* **Value function:**

$$v(t,x) = \operatorname*{Inf}_{\alpha \in \mathcal{A}(t)} J(t,x,\alpha)$$

$$= \operatorname*{Inf}_{\alpha \in \mathcal{A}(t)} \mathbb{E}\Big[\int_t^T f(s, X_s^{t,x,\alpha}, \alpha_s)\,ds + g(X_T^{t,x,\alpha}) \Big]$$

Assumption (+ (H1) on b & σ, + b,σ bd)

+ (H2) $f : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}$, $g : \mathbb{R}^d \to \mathbb{R}$ } continuous & bd

+ Lipschitz in x unif. w.r.t. (t,α)

[Lemma: v is Lipschitz in x uniformly in $t \in [0,T]$.

Proof: (1) Estimate on the controlled SDE:

As b, σ unif. Lipschitz in x, we have

$$\mathbb{E}\Big[\sup_{t \le s \le T} | X_s^{t,x,\alpha} - X_s^{t,y,\alpha} |^2 \Big] \le C|x-y|^2.$$

$\boxed{C \text{ is indep of } \alpha}$

(2) ($f \equiv 0$ for simplicity) Fix $x, y \in \mathbb{R}^d$.

Fix $\varepsilon > 0$. Let α^ε ε-optimal for $v(t,x)$:

$$v(t,x) \geqslant \mathbb{E}\big[g(X_T^{t,x,\alpha^\varepsilon}) \big] - \varepsilon.$$

We estimate

$$\left| \mathbb{E}\left[g(X_T^{t,x,\alpha^\varepsilon}) - g(X_T^{t,y,\alpha^\varepsilon}) \right] \right| \leq$$

$$\mathrm{Lip}(g)\, \mathbb{E}\left[\left| X_T^{t,x,\alpha^\varepsilon} - X_T^{t,y,\alpha^\varepsilon} \right| \right]$$

$$\leq \mathrm{Lip}(g)\, \mathbb{E}^{1/2}\left[\sup_{t\leq s\leq T} \left| X_s^{t,x,\alpha^\varepsilon} - X_s^{t,y,\alpha^\varepsilon} \right|^2 \right]$$

$$\leq \mathrm{Lip}(g)\, C|x-y|.$$

So

$$v(t,x) \geq \mathbb{E}\left[g(X_T^{t,x,\alpha^\varepsilon}) \right] - \varepsilon$$

$$\geq \mathbb{E}\left[g(X_T^{t,y,\alpha^\varepsilon}) \right] - \mathrm{Lip}(g)\, C|x-y| - \varepsilon.$$

$$\geq \inf_{\alpha\in\mathcal{A}(t)} \mathbb{E}\left[g(X_T^{t,y,\alpha}) \right] - C'|x-y| - \varepsilon$$

$$\Rightarrow \boxed{v(t,x) \geq v(t,y) - C'|x-y| - \varepsilon.}$$

As $\varepsilon>0$ is arbitrary, we get

$$v(t,y) - v(t,x) \leq C'|x-y|$$

+ exchange the role of x & y. $\square$

Theo (Dynamic programming (1))

Let $0 \leq t_1 \leq t_2 \leq T$

Then

$$v(t_1, x) = \inf_{\alpha \in \mathcal{A}(t_1)} \mathbb{E}\left[\int_{t_1}^{t_2} f(s, X_s^{t_1,x,\alpha}, \alpha_s)\,ds + v(t_2, X_{t_2}^{t_1,x,\alpha})\right]$$

<u>Proof</u>: difficult!

$$v(t,x) = \inf_{\alpha \in \mathcal{A}(t)} \mathbb{E}\left[\int_{t_1}^{t_2} f() + \int_{t_2}^{T} f() + g(X_T)\right]$$

where

$$(*) = \mathbb{E}\left[\int_{t_2}^{T} f() + g(X_T)\right] = \mathbb{E}\left[\int_{t_2}^{T} f(s, X_s^{t_2, X_{t_2}^{t_1,x,\alpha}, \tilde{\alpha}}, \tilde{\alpha}_s)\,ds + g(X_T^{t_2, X_{t_2}^{t_1,x,\alpha}, \tilde{\alpha}})\right]$$

with $\tilde{\alpha} = \alpha|_{[t_2,T]} \in \mathcal{A}(t_2)$

and by $\frac{1}{2}$ group property

$$X_s^{t_1,x,\alpha} = X_s^{t_2, X_{t_2}^{t_1,x,\alpha}, \tilde{\alpha}}.$$

If we had a Markov property

then $(*) = \mathbb{E}\left[\mathbb{E}\left[\int_{t_2}^{T} f(s, X_s^{t_2,y,\tilde{\alpha}})\,ds + g(X_T^{t_2,y,\tilde{\alpha}})\right]_{y = X_{t_2}^{t_1,x,\alpha}}\right]$

because $\forall \varphi \in C_b^0$

$$\mathbb{E}\left[\varphi\left(X_s^{t_2, X_{t_2}^{t_1,x,\alpha}, \tilde{\alpha}}\right) \,\Big|\, \mathcal{F}_{t_2}\right]$$

$$= \mathbb{E}\left[\varphi\left(X_s^{t_2, y, \tilde{\alpha}}\right)\right]_{y = X_{t_2}^{t_1,x,\alpha}}$$

Pb: α depends on all the past

$\Rightarrow$ not Markov!

