Master M1 - Mido 29th October 2020
Exam: Portfolio Management [1]: 1h30

Exercise 1.[11pts]

We remind that if $X \sim \mathcal{N}(0,1)$ and $Z \sim \chi_2(n)$ are independent then

$$\frac{X}{\sqrt{Z/n}} \sim t(n)$$

where t is a Student law of parameter n

1. Let $R_1, R_2 \cdots R_n$, be the returns of an asset over n consecutive business days. Let R follows a law $\mathcal{N}(m, \sigma^2)$ and let us assume that the R_i are independent and with the same law as R.
   Let $\hat{m}_n = \frac{1}{n}\sum_{i=1}^{n} R_i$ and $\hat{\sigma}_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(R_i - \hat{m}_n)^2}$

   (a) **[0.5pt]** Calculate $\mathbf{E}(\hat{m}_n)$ and $\mathbf{Var}(\hat{m}_n)$
   Correction: $\mathbf{E}(\hat{m}_n) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}(R_i) = m$
   $\mathbf{Var}(\hat{m}_n) = \frac{1}{n^2}\sum_{i=1}^{n}\mathbf{Var}(R_i)$ (because the R_i are independent) $= \frac{\sigma^2}{n}$

   (b) **[0.5pt]** what is the law of $\hat{m}_n$?
   Correction: As the R_i are independent Gaussian variables, any linear combination is also a Gaussian variable. So, $\hat{m}_n \sim \mathcal{N}(m, \frac{\sigma^2}{n})$.

   (c) **[1pt]** calculate $E[\hat{\sigma}_n^2]$
   Correction: $\mathbf{E}((R_i - \hat{m}_n)^2) = \mathbf{E}(((R_i - m) - (\hat{m}_n - m))^2)$
   $= \mathbf{E}((R_i - m)^2 + (\hat{m}_n - m)^2 - 2(R_i - m)(\hat{m}_n - m))$
   now,
   $\mathbf{E}((R_i - m)^2) = \sigma^2$,
   $\mathbf{E}((\hat{m}_n - m)^2) = \frac{1}{n^2}\sum_i\sum_j \mathbf{E}((R_i - m)(R_j - m)) = \frac{\sigma^2}{n}$
   $\mathbf{E}(((R_i - m)(\hat{m}_n - m)) = \mathbf{E}((R_i - m)\frac{(R_i - m)}{n}) = \frac{\sigma^2}{n}$
   so, $\mathbf{E}((R_i - \hat{m}_n)^2) = \sigma^2 + \frac{\sigma^2}{n} - 2\frac{\sigma^2}{n} = \sigma^2 - \frac{\sigma^2}{n}$
   so, $E[\hat{\sigma}_n^2] = \frac{n-1}{n}\sigma^2$.

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(d) **[0.5pt]** gives without proof the law of $\hat{\sigma}_n^2$
Correction: $\frac{n}{\sigma^2}\hat{\sigma}_n^2 \sim \chi_2(n-1)$ or equivalently $\hat{\sigma}_n^2 \sim \frac{\sigma^2}{n}\chi_2(n-1)$

(e) **[1pt]** give without proof the expression of a random variable T_n depending on $\hat{m}_n$, n, m and $\hat{\sigma}_n$ for which the law is a student law $t(n-1)$
Correction: $\hat{m}_n \sim \mathcal{N}(m, \frac{\sigma^2}{n})$ and $\frac{n}{\sigma^2}\hat{\sigma}_n^2 \sim \chi_2(n-1)$ and we know that the two variables are independent, therefore, $\frac{\sqrt{n}}{\sigma}(\hat{m}_n - m) \sim \mathcal{N}(0,1)$ and if we note $T_n = \frac{\frac{\sqrt{n}}{\sigma}(\hat{m}_n - m)}{\frac{\hat{\sigma}_n}{\sigma}\sqrt{n/(n-1)}}$ we get $T_n \sim t(n-1)$ and after simplification

