· FK1

If $u(t,x) = \mathbb{E}\left[\int_t^T f(s, X_s^{t,x})\,ds + g(X_T^{t,x})\right]$

is $C^{1,2}$, then u sol

$$\begin{cases} \partial_t u + \mathcal{L}u + f = 0 \\ u(T) = g \end{cases}$$

· Regularity of u:

① Smooth coef : b, σ, f, g are smooth with bd derivatives.

Space derivative : $x \mapsto X_s^{t,x}$ is $\mathbb{R}^d \to L^2(\Omega, \mathbb{R}^d)$ differentiable and its derivative in direction $v \in \mathbb{R}^d$ is given by Y_s where (Y_s) solves

$$\begin{cases} dY_s = b'(s, X_s^{t,x})(Y_s)\,ds + \sigma'(s, X_s^{t,x})(Y_s)\,dW_s \\ Y_t = v. \end{cases}$$

....

$\Rightarrow$ u is smooth as well.

2) Uniformly elliptic case

- If $\begin{cases} b, \sigma \text{ are as } H1 + \text{Hölder in time} \\ f, g \text{ — } H2 + \text{ — for } f \end{cases}$

$+ \quad \sigma \sigma^T(t,x) \geq c^{-1} I_d$

for some $c > 0$

$\Rightarrow u \in C^{1,2}$ in $[0,T) \times \mathbb{R}^d$.

Ex: $b \equiv 0$, $\sigma \equiv 1$, $f \equiv 0$,

then $u(t,x) = \mathbb{E}\left[g(x + W_T - W_t) \right]$

$$= \frac{1}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} g(y) \, e^{-\frac{|x-y|^2}{2\sigma(T-t)}} dy$$

which C^∞ even if g is just bd.

Theo (Feynman-Kac 2) (Verification thm)

Assume that $u \in C^{1,2}$ is a classical solution to:

$$\text{PDE} \begin{cases} \partial_t u + \mathcal{L} u + f = 0 & \text{in } [0,T) \times \mathbb{R}^d \\ u(T,x) = g(x) & \text{in } \mathbb{R}^d \end{cases}$$

and that u is continuous in $[0,T] \times \mathbb{R}^d$

and bd

Then $u(t,x) = \mathbb{E}\left[\int_t^T f(s, X_s^{t,x})\,ds + g(X_T^{t,x})\right]$.

Consequence:

① There is at most one classical solution to the (PDE)

② If u_1 is associated with f_1, g_1 & u_2 is ———— f_2, g_2 and if $f_1 \leq f_2$, $g_1 \leq g_2$

$\Rightarrow$ $u_1 \leq u_2$. (comparison principle).

Proof: For $n \in \mathbb{N}$ large, set

$$\tau_n = \inf\{ s \in [t,T],\ X_s^{t,x} \notin B(x,n)\}$$

(with $\tau_n = T$ if $X_s^{t,x} \in B(x,n)\ \forall s$)

By Itô's formula, we have:

$$u(\tau_n, X_{\tau_n}) = u(t,x) + \int_t^{\tau_n} (\partial_t u + \mathcal{L}u)(s, X_s)\,ds + \int_t^{\tau_n} \underbrace{\nabla u(s, X_s)\cdot\sigma(s, X_s)}_{bd}\,dW_s.$$

$X_{\tau_n}^{t,x}$

$$\Rightarrow \mathbb{E}\left[u(\tau_n, \overbrace{X_{\tau_n}}^{\text{bd and continuous}}) \right] =$$

$$u(t,x) + \mathbb{E}\left[\int_t^{\tau_n} \underbrace{-f(s, X_s)}_{\substack{\text{bd by assumption} \\ (H2)}} ds \right] + 0$$

As $n \to +\infty$, $\tau_n \to T$ a.s

and so

$$\mathbb{E}\left[u(T, X_T) \right] = u(t,x) + \mathbb{E}\left[\int_t^T -f(s, X_s)\, ds \right]$$

