$R(\Pi)$ is the return of the portfolio of allocation Π, m_Π its expectation and σ_Π its standard deviation.

1_d is the vector of $\mathbf{R}^d$ with all components equal to 1.
An investment portfolio Π satisfies, $\pi_0 + \pi' 1_d = 1$
A self-financing portfolio Π satisfies, $\pi_0 + \pi' 1_d = 0$
We note, $a = 1_d' \Sigma^{-1} 1_d$ and $b = 1_d' \Sigma^{-1} M$.
For any function $L(\pi)$ we note $\frac{\partial L}{\partial \pi}$ the raw vector $(\frac{\partial L}{\partial \pi^1}, \cdots, \frac{\partial L}{\partial \pi^d})$

Exercise 1: [5pts]

1. For an investment portfolio Π, calculate as a function of π, R, r_0, M and Σ the quantities
   (a) **[0.5pt]** $R(\Pi)$,
   (b) **[0.5pt]** $\mathbf{E}(R(\Pi))$,
   (c) **[0.5pt]** $\mathbf{Var}(R(\Pi))$.

   Correction:
   $R(\Pi) = r_0 + \pi'(R - r_0 1_d)$
   $\mathbf{E}(R(\Pi)) = r_0 + \pi'(M - r_0 1_d)$
   $\mathbf{Var}(R(\Pi)) = \pi' \Sigma \pi$

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2. For a self-financing portfolio Π, calculate as a function of π, R, r_0, M and Σ the quantities

   (a) **[0.5pt]** $R(\Pi)$,

   (b) **[0.5pt]** $\mathbf{E}(R(\Pi))$,

   (c) **[0.5pt]** $\mathbf{Var}(R(\Pi))$.

3. **[1pt]** If Π_P, and Π_Q are two risky investment portfolios of return R_P and R_Q calculate $\min_{\alpha \in \mathbf{R}} \mathbf{Var}(R_P - \alpha R_Q)$ as a function of $\mathbf{Var}(R_P)$ and of the correlation ρ between R_P and R_Q.
   Correction:
   $\mathbf{Var}(R_P - \alpha R_Q) = \mathbf{Var}(R_P) + \alpha^2 \mathbf{Var}(R_Q) - 2\alpha \mathbf{Cov}(R_P, R_Q)$
   minimum when the derivatives cancels i.e $\alpha = \frac{\mathbf{Cov}(R_P, R_Q)}{\mathbf{Var}(R_Q)} = \rho \frac{\sigma_P}{\sigma_Q}$ and the minimum is $\sigma_P^2 + \rho^2 \sigma_P^2 - 2\rho^2 \sigma_P^2 = (1 - \rho^2)\sigma_P^2$

4. **[1pt]** If A is a matrix of $\mathbf{R}^{n \times d}$ and B is a matrix of $\mathbf{R}^{m \times d}$ prove that $\mathbf{Cov}(AR, BR) = A\Sigma B'$

   Correction:
   $\mathbf{Cov}(AR, BR) = \mathbf{E}(AR(BR)') - \mathbf{E}(AR)(\mathbf{E}(BR))'$
   $= A\mathbf{E}(RR')B' - A\mathbf{E}(R)(\mathbf{E}(R))'B'$
   $= A(\mathbf{E}(RR') - \mathbf{E}(R)(\mathbf{E}(R))')B'$
   $= A\Sigma B'$

Exercise: [5pts]

We consider the economy with only the d risky assets and we consider the optimisation problem,

$$(P_\lambda) \left\{ \begin{array}{l} \sup_{\pi \in \mathbf{R}^d} \mathbf{E}(R(\pi)) - \lambda \mathbf{Var}(R(\pi)) \\ \qquad \pi' 1_d = 1 \end{array} \right.$$

1. **[0.50pt]** Without any calculation explain why the sup of (P_λ) is reached. We note π_λ the π solution of (P_λ)
   Correction:
   Σ is inversible and $\lambda > 0$ so the negative quadratic terms $-\lambda \mathbf{Var}(R^\pi)$ is the most important term and tends to $-\infty$ when $\|\pi\|$ tend to $+\infty$. The Sup is to be searched therefore inside a closed ball of $\mathbf{R}^d$ which is compact and a function continuous on a compact reaches its extremum.