$$T_n = \frac{\sqrt{n-1}(\hat{m}_n - m)}{\hat{\sigma}_n}$$

(f) **[1pt]** deduct from the previous questions a way to build a confidence interval at level 95% for m based on $\hat{m}_n$, $\hat{\sigma}_n$, n, and α_n, when α_n verifies $P(-\alpha_n < T_n < \alpha_n) = 95\%$
Correction: $P(-\alpha_n < T_n < \alpha_n) = 95\%$
$\Longrightarrow P(-\alpha_n < \frac{\sqrt{n-1}(\hat{m}_n - m)}{\hat{\sigma}_n} < \alpha_n) = 95\%$
$\Longrightarrow P(\hat{m}_n - \alpha_n\frac{\hat{\sigma}_n}{\sqrt{n-1}} < m < \hat{m}_n + \alpha_n\frac{\hat{\sigma}_n}{\sqrt{n-1}}) = 95\%$
$\Longrightarrow P(m \in]\hat{m}_n - \alpha_n\frac{\hat{\sigma}_n}{\sqrt{n-1}}, \hat{m}_n + \alpha_n\frac{\hat{\sigma}_n}{\sqrt{n-1}}[) = 95\%$
so, $]\hat{m}_n - \alpha_n\frac{\hat{\sigma}_n}{\sqrt{n-1}}, \hat{m}_n + \alpha_n\frac{\hat{\sigma}_n}{\sqrt{n-1}}[$ is a confidence interval at level 95% for m.

(g) **[1pt]** how would you build a confidence interval at level 95% for m if σ was known?
Correction: using the fact that $\hat{m}_n \sim \mathcal{N}(m, \frac{\sigma^2}{n})$ we get that $\frac{\sqrt{n}}{\sigma}(\hat{m}_n - m) \sim \mathcal{N}(0,1)$ and consequently
$P(-1.96 < \frac{\sqrt{n}}{\sigma}(\hat{m}_n - m) < 1.96) = 95\%$ and therefore,
$P(m \in]\hat{m}_n - 1.96\frac{\sigma}{\sqrt{n}}, \hat{m}_n + 1.96\frac{\sigma}{\sqrt{n}}[) = 95\%$
so, $]\hat{m}_n - 1.96\frac{\sigma}{\sqrt{n}}, \hat{m}_n + 1.96\frac{\sigma}{\sqrt{n}}[$ is a confidence interval at level 95%

(h) **[0.5pt]** is the confidence interval obtained in f) very different from the confidence interval obtained in g) when n is large? explain briefly your answer.
Correction: The confidence intervals will look similar because

$t(n-1)$ converges toward a $\mathcal{N}(0,1)$ when n tends to infinity and $\frac{\hat{\sigma}_n}{\sqrt{n-1}} \sim \frac{\sigma}{\sqrt{n}}$ when n tends to infinity .

2. Let $R_1 = (R_1^P, R_1^Q), R_2 = (R_2^P, R_2^Q) \cdots R_n = (R_n^P, R_n^Q)$ be the returns of two assets P and Q over n consecutive business days. Let R follows a law $\mathcal{N}(M, \Sigma)$ in $\mathbb{R}^2$, with Σ invertible and let us assume that the R_i are independent and with the same law as R.

   (a) **[0.25pt]** Exhibit and estimator $\hat{M}_n$ for M such that $E(\hat{M}_n) = M$
   Correction: $\hat{M}_n = \frac{1}{n}\sum_{i=1}^{n} R_i$ satisfies the condition

   (b) **[0.25pt]** calculate $Var(\hat{M}_n)$
   Correction: the R_i are independent so,
   $\mathbf{Var}(\hat{M}_n) = \frac{1}{n^2}\sum_{i=1}^{n} \mathbf{Var}(R_i) = \frac{1}{n}\Sigma$

   (c) **[0.5pt]** what is the law of $\hat{M}_n$?
   Correction: $\hat{M}_n$ is Gaussian as it is a linear combination of independent Gaussian variables so $\hat{M}_n \sim \mathcal{N}(M, \frac{1}{n}\Sigma)$

   (d) **[1pt]** exhibit an empirical estimator $\hat{\Sigma}_n$ for Σ such that:

   $$E(\hat{\Sigma}_n) = \Sigma$$

   Correction: Let $\hat{\Sigma}_n = \frac{1}{n-1}\sum_{i=1}^{n}(R_i - \hat{M})'(R_i - \hat{M})$ then $E(\hat{\Sigma}_n) = \Sigma$

   (e) **[2pt]** show that when $\hat{M}_n$ is the empirical estimator of M we have:

   $$(\hat{M}_n - M)'\Sigma^{-1}(\hat{M}_n - M) \sim \frac{1}{n}\chi_2(2)$$

   Correction: Let Δ be symmetric such that $\Delta'\Delta = \Sigma^{-1}$ then $(\hat{M}_n - M)'\Sigma^{-1}(\hat{M}_n - M) = Z_n' Z_n$ where $Z_n = \Delta(\hat{M}_n - M)$.
   Now, Z_n is Gaussian of expectation zero and matrix of variance covariance $\Delta\frac{1}{n}\Sigma\Delta'$ but $\Sigma = \Delta^{-1}(\Delta')^{-1}$ therefore $Z_n \sim \frac{1}{\sqrt{n}}\mathcal{N}(0, Id_2)$ and therefore $Z_n' Z_n \sim \frac{1}{n}\chi_2(2)$

