Master M1 - Mido 29th October 2020
Exam: Portfolio Management [1]: 1h30

Exercise 1.[11pts]

We remind that if $X \sim \mathcal{N}(0,1)$ and $Z \sim \chi_2(n)$ are independent then

$$\frac{X}{\sqrt{Z/n}} \sim t(n)$$

where t is a Student law of parameter n

1. Let $R_1, R_2 \cdots R_n$, be the returns of an asset over n consecutive business days. Let R follows a law $\mathcal{N}(m, \sigma^2)$ and let us assume that the R_i are independent and with the same law as R.
   Let $\hat{m}_n = \frac{1}{n}\sum_{i=1}^{n} R_i$ and $\hat{\sigma}_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(R_i - \hat{m}_n)^2}$.

   (a) **[0.5pt]** Calculate $\mathbf{E}(\hat{m}_n)$ and $\mathbf{Var}(\hat{m}_n)$.

   (b) **[0.5pt]** what is the law of $\hat{m}_n$?

   (c) **[1pt]** calculate $\mathbf{E}[\hat{\sigma}_n^2]$.

   (d) **[0.5pt]** give without proof the law of $\hat{\sigma}_n^2$.

   (e) **[1pt]** give without proof the expression of a random variable T_n depending on $\hat{m}_n$, n, m and $\hat{\sigma}_n$ for which the law is a Student law $t(n-1)$.

   (f) **[1pt]** deduct from the previous questions a way to build a confidence interval at level 95% for m based on $\hat{m}_n$, $\hat{\sigma}_n$, n, and α_n, when α_n verifies $P(-\alpha_n < T_n < \alpha_n) = 95\%$

   (g) **[1pt]** how would you build a confidence interval at level 95% for m if σ is known?

   (h) **[0.5pt]** is the confidence interval obtained in f) very different from the confidence interval obtained in g) when n is large? explain briefly your answer.

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2. Let $R_1 = (R_1^P, R_1^Q), R_2 = (R_2^P, R_2^Q) \cdots R_n = (R_n^P, R_n^Q)$ be the returns of two assets P and Q over n consecutive business days. Let R follows a law $\mathcal{N}(M, \Sigma)$ in $\mathbb{R}^2$, with Σ invertible and let us assume that the R_i are independent and with the same law as R.

   (a) **[0.25pt]** Exhibit an estimator $\hat{M}_n$ of M such that $\mathbf{E}(\hat{M}_n) = M$

   (b) **[0.25pt]** calculate $\mathbf{Var}(\hat{M}_n)$

   (c) **[0.5pt]** what is the law of $\hat{M}_n$?

   (d) **[1pt]** exhibit the empirical estimator $\hat{\Sigma}_n$ for Σ such that:

   $$\mathbf{E}(\hat{\Sigma}_n) = \Sigma$$

   (e) **[2pts]** show that when $\hat{M}_n$ is the empirical estimator of M we have:

   $$(\hat{M}_n - M)'\Sigma^{-1}(\hat{M}_n - M) \sim \frac{1}{n}\chi_2(2)$$

   (f) **[1pt]** deduct from e) a way to build a 95% confidence domain $\mathcal{D}_n$ for M when Σ is known and β_n satisfies:

   $$P(\frac{1}{n}\chi_2(2) < \beta_n) = 95\%$$

Exercise 2: [9pts]

We consider an economy with risky assets only. Let R be the random vector of $\mathbb{R}^d$ of returns of the assets, π be an allocation and 1_d be the vector of $\mathbb{R}^d$ with all components equal to 1.

We assume that the expectation of R is M and that its matrix of variance-covariance is Σ.

1. (a) **[0.25pt]** How is called a portfolio π for which $\pi' 1_d = 1$?

   (b) **[0.25pt]** how is called a portfolio π for which $\pi' 1_d = 0$?

   (c) **[0.25pt]** express without any justification the return of a portfolio of allocation π as a function of π and R

   (d) **[0.25pt]** express without any justification the expected return of a portfolio of allocation π as a function of π and M

   (e) **[0.25pt]** express without any justification the variance of the returns a portfolio of allocation π as a function of π and Σ

(f) **[0.25pt]** express without any justification the covariance of the returns of two portfolios P and Q of allocations π_P and π_Q as a function of π_P, π_Q and Σ.

2. The aim here is to calculate the allocation between two risky assets P and Q which enables to build a portfolio of minimum variance of returns.

   (a) **[0.5pt]** Express $\mathbf{Var}(\pi R_P + (1-\pi)R_Q)$ as a function of π, σ_P, σ_Q and ρ the correlation between R_P and R_Q

   (b) **[1pt]** discuss $\min_{\pi\in\mathbb{R}} \mathbf{Var}(\pi R_P + (1-\pi)R_Q)$ for the particular case $\rho = 1$

   (c) **[0.5pt]** show that if $\rho \neq 1$, $\sup_{\pi\in\mathbb{R}} \mathbf{Var}(\pi R_P + (1-\pi)R_Q) = +\infty$

   (d) **[1pt]** calculate $\arg\min_{\pi\in\mathbb{R}} \mathbf{Var}(\pi R_P + (1-\pi)R_Q)$ when $\rho \neq 1$

   (e) **[1pt]** if you are obliged to hold x_P millions of asset P and are allowed to take an exposure to asset Q to reduce your risk (which is defined here as the variance of the final wealth of the portfolio formed by the assets P and Q), what position do you take in asset Q (as a function of x_P, σ_P, σ_Q and the correlation of the returns ρ) to reduce your risk to the minimum?

   (f) **[0.5pt]** is there a link between the questions d) and e)?

3. The aim of these questions is to study the implications of some properties of the matrix Σ of variance-covariance.

   (a) **[0.5pt]** Show that if Σ is not invertible it is possible to build with the risky assets either an investment portfolio or a self-financing portfolio which is risk-free

   (b) **[0.5pt]** show that if it is possible to build with the risky assets a portfolio which is risk-free then Σ is not invertible

   (c) **[1pt]** explain why, in the absence of arbitrage opportunity, any self-financing portfolio without risk should have a return of zero

   (d) **[1pt]** show that if it is possible to build a self-financing portfolio without risk and if there is no arbitrage then it is possible to find

$d - 1$ risky assets which can replicate all the investment and self-financing portfolios of the economy (implying that one of the risky asset is redundant).