Programmation dynamique : $0 \leq t_1 \leq t_2 \leq T$

$$v(t_1, x) = \inf_{\alpha \in \mathcal{A}(t_1)} \mathbb{E}\left[\int_{t_1}^{t_2} f(s, X_s^{t_1,x,\alpha}, \alpha_s)\,ds + v(t_2, X_{t_2}^{t_1,x,\alpha})\right]$$

[**Lemma**: v is $\frac{1}{2}$ Hölder in t ~~unif.~~ unif. in x

Proof : postponed.

Theo (DPP 2) If τ stopping τ between t and T, then

$$v(t,x) = \inf_{\alpha \in \mathcal{A}(t)} \mathbb{E}\left[\int_t^\tau f(s, X_s^{t,x,\alpha}, \alpha_s)\,ds + v(\tau, X_\tau^{t,x,\alpha})\right]$$

[Not proved]

Heuristic for the PDE : if $v \in C_b^{1,2}$

For $\tau = t+h$ ($h > 0$ small)

$$0 \cancel{v(t,x)} = \inf_{\alpha \in \mathcal{A}(t)} \mathbb{E}\Big[\int_t^{t+h} f\,ds + \cancel{v(t,x)} + \int_t^{t+h} \Big(\partial_t v + b(s, X_s, \alpha_s) \cdot \nabla v(s, X_s) + \frac{1}{2} \mathrm{Tr}\big(\sigma\sigma^T(s, X_s, \alpha_s) D^2 v(s, X_s)\big)\Big)\,ds + \cancel{\int_t^{t+h} \nabla v(s, X_s) \cdot \sigma(s, X_s, \alpha_s)\, dW_s}\Big]$$

martingale

As h small, $s \simeq t$ & $X_s \simeq x$ on $[t, t+h]$

So $\quad 0 = \inf_{\alpha \in \mathcal{A}(t)} \mathbb{E}\Big[\int_t^{t+h} \big(f + \partial_t u + b \cdot Du$

$+ \frac{1}{2} \mathrm{Tr}(\sigma\sigma^T D^2 u)\big)(t, x, \alpha_s)\, ds \Big] + o(h)$

$0 \text{ "=" } \inf_{a \in A} \cancel{\mathbb{E}}\Big[\cancel{\frac{1}{h}\int_t^{t+h}} \big(f + \partial_t u + b \cdot Du$

$+ \frac{1}{2} \mathrm{Tr}(\sigma\sigma^T D^2 u)\big)(t, x, a)\, \cancel{ds} \Big] + o(1)$

$\Rightarrow$ let $h \to 0$ $\quad 0 = \inf_{a \in A} \{ f + \partial_t u + b \cdot Du + \frac{1}{2} \mathrm{Tr}(\sigma\sigma^T D^2 u) \}(t, x, a)$

$$(HJ) \begin{cases} -\partial_t u(t,x) + \sup_{a \in A} \{ -b(t,x,a) \cdot Du(t,x) \\ \qquad - \frac{1}{2} \mathrm{Tr}(\sigma\sigma^T(t,x,a) D^2 u(t,x)) \\ \qquad - f(t,x,a) \} = 0 \quad \text{in } (0,T) \times \mathbb{R}^d \\ u(T,x) = g(x) \quad \text{in } \mathbb{R}^d \end{cases}$$

Hamilton-Jacobi eq.

<u>Hamiltonian</u> :

$$H(t,x,p,A) = \sup_{a \in A} \{ -b \cdot p - \frac{1}{2} \mathrm{Tr}(\sigma\sigma^T A) - f \}.$$

and HJ becomes

$$(HJ) \begin{cases} -\partial_t u + H(t,x,Du,D^2u) = 0 & \text{in } (0,T) \times \mathbb{R}^d \\ u(T,x) = g(x) & \text{in } \mathbb{R}^d \end{cases}$$

Interest of HJ eq

① Numerical reason : "finite difference method" to compute the sol of HJ.

② Analytic tools to analyse $\exists$, ! and regularity of the sol of HJ

③ Verification principle (optimal synthesis).

