Stochastic control (course 2)

Chap I Feynman - Kac

- SDE: in $\mathbb{R}^d$ $(d \geq 1)$

$$\begin{cases} dX_s = b(s, X_s)\,ds + \sigma(s, X_s)\,dW_s \\ X_t = x \end{cases}$$

(H1)
- $(t, x) \in [0,T] \times \mathbb{R}^d$ initial condition
- $(W_s)_{0 \leq s \leq T}$ Brownian motion, m-d.
- $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$
- $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ } continuous
- Hyp1 $\exists L > 0$ s.t. $\forall s \in [0,T]$, $\forall x, y \in \mathbb{R}^d$

$$|b(s,x) - b(s,y)| + |\sigma(s,x) - \sigma(s,y)| \leq L|x-y|$$

Theo (Cauchy - Lipschitz)
$\exists!$ sol $(X_s^{t,x})_{0 \leq s \leq T}$ sol to (SDE)

ie
$$X_s^{t,x} = x + \int_t^s b(u, X_u^{t,x})\,du + \int_t^s \sigma(u, X_u^{t,x})\,dW_u \qquad \forall s \in [t,T].$$

- Cost.

$$v(t,x) = \mathbb{E}\Big[\underbrace{\int_t^T f(s, X_s^{t,x})\,ds}_{\text{running cost}} + \underbrace{g}_{\text{terminal cost}}(X_T^{t,x}) \Big]$$

where: $\cdot\ f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$
(H2) $\quad \cdot\ g : \mathbb{R}^d \to \mathbb{R}$ $\Big\}$ continuous & bounded

and $|f(s,x) - f(s,y)| + |g(x) - g(y)| \leq L|x-y|$

$$\forall s \in [0,T],\ x, y \in \mathbb{R}^d.$$

Lemma 1: v is Lipschitz in x uniformly in t.

because:

- $\exists\, C > 0$ s.t.

$$\mathbb{E}\Big[\sup_{s \in [t,T]} |X_s^{t,x} - X_s^{t,y}|^2 \Big] \leq C|x-y|^2.$$

(see course 1)

- So

$$|v(t,x) - v(t,y)| \leq \mathbb{E}\Big[\int_t^T |f(s, X_s^{t,x}) - f(s, X_s^{t,y})|\,ds + |g(X_T^{t,x}) - g(X_T^{t,y})| \Big]$$

$$\leq \mathbb{E}\Big[L \int_t^T |X_s^{t,x} - X_s^{t,y}|\,ds + L\,|X_T^{t,x} - X_T^{t,y}| \Big]$$

$$\leq L(T+1)\, \mathbb{E}\Big[\sup_{0\leq s\leq T} |X_s^{t,x} - X_s^{t,y}|\Big]$$

$$\leq L(T+1)\, C^{1/2} |x-y| . \qquad \square$$

Lemma 2: u is $\frac{1}{2}$ Hölder continuous in t independently of x.

Proof: $0 \leq t_1 < t_2 \leq T$,

$$|u(t_1, x) - u(t_2, x)| =$$

$$\Big| \mathbb{E} \int_{t_1}^{T} f(s, X_s^{t_1,x})\, ds + g(X_T^{t_1,x}) - \int_{t_2}^{T} f(s, X_s^{t_2,x})\, ds - g(X_T^{t_2,x}) \Big|$$

$$\leq \mathbb{E}\Big[\int_{t_1}^{t_2} |f(s, X_s^{t_1,x})|\, ds\Big] + \mathbb{E}\Big[\int_{t_2}^{T} |f(s, X_s^{t_1,x}) - f(s, X_s^{t_2,x})|\, ds\Big] + \mathbb{E}\Big[\, |g(X_T^{t_1,x}) - g(X_T^{t_2,x})|\Big]$$

$$\leq \|f\|_\infty (t_2 - t_1) + L(T+1)\, \mathbb{E}\Big[\sup_{t_2\leq s\leq T} |X_s^{t_1,x} - X_s^{t_2,x}|\Big]$$

$$\leq \|f\|_\infty (t_2 - t_1) + L(T+1)\, C\, |t_2 - t_1|^{1/2}$$

$$\leq \Big(\|f\|_\infty T^{1/2} + L(T+1) C \Big) |t_2 - t_1|^{1/2} .$$

Lemma 3: u is bd by

$$|u(t,x)| \leq (T\|f\|_\infty + \|g\|_\infty).$$

Theo (Dynamic programming) Fix $t\in[0,T]$. For any stopping time $\tau\in[t,T]$, we have

$$u(t,x) = \mathbb{E}\left[\int_t^\tau f(s, X_s^{t,x})ds + u(\tau, X_\tau^{t,x})\right]$$

Proof: $f\equiv 0$, $\tau = \underbrace{t_2}_{\text{deterministic.}} \in [t,T]$

$$u(t,x) = \mathbb{E}\left[g(X_T^{t,x})\right]$$

$$= \mathbb{E}\left[g(X_T^{t_2, X_{t_2}^{t,x}})\right]$$

$$\underset{\text{(Markov prty)}}{=} \mathbb{E}\left[\mathbb{E}\left[g(X_T^{t_2, X_{t_2}^{t,x}}) \,\middle|\, \mathcal{F}_{t_2}\right]\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[g(X_T^{t_2,y})\right]_{y = X_{t_2}^{t,x}}\right]$$

$$= \mathbb{E}\left[u(t_2, X_{t_2}^{t,x})\right].$$

- Set $\mathcal{L}\varphi(t,x) =$
$$b(t,x)\cdot\nabla\varphi(t,x) + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^T(t,x)D^2\varphi(t,x)\right)$$

Theo (Feynman-Kac 1)

Assume that
$$u(t,x) = \mathbb{E}\left[\int_t^T f(s,X_s^{t,x})\,ds + g(X_T^{t,x})\right]$$
is $C^{1,2}$.