   We remind that the Lagrangian for (P_λ) is,
   $L_\lambda(\pi, \mu) = \mathbf{E}(R^\pi) - \lambda \mathbf{Var}(R^\pi) - \mu(\pi' 1_d - 1)$

2. **[1.00pt]** Show that, $\frac{\partial L_\lambda}{\partial \pi}(\pi, \mu) = 0 \Leftrightarrow \pi = \frac{1}{2\lambda}\Sigma^{-1}(M - \mu 1_d)$
   Correction:
   $L_\lambda(\pi, \mu) = M'\pi - \lambda \pi' \Sigma \pi - \mu(\pi' 1_d - 1) \Longrightarrow (\frac{\partial L_\lambda}{\partial \pi})'(\pi, \mu) = M - 2\lambda \Sigma \pi - \mu 1_d$ which proves the result.

3. **[1.00pt]** Calculate the expression of π_λ
Correction:
$\pi' 1_d = 1 \Rightarrow \frac{1}{2\lambda} 1_d' \Sigma^{-1}(M - \mu 1_d) = 1 \Rightarrow b - \mu a = 2\lambda \Rightarrow \mu = \frac{b-2\lambda}{a}$
$\Longrightarrow \pi_\lambda = \frac{1}{2\lambda}\Sigma^{-1}(M - \frac{b-2\lambda}{a} 1_d) = \frac{1}{a}\Sigma^{-1} 1_d + \frac{1}{2\lambda}\Sigma^{-1}(M - \frac{b}{a} 1_d)$.

4. **[1.00pt]** Express $\mathbf{E}(R(\pi_\lambda))$ as a function of a, b et λ et $\|M - \frac{b}{a} 1_d\|_{\Sigma^{-1}}$
Correction:
$\mathbf{E}(R(\pi_\lambda)) = M'\pi_\lambda = \frac{b}{a} + \frac{1}{2\lambda}\|M - \frac{b}{a} 1_d\|^2_{\Sigma^{-1}}$ as $1_d'\Sigma^{-1}(M - \frac{b}{a} 1_d) = 0$.

5. **[1.00pt]** Express $\mathbf{Var}(R(\pi_\lambda))$ as a function of a, b λ et $\|M - \frac{b}{a} 1_d\|_{\Sigma^{-1}}$.
Correction:
$\pi_\lambda' \Sigma \pi = \|\frac{1}{a} 1_d + \frac{1}{2\lambda}(M - \frac{b}{a} 1_d)\|^2_{\Sigma^{-1}} = \frac{1}{a} + \frac{1}{4\lambda^2}\|(M - \frac{b}{a} 1_d)\|^2_{\Sigma^{-1}}$.

6. **[0.50pt]** What is the value of $\lim_{\lambda \to +\infty} \pi_\lambda$? what can you say about of this portfolio ?

Correction:
$\frac{1}{a}$ the variance of the minimum variance portfolio.

Problem: [12pts]

If π is the risky allocation of a portfolio, different from the risk-free asset, the Diversification Ratio for π is defined as,

$$D(\pi) = \frac{\pi' \boldsymbol{\sigma}}{\sigma(\pi)}$$

where,
$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_d)'$ is the vector of the standard deviation of the assets returns and
$\sigma(\pi) = \sqrt{\pi' \Sigma \pi}$ is the standard deviation of the portfolio of risky allocation π.
let, Σ be the matrix of variance-covariance for the assets
Λ be the matrix of correlation for the returns of general teem $\rho_{i,j}$ and
$diag(\sigma_i)$ be the diagonal matrix of diagonal term i, σ_i
We note, $\langle ., . \rangle_{\Sigma^{-1}}$ the scalar product defined by $\langle x, y \rangle_{\Sigma^{-1}} = x'\Sigma^{-1} y$

1. **[0.5pt]** find $z(\pi)$ such that $\pi' \boldsymbol{\sigma} = \langle z(\pi), \boldsymbol{\sigma} \rangle_{\Sigma^{-1}}$ and $\pi' \Sigma \pi = \langle z(\pi), \Sigma \pi \rangle_{\Sigma^{-1}}$.
Correction:
$z = \Sigma \pi$

2. (Geometric method) Show that,

   (a) **[1pt]**

$$D(\pi^*) = \sup_{\sigma(\pi)=1} D(\pi) \Longleftrightarrow \pi^* = \frac{\Sigma^{-1}\boldsymbol{\sigma}}{\|\boldsymbol{\sigma}\|_{\Sigma^{-1}}}$$