(f) **[1pt]** deduct from e) a way to build a 95% confidence domain $\mathcal{D}_n$ for M when Σ is known and β_n satisfies:

$$P(\frac{1}{n}\chi_2(2) < \beta_n) = 95\%$$

Correction: Let $\mathcal{D}_n = \{x \in \mathbb{R}^2, (\hat{M}_n - x)'\Sigma^{-1}(\hat{M}_n - x) < \beta_n\}$
then $P(M \in \mathcal{D}_n) = P((\hat{M}_n - M)'\Sigma^{-1}(\hat{M}_n - M) < \beta_n))$
$= P(\frac{1}{n}\chi_2(2) < \beta_n) = 95\%$
so, $\mathcal{D}_n$ is a confidence interval for M at level 95%.

Exercise 2: [9pts]

We consider an economy with risky assets only. Let R be the random vector of $\mathbb{R}^d$ of returns of the assets, π be an allocation and 1_b be the vector of $\mathbb{R}^d$ with all components equal to 1.
We assume that the expectation of R is M and that its matrix of variance-covariance is Σ.

1. (a) **[0.25pt]** how is called a portfolio π for which $\pi' 1_d = 1$?
Correction: an investment portfolio.

   (b) **[0.25pt]** how is called a portfolio π for which $\pi' 1_d = 0$?
Correction: a self-financing portfolio.

   (c) **[0.25pt]** express without any justification the return of a portfolio of allocation π as a function of π and R
Correction: $\pi' R$.

   (d) **[0.25pt]** express without any justification the expected return of a portfolio of allocation π as a function of π and M
Correction: $\pi' M$.

   (e) **[0.25pt]** express without any justification the variance of the returns a portfolio of allocation π as a function of π and Σ
Correction: $\pi'\Sigma\pi$.

   (f) **[0.25pt]** express without any justification the covariance of the returns of two portfolios P and Q of allocations π_P and π_Q as a function of π_P, π_Q and Σ
Correction: $\pi_P'\Sigma\pi_Q$.

2. The aim here is to calculate the allocation between two risky assets P and Q which enables to build a portfolio of minimum variance of returns.

   (a) **[0.5pt]** calculate the expression of $\mathbf{Var}(\pi R_P + (1-\pi)R_Q)$ as a function of π, σ_P, σ_Q and ρ the correlation between R_P and R_Q
   Correction:
   $\mathbf{Var}(\pi R_P + (1-\pi)R_Q) = \pi^2\sigma_P^2 + (1-\pi)^2\sigma_Q^2 + 2\pi(1-\pi)\rho\sigma_P\sigma_Q$.

   (b) **[1pt]** discuss $\min_{\pi\in\mathbb{R}} \mathbf{Var}(\pi R_P + (1-\pi)R_Q)$ for the particular case $\rho = 1$
   Correction: if $\rho = 1$,
   $\mathbf{Var}(\pi R_P + (1-\pi)R_Q) = (\pi\sigma_P + (1-\pi)\sigma_Q)^2 = (\pi(\sigma_P - \sigma_Q) + \sigma_Q)^2$.
   so,
   ∘ if $\sigma_P = \sigma_Q$ the **Var** is constant for all values of π and equal to $\sigma_P = \sigma_Q$
   ∘ if $\sigma_P \neq \sigma_Q$ the **Var** cancels and therefore reaches the minimum value of zero for $\pi(\sigma_P - \sigma_Q) + \sigma_Q = 0$ i.e $\pi = \frac{\sigma_Q}{\sigma_Q - \sigma_P}$.

   (c) **[0.5pt]** show that if $\rho \neq 1$, $\sup_{\pi\in\mathbb{R}} \mathbf{Var}(\pi R_P + (1-\pi)R_Q) = +\infty$
   Correction: The term in π^2 in $\mathbf{Var}(\pi R_P + (1-\pi)R_Q)$ is,
   $\sigma_P^2 + \sigma_Q^2 - 2\rho\sigma_P\sigma_Q = (\sigma_P - \sigma_Q)^2 + 2(1-\rho)\sigma_P\sigma_Q$. So if $\rho \neq 1$ the coefficient is strictly positive and therefore the **Var** when $|\pi|$ tends to $+\infty$ tends to $+\infty$. Q.E.D.