<u>Lemma</u> (difficult part - assumed)

$\forall\, Z \in \mathcal{F}_{t_2}$, we have

$$v(t, Z) = \operatorname{ess-Inf}_{\alpha \in \mathcal{A}(t_2)} \mathbb{E}\left[\int_{t_2}^T f(s, X_s^{t_2, Z, \alpha})\, ds + g(X_T^{t_2, Z, \alpha}) \,\Big|\, \mathcal{F}_{t_2}\right]$$

<u>End of the proof:</u>

- Let $w(t_1, x) = \operatorname{Inf}_{\alpha \in \mathcal{A}(t_1)} \mathbb{E}\left[\int_{t_1}^{t_2} f(s, X_s^{t_1,x,\alpha}, \alpha_s)\, ds + v(t_2, X_{t_2}^{t_1,x,\alpha})\right]$.

- <u>$w \geq v$</u>:

Fix $\varepsilon > 0$ & $\alpha_1^\varepsilon \in \mathcal{A}(t_1)$ ε-optimal for $w(t_1, x)$:

$$\varepsilon + w(t_1, x) \geq \mathbb{E}\left[\int_{t_1}^{t_2} f(s, X_s^{t_1,x,\alpha_1^\varepsilon}, \alpha_{1,s}^\varepsilon)\, ds + v(t_2, \underbrace{X_{t_2}^{t_1,x,\alpha_1^\varepsilon}}_{=Z})\right]$$

Let $\alpha_2^\varepsilon \in \mathcal{A}(t_2)$ ε-optimal for $v(t_2, Z)$

$$\varepsilon + v(t_2, \quad Z \quad) \geq$$
$$\mathbb{E}\left[\int_{t_2}^{T} f(s, X_s^{t_2, Z, \alpha_2^\varepsilon}, \alpha_{2,s}^\varepsilon)\,ds + g(X_T^{t_2, Z, \alpha_2^\varepsilon}) \,\Big|\, \mathcal{F}_{t_2}\right]$$

where $Z = X_{t_2}^{t_1, x, \alpha_1^\varepsilon}$. ($\exists$ given by the Lemma).

Set $\alpha_s^\varepsilon = \begin{cases} \alpha_{1,s}^\varepsilon & \text{if } s \in (t_1, t_2] \\ \alpha_{2,s}^\varepsilon & \text{if } s \in [t_2, T] \end{cases}$.

Then $\alpha^\varepsilon \in \mathcal{A}(t_1)$ and

$$J(t, x, \alpha^\varepsilon) = \mathbb{E}\left[\int_{t_1}^{T} f(\,) + g(\,)\right]$$
$$= \mathbb{E}\Big[\int_{t_1}^{t_2} f(s, X_s^{t_1, \alpha_\varepsilon^1}, \alpha_{1,s}^\varepsilon)\,ds$$
$$+ \int_{t_2}^{T} f(s, X_s^{t_1, x, \alpha^\varepsilon}, \alpha_{2,s}^\varepsilon)\,ds$$
$$+ g(X_T^{t_1, x, \alpha^\varepsilon})\Big]$$

where $(X_s^{t_1 x, \alpha^\varepsilon})$ is a sol of the SDE with control α_1^ε on $[t_1, t_2]$

$$\Rightarrow X_s^{t_1, x, \alpha^\varepsilon} = X_s^{t_1, x, \alpha_1^\varepsilon} \text{ on } [t_1, t_2]$$

$$\Rightarrow X_{t_2}^{t_1, x, \alpha^\varepsilon} = X_{t_2}^{t, x, \alpha_1^\varepsilon} = Z.$$

So

So $X_s^{t_1,x,\alpha^\varepsilon} = X_s^{t_2, Z, \alpha_2^\varepsilon}$ on $(t_2, T]$.

and $J(t_1, x, \alpha^\varepsilon) =$

$$\mathbb{E}\Big[\int_{t_1}^{t_2} f(s, X_s^{t_1,x,\alpha_1^\varepsilon}, \alpha_{1,s}^\varepsilon)\, ds$$

$$+ \mathbb{E}\int_{t_1}^{t_2} f(s, X_s^{t_2,Z,\alpha_2^\varepsilon}, \alpha_{2,s}^\varepsilon)\, ds + g(X_T^{t_2,Z,\alpha_2^\varepsilon}) | \mathcal{F}_{t_2}\Big]\Big]$$

$$\leq \mathbb{E}\Big[\int_{t_1}^{t_2} f(\,)\, ds + v(t_2, X_{t_2}^{t_1,x,\alpha_1^\varepsilon})\Big] + \varepsilon$$

from def of α_ε^2

$$\leq v(t_1, x) + \varepsilon + \varepsilon.$$

$$\Rightarrow \quad v(t_1, x) \leq J(t_1, x, \alpha^\varepsilon) \leq v(t_1, x) + 2\varepsilon.$$

$$\Rightarrow \quad v \leq v.$$

- $\underline{v \geq v}$

Fix $\varepsilon > 0$. α^ε ε-optimal for $v(t_1, x)$:

$$\varepsilon + v(t_1, x) \geq \mathbb{E}\Big[\int_{t_1}^{t_2} f(s, X_s^{t_1,x,\alpha^\varepsilon}, \alpha_s^\varepsilon)\, ds$$

$$+ \mathbb{E}\Big[\int_{t_2}^{T} f(s, X_s^{t_2,Z,\alpha^\varepsilon}, \alpha_s^\varepsilon)\, ds + g(X_T^{t_2,Z,\alpha^\varepsilon}) | \mathcal{F}_{t_2}\Big]\Big]$$

where $Z = X_{t_2}^{t_1,x,\alpha^\varepsilon}$

$$\geq \mathbb{E}\left[\int_{t_1}^{t_2} f(\ )\, ds + u(t_2, z) \right]$$

Lemma "Ganzo"

$$\overset{\geq}{\text{def } v} \quad v(t_1, x)$$

$$\Rightarrow \quad u + \varepsilon \geq v \quad \Rightarrow \quad \underline{u = v}.$$