Since $u(T, \cdot) = g$,

$$u(t,x) = \mathbb{E}\left[\int_t^T f\, ds + g \right].$$

<u>Rk:</u> Growth assumption on u are mandatory because in Evans book on PDE, one can find a nontrivial sol to

$$\begin{cases} \partial_t u + u_{xx} = 0 & \text{in } (0,T) \times \mathbb{R} \\ u(T,x) = 0 & \text{in } \mathbb{R} \end{cases}$$

- <u>Trivial example</u>: $d=1$, $\sigma>0$

$$\begin{cases} dX_s = -X_s\,ds + \sigma\,dW_s \\ X_t = x \end{cases}$$

$$v(t,x) = \mathbb{E}\left[\int_t^T X_s^2\,ds + A X_T^2\right]$$

$(A>0)$

<u>Compute v</u>.

FK formula: v solves the PDE

$$(*)\begin{cases} \partial_t v + \mathcal{L}v + x^2 = 0 \\ v(T,x) = A x^2. \end{cases}$$

and $\mathcal{L}\varphi(t,x) = -x \cdot \partial_x \varphi(t,x) + \frac{\sigma^2}{2}\partial_{xx}\varphi(t,x)$.

Idea: look for v under the form:

$$v(t,x) = \frac{a(t)}{2}x^2 + b(t)x + c(t).$$

where a, b, c are C^1.

v solves $(*)$ $\Leftrightarrow$

$$\begin{cases} \left(a'\frac{x^2}{2} + b'x + c'\right) - x\left(ax + b\right) + \frac{\sigma^2}{2}a \\ \quad + x^2 = 0 \\ a(T)x^2 + b(T)x + c(T) \equiv A x^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{a'}{2} - a + 1 = 0 & (\text{in } x^2) \\ b' - b = 0 & (\text{in } x) \\ c' + \frac{\sigma^2}{2}a = 0 & \end{cases}$$

$$\begin{cases} a(T) = A \\ c(T) = b(T) = 0 \end{cases}$$

$\Rightarrow$ $b \equiv 0$, + computation of a and c easy..

Chap II Examples of control pb

① Bolza problem.

- Controlled SDE:

$$(SDE) \begin{cases} dX_s = b(s, X_s, \alpha_s)ds + \sigma(s, X_s, \alpha_s)dW_s \\ X_t = x \end{cases}$$

where:

- (t, x) initial condition, $\in [0,T] \times \mathbb{R}^d$
- $(W_s)_{0 \le s \le T}$ B.M. def. on proba basis $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{0 \le s \le T}, \mathbb{P})$ which is m-dim.

(H1.1) $\left.\begin{array}{l} \cdot\ b : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}^d \\ \cdot\ \sigma : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}^{d \times m} \end{array}\right\}$ continuous

where A compact subset of some euclidean space.

- The control $\alpha : [0,T] \times \Omega \to A$ which is adapted to $(\mathcal{F}_s)_{t \le s \le T}$.

(H1.2) $\exists L > 0$, s.t. $\forall t \in [0,T]$, $x, y \in \mathbb{R}^d$, $a \in A$

$$|b(t,x,a) - b(t,y,a)| + |\sigma(t,x,a) - \sigma(t,y,a)| \le L|x-y|$$

<u>Theo</u> (Cauchy-Lipschitz with random coef)

For any control (α_s), there is a unique sol $(X_s^{t,x,\alpha})_{0 \le s \le T}$ to SDE

- <u>Cost and value function:</u>

Given $(t,x) \in [0,T] \times \mathbb{R}^d$, and a control α, let

$$J(t,x,\alpha) = \mathbb{E}\left[\int_t^T f(s, X_s^{t,x,\alpha}, \alpha_s)\, ds + g(X_T^{t,x,\alpha})\right]$$

Let $\mathcal{A}(t)$ be the set of controls and $v(t,x)$ the value function:

$$v(t,x) = \inf_{\alpha \in \mathcal{A}(t)} J(t,x,\alpha).$$

<u>To understand:</u>

- How compute v ? (PDE)
- $\exists$? optimal control ?
- How compute this control ?