   Correction:
   $\sigma(\pi) = 1 \Longrightarrow D(\sigma) = \langle \Sigma\pi, \boldsymbol{\sigma} \rangle_{\Sigma^{-1}}$ and the maximum is attained when the vectors are colinear i.e $\Sigma \pi^* = \alpha \boldsymbol{\sigma}$ with $\alpha > 0$. Now the condition $\sigma(\pi) = 1$ implies that $\alpha = \frac{1}{\|\boldsymbol{\sigma}\|_{\Sigma^{-1}}}$ and therefore $\pi^* = \frac{\Sigma^{-1}\boldsymbol{\sigma}}{\|\boldsymbol{\sigma}\|_{\Sigma^{-1}}}$

(b) **[1pt]**

$$D(\pi^*) = \sup_{\pi \in \mathbf{R}^d \setminus \{0\}} D(\pi) \Longleftrightarrow \exists \alpha > 0, \pi^* = \alpha \Sigma^{-1} \boldsymbol{\sigma}$$

Correction:
The sup is attained on all the spheres $\sigma(\pi) = \sigma$ with the same values, and all the sup are of the form $\pi^* = \alpha\Sigma^{-1}\boldsymbol{\sigma}$ according to a)

3. (Analytic method)

   (a) **[1pt]** For $\pi \in \mathbf{R}^d \setminus \{0\}$ calculate $\frac{\partial D}{\partial \pi}(\pi)$
   Correction:
   By composition of the derivatives, $\frac{\partial D}{\partial \pi}(\pi) = \frac{1}{\sigma^2(\pi)}\Big(\boldsymbol{\sigma}\sigma(\pi) - (\pi'\boldsymbol{\sigma})\frac{2\Sigma\pi}{2\sigma(\pi)}\Big)$.

   (b) **[1pt]** Find $\{\pi \in \mathbf{R}^d \setminus \{0\}, \frac{\partial D}{\partial \pi}(\pi) = 0\}$
   Correction:
   It is easy to see that $\frac{\partial D}{\partial \pi}(\pi) = 0 \Longrightarrow \Big(\pi'\boldsymbol{\sigma} \neq 0 \text{ and } \pi = \frac{\sigma^2(\pi)}{(\pi'\boldsymbol{\sigma})}\Sigma^{-1}\boldsymbol{\sigma}\Big)$
   so, π is of the form $\pi = \alpha\Sigma^{-1}\boldsymbol{\sigma}$.
   Conversely, it is easy to prove that if π is of the form $\pi_\alpha = \alpha\Sigma^{-1}\boldsymbol{\sigma}$ then π_α cancels the derivatives.

From now on you can admit that the solutions of

$$\sup_{\pi \in \mathbf{R}^d \setminus \{0\}} D(\pi)$$

are the vectors $\pi_\alpha = \alpha\Sigma^{-1}\boldsymbol{\sigma}$.

4. **[1pt]** Calculate $D(\pi_\alpha)$ for $\alpha > 0$.
   Correction:
   $\pi'_\alpha\boldsymbol{\sigma} = \alpha\boldsymbol{\sigma}'\Sigma^{-1}\Sigma = \alpha\|\boldsymbol{\sigma}\|^2_{\Sigma^{-1}}$ and
   $\sigma(\pi_\alpha) = \sqrt{\alpha^2\boldsymbol{\sigma}'\Sigma^{-1}\Sigma\Sigma^{-1}\boldsymbol{\sigma}} = \alpha\|\boldsymbol{\sigma}\|_{\Sigma^{-1}}$
   so, $D(\pi_\alpha) = \|\boldsymbol{\sigma}\|_{\Sigma^{-1}}$

5. **[1pt]** Establish the relationship between Σ, Λ and $diag(\sigma_i)$.
   Correction:
   $\Sigma = diag(\sigma_i)\Lambda diag(\sigma_i)$

6. **[1pt]** Show that, $\sup_{\pi \in \mathbf{R}^d \setminus \{0\}} D(\pi) = \sqrt{1'_d\Lambda^{-1}1_d}$.
   Correction:
   $D(\pi_\alpha)$ reaches the sup which therefore is equal to $\|\boldsymbol{\sigma}\|_{\Sigma^{-1}}$
   Now, $\|\boldsymbol{\sigma}\|^2_{\Sigma^{-1}} = \boldsymbol{\sigma}'\Sigma^{-1}\boldsymbol{\sigma}. = \boldsymbol{\sigma}' diag(\frac{1}{\sigma_i})\Lambda^{-1}diag(\frac{1}{\sigma_i})\boldsymbol{\sigma}$
   but, $diag(\frac{1}{\sigma_i})\boldsymbol{\sigma} = 1_d$ which proves the result.

