

**Master M1 - Mido 28th October 2019**

Exam: Portfolio Management <sup>1</sup>: 2h

**Notations:** We consider a risk-free asset  $S^0$  of risk-free rate  $r_0$  and  $d$  risky assets  $S^1, S^2, \dots, S^d$  of vector of returns  $R$  which verifies :

$$R = M + \epsilon \text{ with } R = \begin{pmatrix} R^1 \\ \vdots \\ R^d \end{pmatrix}, M = \begin{pmatrix} m^1 \\ \vdots \\ m^d \end{pmatrix} \text{ et } \epsilon = \begin{pmatrix} \epsilon^1 \\ \vdots \\ \epsilon^d \end{pmatrix}$$

where  $M$  is a vector of  $\mathbf{R}^d$ , and  $\epsilon$  is a Gaussian vector of expectation zero and of variance-covariance matrix  $\Sigma$  invertible.

We note,

$$\Pi = \begin{pmatrix} \pi_0 \\ \pi \end{pmatrix} \text{ a portfolio allocation for which,}$$

$\pi_0$  is the allocation in the risk-free asset  $S^0$  and

$$\pi = \begin{pmatrix} \pi^1 \\ \vdots \\ \pi^d \end{pmatrix} \text{ is the allocation in the risky assets } S^i.$$

$R(\Pi)$  is the return of the portfolio of allocation  $\Pi$ ,  $m_\Pi$  its expectation and  $\sigma_\Pi$  its standard deviation.

$1_d$  is the vector of  $\mathbf{R}^d$  with all components equal to 1.

An investment portfolio  $\Pi$  satisfies,  $\pi_0 + \pi'1_d = 1$

A self-financing portfolio  $\Pi$  satisfies,  $\pi_0 + \pi'1_d = 0$

We note,  $a = 1'_d \Sigma^{-1} 1_d$  and  $b = 1'_d \Sigma^{-1} M$ .

For any function  $L(\pi)$  we note  $\frac{\partial L}{\partial \pi}$  the row vector  $(\frac{\partial L}{\partial \pi^1}, \dots, \frac{\partial L}{\partial \pi^d})$

**Exercise 1: [5pts]**

1. For an investment portfolio  $\Pi$ , calculate as a function of  $\pi$ ,  $R$ ,  $r_0$ ,  $M$  and  $\Sigma$  the quantities

- (a) [0.5pt]  $R(\Pi)$ ,
- (b) [0.5pt]  $\mathbf{E}(R(\Pi))$ ,
- (c) [0.5pt]  $\mathbf{Var}(R(\Pi))$ .

2. For a self-financing portfolio  $\Pi$ , calculate as a function of  $\pi$ ,  $R$ ,  $r_0$ ,  $M$  and  $\Sigma$  the quantities

- (a) [0.5pt]  $R(\Pi)$ ,

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- (b) [0.5pt]  $\mathbf{E}(R(\Pi))$ ,
- (c) [0.5pt]  $\mathbf{Var}(R(\Pi))$ .
3. [1pt] If  $\Pi_P$ , and  $\Pi_Q$  are two risky investment portfolios of return  $R_P$  and  $R_Q$  calculate  $\min_{\alpha \in \mathbf{R}} \mathbf{Var}(R_P - \alpha R_Q)$  as a function of  $\mathbf{Var}(R_P)$  and of the correlation  $\rho$  between  $R_P$  and  $R_Q$ .
4. [1pt] If  $A$  is a matrix of  $\mathbf{R}^{n \times d}$  and  $B$  is a matrix of  $\mathbf{R}^{m \times d}$  prove that  $\mathbf{Cov}(AR, BR) = A \Sigma B'$

**Exercise: [6pts]**

We consider the economy with only the  $d$  risky assets and we consider the optimisation problem,

$$(P_\lambda) \begin{cases} \sup_{\pi \in \mathbf{R}^d} \mathbf{E}(R(\pi)) - \lambda \mathbf{Var}(R(\pi)) \\ \pi' \mathbf{1}_d = 1 \end{cases}$$

1. [1.00pt] Without any calculation explain why the sup is reached. We note  $\pi_\lambda$  the allocation which maximises  $(P_\lambda)$ .