Then u solves
$$\begin{cases} \partial_t u(t,x) + \mathcal{L}u(t,x) + f(t,x) = 0 \quad \text{in } [0,T)\times\mathbb{R}^d \\ u(T,x) = g(x) \quad \text{in } \mathbb{R}^d \end{cases}$$

Ex: $d=1$, $b\equiv 0$, $\sigma\equiv 1$, $f\equiv 0$

$$\begin{cases} dX_s = dW_s \\ X_t = x \end{cases} \Rightarrow X_s = x + W_s - W_t.$$

If $u(t,x) = \mathbb{E}\left[g(x+W_T-W_t)\right]$ is $C^{1,2}$, then u solves the heat eq:
$$\begin{cases} \partial_t u + \frac{1}{2}\partial_{xx}u = 0 \quad \text{in } (0,T)\times\mathbb{R} \\ u(T,x) = g(x) \quad \text{in } \mathbb{R}. \end{cases}$$

Proof of F-K formula 1:

- Hyp #. $f \equiv 0$, v is $C_b^{1,2}$ (v, $\partial_t v$, ∇v, $D^2 v$ bd)

Fix $t \in [0,T)$, $h>0$ small,

use the dynamic programming:

$$\cancel{v(t,x)} = \mathbb{E}\left[v(t+h, X_{t+h}^{t,x}) \right]$$

$$\overset{=}{\text{Itô's formula}} = \mathbb{E}\left[\cancel{v(t,x)} + \int_t^{t+h} (\partial_t v + \mathcal{L} v)(s, X_s) ds \right.$$
$$\left. + \int_t^{t+h} \cancel{\nabla v(s, X_s^{t,x}) \sigma(s, X_s)} dW_s \right]$$

$$0 = \mathbb{E}\left[\int_t^{t+h} (\partial_t v + \mathcal{L} v)(s, X_s) ds \right]$$

Divide by $h>0$ and let $h \to 0^-$

Then $X_s^{t,x} \to x$

$$\Rightarrow \quad 0 = (\partial_t v + \mathcal{L} v)(t,x).$$

- General case ($f \not\equiv 0$, $v \in C^{1,2}$).

Let $\tau = \inf\{ s \in [t,T], X_s^{t,x} \notin B(x,1) \}$.

Then τ is a stopping time (with $\tau = T$ if $X_s^{t,x} \in B(x,1)$ $\forall s \in (t,T]$).

For $h>0$ small, set $\tau_h = \inf\{ t+h, \tau \}$.

DPP $\Rightarrow$

$$\cancel{u(t,x)} = \mathbb{E}\left[\int_t^{\tau_h} f(s, X_s^{t,x})\, ds + u(\tau_h, X_{\tau_h}^{t,x}) \right]$$

$$\underset{\text{Itô's f.}}{=} \mathbb{E}\Big[\cancel{u(t,x)} + \int_t^{\tau_h} (\partial_t u + \mathcal{L}u + f)(s, X_s)\, ds + \underbrace{\int_t^{\tau_h} \underbrace{\nabla u(s, X_s)\, \sigma(s, X_s)}_{\text{bd in } [0,T]\times B(x,1)}\, dW_s}_{=0 \text{ martingale}} \Big]$$

So $0 = \mathbb{E}\left[\int_t^{\tau_h} (\partial_t u + \mathcal{L}(u) + f)(s, X_s)\, ds \right]$

$$(*) \quad 0 = \mathbb{E}\left[\frac{1}{h} \int_t^{\tau_h} (\partial_t u + \mathcal{L}u + f)(s, X_s)\, ds \right] \xrightarrow[h \to 0^+]{} (\partial_t u + \mathcal{L}u + f)(t,x)$$

Rk: $\mathbb{P}[\tau_h \neq h] \underset{h \to 0^+}{\longrightarrow} 0$.

because $\tau_h = \inf\{\tau, t+h\}$

where $\tau = \inf\{s \in [t,T] \,/\, X_s \notin B(x,1)\}$

As $X_t = x$ and $s \to X_s$ is continuous. So $\tau > t$ a.s.

Coming back to $(*)$:

$$0 = \mathbb{E}\Big[\frac{1}{h} \int_t^{t+h} (\partial_t u + \mathcal{L}(u) + f)\, ds \, \underbrace{\mathbb{1}_{\{\tau_h = t+h\}}}_{\to 1 \text{ a.s.}} \Big]$$

$$+ \frac{1}{h} \int_t^{\tau^{\varepsilon}(\tau \le t+h)} \underbrace{(\partial_t u + \mathcal{L} u + f)}_{bd} \, ds \; \underbrace{\mathbb{1}_{\{\tau_h \neq t+h\}}}_{\downarrow \, 0 \ a.s} \Big]$$

$$\longrightarrow (\partial_t u + \mathcal{L} u + f)(t, x).$$

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$$0 = \mathbb{E}\left[\frac{1}{h} \int_t^{t+h} (\partial_t u + \mathcal{L} u)(s, X_s) \, ds \right]$$

$$s = t + uh$$

$$= \mathbb{E}\Big[\int_0^1 \underbrace{(\partial_t u + \mathcal{L} u)}_{bd} \big(\underset{\substack{\downarrow h \to 0^+ \\ t}}{t + uh}, \; X_{\underbrace{t+uh}_{\substack{\downarrow h \to 0^+ \\ X_t = x}}} \big) \, du \Big]$$

$$\underset{h \to 0}{\longrightarrow} (\partial_t u + \mathcal{L} u)(t, x)$$