7. (a) **[1pt]** Show that, Σ invertible $\Longrightarrow \forall i \neq j \rho_{i,j} \neq 1$.
   Correction:
   by the absurd, if we assume that $\rho_{i,j} = 1$ then,
   $\mathbf{Var}(R_i - \lambda R_j) = \sigma_i^2 + \lambda^2\sigma_j^2 - 2\lambda\sigma_i\sigma_j = (1 - \lambda\sigma_j)^2$ which cancels for $\lambda = \frac{\sigma_i}{\sigma_j}$ which contredicts that Σ in invertible.

(b) **[1pt]** Show that, $\forall \pi \in (\mathbf{R}^+)^d \setminus \{0\}, D(\pi) \geq 1$.
Correction:
$(\pi'\boldsymbol{\sigma})^2 = (\sum_{i=1}^{n} \pi_i\sigma_i)^2 = \sum_{i,j=1}^{n} \pi_i\pi_j\sigma_i\sigma_j$ and
$\sigma^2(\pi) = \sum_{i,j=1}^{n} \pi_i\pi_j \mathbf{Var}(R_i, R_j) \sum_{i,j=1}^{n} \pi_i\pi_j\rho_{i,j}\sigma_j\sigma_i$ and as
$0 \leq \pi_i\pi_j\sigma_i\sigma_j \leq \pi_i\pi_j\rho_{i,j}\sigma_j\sigma_i$ we get the result.

(c) **[0.5pt]** Show that, $\forall \pi \in (\mathbf{R}^+)^d \setminus \{0\}, D(\pi) = 1$ if and only if the allocation π is reduced to a single asset.
Correction:
$D(\pi) = 1 \iff \forall i, j\ \pi_i\pi_j\sigma_i\sigma_j = \pi_i\pi_j\rho_{i,j}\sigma_j\sigma_i$ but as Σ is invertible $\forall i \neq j,\ \rho_{i,j} < 1$. So, for the equality to be verified we need only one π_i to be non zero. Q.E.D.

8. (a) **[2pt]** Let π_P be the risky allocation of an investment portfolio P. Show that,

$$correl(R(\pi_\alpha), R(\pi_P)) = \frac{D(\pi_P)}{D(\pi_\alpha)}.$$

Correction:
If P is the risk-free asset the correlation is zero and the formula is valid.
If P is not the risk-free asset then $\pi_P \neq 0$ and we have,

$$correl(R(\pi_\alpha), R(\pi_P)) = \frac{\pi_P'\Sigma\pi_\alpha}{\sigma(\pi_P)\sigma(\pi_\alpha)}$$

as $\pi_P'\Sigma\pi_\alpha = \alpha\pi_P'\Sigma\Sigma^{-1}\boldsymbol{\sigma} = \alpha\pi_P'\boldsymbol{\sigma}$ we get,

$$correl(R(\pi_\alpha), R(\pi_P)) = \alpha\frac{\pi_P'\boldsymbol{\sigma}}{\sigma(\pi_P)\sigma(\pi_\alpha)} = \alpha\frac{D(\pi_P)}{\sigma(\pi_\alpha)} \quad (1)$$

Now, applying this formula to $\pi_P = \pi_\alpha$ we get,

$$correl(R(\pi_\alpha), R(\pi_\alpha)) = \alpha\frac{D(\pi_\alpha)}{\sigma(\pi_\alpha)}$$

which implies that $\alpha\frac{D(\pi_\alpha)}{\sigma(\pi_\alpha)} = 1$ and that $\alpha = \frac{\sigma(\pi_\alpha)}{D(\pi_\alpha)}$. Replacing now α in equation 1 we get,

$$correl(R(\pi_\alpha), R(\pi_P)) = \frac{\sigma(\pi_\alpha)}{D(\pi_\alpha)}\frac{D(\pi_P)}{\sigma(\pi_\alpha)} = \frac{D(\pi_P)}{D(\pi_\alpha)}$$

which finishes the proof.

(b) **[0.5pt]** Show that, for any risky asset $Si, i \in [\![1, d]\!]$

$$correl(R(\pi_\alpha), R_i) = \frac{1}{D(\pi_\alpha)}$$

Correction:
For a single risky asset allocation π we have $D(\pi) = 1$, which proves the result.