Theo (Verification principle)

① Assume that $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ of class $C^{1,2}$ in $[0,T) \times \mathbb{R}^d$, continuous, bd, in $[0,T] \times \mathbb{R}^d$ sol of HJ, then

$$v \leq u$$

② Assume in addition that there is $\alpha^* : [0,T] \times \mathbb{R}^d \to A$, lipsch. in x, indep of t, such that

$$H(t, x, Dv(t,x), D^2 v(t,x)) =$$

$$- b(t, x, \alpha^*(t,x)) \cdot Dv(t, x) - \frac{1}{2} Tr(\sigma\sigma^T(t, x, \alpha^*(t,x)) \times D^2 v(t,x))$$

$$- f(t, x, \alpha^*(t,x)) .$$

Then $v = v$ and α^* is optimal.

Meaning of "α^* optimal": For any t, x, let X^*_s the sol to
$$\begin{cases} dX_s = b(s, X_s, \alpha^*(s, X_s))\,ds \\ \quad + \sigma(s, X_s, \alpha^*(s, X_s))\,dW_s \\ X_t = x \end{cases}$$

Then $v(t,x) = J(t, x, \underbrace{\alpha^*(\cdot, X^*_\cdot)}_{\in \mathcal{A}(t)})$.

$\alpha^* = \alpha^*(t,x)$ is called a feedback..

Ex: $\begin{cases} dX_t = \alpha_s\, ds + dW_s \\ X_t = x. \end{cases}$ $\quad d = 1$, $\quad A = \mathbb{R}$

$$v(t,x) = \inf_{\alpha \in \mathcal{A}(t)} \mathbb{E}\left[\int_t^T \frac{1}{2}|\alpha_s|^2 ds + g(X_T)\right]$$

where g Lipschitz & bd.

$b(s,x,a) = a$

$\sigma = 1$

$f = \frac{1}{2}a^2$

$$H(t, x, p, A) = \sup_{a \in \mathbb{R}} \left\{ -a \cdot p - \frac{1}{2}A - \frac{1}{2}a^2 \right\}.$$

$$= -\frac{1}{2}A + \sup_{a \in \mathbb{R}} \left\{ -a \cdot p - \frac{1}{2}a^2 \right\}$$

$$= -\frac{1}{2}A + \frac{p^2}{2}. \quad \text{for } \bar{a}(t,x,p) = -p.$$

HJ eq

$$(HJ)\quad \begin{cases} -\partial_t v - \frac{1}{2}\partial_{xx} v + \frac{1}{2}|\partial_x v|^2 = 0 \\ v(T,x) = g(x) \end{cases}$$

If v is a smooth sol of (HJ), then
$$\alpha^*(t,x) = -\partial_x v(t,x).$$

<u>Proof of the verification thm</u>: (Hyp $v \in C^{1,2}_b$)

① Fix (t,x), let $\alpha \in \mathcal{A}(t)$. $\quad (X_s = X_s^{t,x,\alpha})$

$$g(X_T) = v(T, X_T^{t,x,\alpha})$$
$$= v(t,x) + \int_t^T \left(\partial_t v + b \cdot \nabla v + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2 v)\right) ds$$
$$+ \int_t^T \nabla v \cdot \sigma \, dW_s$$

where all terms are computed at (s, X_s, α_s)

and where $-\partial_t v - b\nabla v - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T D^2 v) - f$
$$\leq -\partial_t v + H(t,x,Dv,D^2v) = 0.$$

$$\Rightarrow \mathbb{E}[g(X_T)] \geq v(t,x) + \mathbb{E}\left[\int_t^T -f(s,X_s,\alpha_s)ds\right].$$

$$\Rightarrow v(t,x) \leq \inf_{\alpha\in\mathcal{A}(t)} J(t,x,\alpha) = u(t,x).$$

② Let X_s^* solve the SDE

$$\begin{cases} dX_s = b(s, X_s, \alpha^*(s, X_s))\,ds \\ \qquad\quad + \sigma(s, X_s, \alpha^*(s, X_s))\,dW_s \\ X_t = x \end{cases}$$

$(\exists!$ sol$)$

let $\bar{\alpha}_s = \alpha^*(s, X_s)$.

Then $X_s^* = X_s^{t,x,\bar{\alpha}}$ $\forall s \in [t,T]$.

Using $\boxed{\bar{\alpha} \text{ in the computation above}}$

Then $v(t,x) = J(t,x,\bar{\alpha})$ $(\geq u(t,x))$

$\leq u(t,x)$

previous proof

$\Rightarrow \bar{\alpha}$ optimal for $u(t,x)$

and $v(t,x) = J(t,x,\bar{\alpha}) = u(t,x)$.

Exo Compute $v(t,x)$ for the 1d example when g quadratic and justify the verification thm in this setting.