   (d) **[1pt]** calculate $\arg\min_{\pi\in\mathbb{R}} \mathbf{Var}(\pi R_P + (1-\pi)R_Q)$ when $\rho \neq 1$
   Correction: The derivatives in π is $2\pi\sigma_P^2 - 2(1-\pi)\sigma_Q^2 + 2\rho\sigma_P\sigma_Q - 4\pi\rho\sigma_P\sigma_Q$ and is equal to zero if and only if $\pi(\sigma_P^2 + \sigma_Q^2 - 2\rho\sigma_P\sigma_Q) + \rho\sigma_P\sigma_Q - \sigma_Q^2 = 0$. The coefficient of π is not zero here as $\rho \neq 1$ and the derivative cancels for $\pi = \frac{\sigma_Q(\sigma_Q - \rho\sigma_P)}{\sigma_P^2 + \sigma_Q^2 - 2\rho\sigma_P\sigma_Q}$ which necessarily corresponds to a minimum as the function tends to infinity when $|\pi|$ tends to ∞. Q.E.D.

   (e) **[1pt]** if you are obliged to hold x_P millions of asset P and are allowed to take an exposure to asset Q to reduce your risk (which is defined as the variance of the final wealth of the portfolio formed by the assets P and Q), what position do you take in asset Q (as a function of x_P, σ_P, σ_Q and the correlation of the returns ρ) to reduce your risk to the minimum?

Correction: the quantity to minimize here is $\mathbf{Var}(x_P(1+R_P)+x_Q(1+R_Q))$. So we minimise in x_Q the quantity: $x_p^2\sigma_P^2+x_Q^2\sigma_Q^2+2\rho x_P x_Q\sigma_P\sigma_Q$ and by derivation the minimum is obtained for $x_Q=-\rho x_P\frac{\sigma_P}{\sigma_Q}$

(f) **[0.5pt]** is there a link between the questions d and e?
Correction: in questions d the notional of the portfolio hedged is fixed whereas in question e only the notional of the asset P is fixed, therefore the results differ.

3. The aim of these questions is to study the implications of some properties of the matrix Σ of variance-covariance.

(a) **[0.5pt]** Show that if Σ is not invertible it is possible to build with the risky assets either an investment portfolio or a self-financing portfolio which is risk-free
Correction: Σ not invertible $\Longrightarrow \exists x\in\mathbb{R}^d$ such that $\Sigma x=0$
∘ if $1_d'x=0$ then x represents a self-financing portfolio and $\Sigma x=0\Longrightarrow x'\Sigma x=0\Longrightarrow$ the self-financing portfolio x is without risk
∘ if $1_d'x\neq 0$ then $\pi=\frac{x}{1_d'x}$ is an investment portfolio which verifies $\pi'\Sigma\pi=0$ and which therefore is without risk.

(b) **[0.5pt]** show that if it is possible to build with the risky assets a portfolio which is risk-free then Σ is not invertible
Correction: a portfolio π without risk verifies $\pi'\Sigma\pi=0$ which implies that Σ is not invertible (a portfolio π by definition is not null.)

(c) **[1pt]** explain why, in the absence of arbitrage opportunity, any self-financing portfolio without risk should have a return of zero.
Correction: If it was possible to build a self-financing portfolio without risk with a non zero return c_0 then it would be possible to generate money for sure out of nothing by getting long (if $c_0>0$) or short (if $c_0<0$) this self-financing portfolio.

(d) **[1pt]** show that if it is possible to build a self-financing portfolio without risk and if there is no arbitrage then it is possible to find $d-1$ risky assets which can replicate all the investment and self-financing portfolios of the economy (implying that one of the risky asset is redundant).

Correction: Let β be a self-financing portfolio such that $\beta' R = 0$. We can assume without loss of generality that $\beta_1 = 1$ and we are going to show that the risky asset 1 is redundant. Let π be a portfolio. $\pi' R = \pi_1 R_1 + \sum_{i>1} \pi_i R_i = \pi_1 (-\sum_{i>1} \beta_i R_i) + \sum_{i>1} \pi_i R_i$ and the sum of the coefficients of this portfolio made of all the assets except the first one is still equal to $\pi' 1_d$. So, the returns of any investment or self-financing portfolio made of the d assets can be expressed as the returns of portfolio made of all the assets excluding asset 1. Q.E.D.