We remind that the Lagrangian for  $(P_\lambda)$  is,  
 $L_\lambda(\pi, \mu) = \mathbf{E}(R(\pi)) - \lambda \mathbf{Var}(R(\pi)) - \mu(\pi' \mathbf{1}_d - 1)$

2. [1.00pt] Show that,  $\frac{\partial L_\lambda}{\partial \pi}(\pi, \mu) = 0 \Leftrightarrow \pi = \frac{1}{2\lambda} \Sigma^{-1}(M - \mu \mathbf{1}_d)$ .
3. [1.00pt] Calculate the expression of  $\pi_\lambda$ .
4. [1.00pt] Express  $\mathbf{E}(R(\pi_\lambda))$  as a function of  $a, b$  and  $\lambda$  and  $\|M - \frac{b}{a} \mathbf{1}_d\|_{\Sigma^{-1}}$
5. [1.00pt] Express  $\mathbf{Var}(R(\pi_\lambda))$  as a function of  $a, b, \lambda$  and  $\|M - \frac{b}{a} \mathbf{1}_d\|_{\Sigma^{-1}}$ .
6. [1.00pt] What is the value of  $\lim_{\lambda \rightarrow +\infty} \pi_\lambda$ ? what can you say about of this portfolio? Was the result predictable?

**Problem: [12pts]**

If  $\pi$  is the risky allocation of a portfolio, different from the risk-free asset, the Diversification Ratio for  $\pi$  is defined as,

$$D(\pi) = \frac{\pi' \boldsymbol{\sigma}}{\sigma(\pi)}$$

where,  
 $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_d)'$  is the vector of the standard deviation of the assets returns

and

$\sigma(\pi) = \sqrt{\pi' \Sigma \pi}$  is the standard deviation of the portfolio of risky allocation  $\pi$ .

Let,

$\Sigma$  be the matrix of variance-covariance for the returns of the assets

$\Lambda$  be the matrix of correlation for the returns, of general term  $\rho_{i,j}$  and

$diag(\sigma_i)$  be the diagonal matrix of diagonal term  $i$  the quantity  $\sigma_i$

We note,  $\langle \cdot, \cdot \rangle_{\Sigma^{-1}}$  the scalar product defined by  $\langle x, y \rangle_{\Sigma^{-1}} = x' \Sigma^{-1} y$

1. **[0.5pt]** Find  $z(\pi)$  such that  $\pi' \sigma = \langle z(\pi), \sigma \rangle_{\Sigma^{-1}}$  and  $\pi' \Sigma \pi = \langle z(\pi), \Sigma \pi \rangle_{\Sigma^{-1}}$ .
2. (Geometric method) Show that,

(a) **[1pt]**

$$D(\pi^*) = \sup_{\sigma(\pi)=1} D(\pi) \iff \pi^* = \frac{\Sigma^{-1} \sigma}{\|\sigma\|_{\Sigma^{-1}}}$$

(b) **[1pt]**

$$D(\pi^*) = \sup_{\pi \in \mathbf{R}^d \setminus \{0\}} D(\pi) \iff \exists \alpha > 0, \pi^* = \alpha \Sigma^{-1} \sigma$$

3. (Analytic method)

(a) **[1pt]** For  $\pi \in \mathbf{R}^d \setminus \{0\}$  calculate  $\frac{\partial D}{\partial \pi}(\pi)$ .

(b) **[1pt]** Find  $\{\pi \in \mathbf{R}^d \setminus \{0\}, \frac{\partial D}{\partial \pi}(\pi) = 0\}$ .

From now on you can admit that the solutions of

$$\sup_{\pi \in \mathbf{R}^d \setminus \{0\}} D(\pi)$$

are the vectors  $\pi_\alpha = \alpha \Sigma^{-1} \sigma$  with  $\alpha > 0$ .

4. **[0.5pt]** Calculate  $D(\pi_\alpha)$  for  $\alpha > 0$ .
5. **[1pt]** Establish the relationship between  $\Sigma$ ,  $\Lambda$  and  $diag(\sigma_i)$ .
6. **[1pt]** Show that,  $\sup_{\pi \in \mathbf{R}^d \setminus \{0\}} D(\pi) = \sqrt{1'_d \Lambda^{-1} 1_d}$ .
7. (a) **[1pt]** Show that,  $\Sigma$  invertible  $\implies \forall i \neq j \rho_{i,j} \neq 1$ .  
 (b) **[1pt]** Show that,  $\forall \pi \in (\mathbf{R}^+)^d \setminus \{0\}, D(\pi) \geq 1$ .  
 (c) **[0.5pt]** Show that,  $\forall \pi \in (\mathbf{R}^+)^d \setminus \{0\}, D(\pi) = 1$  if and only if the allocation  $\pi$  is reduced to a single asset.
8. (a) **[2pt]** Let  $\pi_P$  be the risky allocation of an investment portfolio  $P$ . Show that,

$$correl(R(\pi_\alpha), R(\pi_P)) = \frac{D(\pi_P)}{D(\pi_\alpha)}.$$

(b) **[0.5pt]** Show that, for any risky asset  $S^i, i \in \llbracket 1, d \rrbracket$

$$correl(R(\pi_\alpha), R^i) = \frac{1}{D(\pi_\alpha)}$$