Other example:

1) • Infinite horizon pbs

2) • Exit time pbs.

1) Infinite horizon pb

Dynamics: $\begin{cases} dX_t = b(X_t, \alpha_t)\,dt + \sigma(X_t, \alpha_t)\,dW_t \\ X_0 = x_0 \end{cases}$

where b, σ as before.

Cost: (for $r > 0$ rate)

$$J(x_0, \alpha) = \mathbb{E}\left[\int_0^{+\infty} e^{-rt} f(X_t, \alpha_t)\,dt\right]$$

Value function:

$$v(x_0) = \inf_{\alpha \in \mathcal{A}} J(x_0, \alpha)$$

where $\mathcal{A} = \{\alpha : [0, +\infty[\times \Omega \to A \text{ adapted}\}$

Here $f : \mathbb{R}^d \times A \to \mathbb{R}$ is bd and Lipschitz unif in $a \in A$.

Lemma: v is bounded and uniformly continuous (Hölder)

Proof: • $\forall \alpha \in \mathcal{A}$,

$$\left| \mathbb{E}\left[\int_0^{+\infty} e^{-rt} f(X_t^\alpha, \alpha_t)\,dt\right] \right| \leq$$

$$\leq \mathbb{E}\left[\int_0^{+\infty} e^{-rt} |f(X_t^\alpha, \alpha_t)| \, dt\right]$$

$$\leq \|f\|_\infty \int_0^{+\infty} e^{-rt} dt < +\infty.$$

$$= r^{-1} \|f\|_\infty.$$

$$\Rightarrow \quad |v(x_0)| = \left|\inf_\alpha J(x_0, \alpha)\right| \leq r^{-1} \|f\|_\infty.$$

- Hölder continuity: exo.

- Dynamic programming: $0 \leq t_1 \leq +\infty$

$$v(x_0) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^{t_1} e^{-rt} f(X_t^\alpha, \alpha_t) \, dt + \int_{t_1}^{+\infty} e^{-rt} f(X_t^\alpha, \alpha_t) \, dt\right]$$

$$= \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^{t_1} e^{-rt} f \, dt + \int_0^{+\infty} e^{-r(t+t_1)} f(X_{t+t_1}^\alpha, \alpha_{t+t_1}) dt\right]$$

Key remark: set $\tilde{\alpha}_t = \alpha_{t+t_1}$, $t \geq 0$

then $\mathcal{L}\left((X_{t+t_1}^\alpha, \alpha_{t+t_1})_{t \geq 0} \right)$

$$= \mathcal{L}\left((X_t^{X_{t_1}^{x_0,\alpha}, \tilde{\alpha}}, \tilde{\alpha}_t)_{t \geq 0} \right)$$

$$= \mathcal{L}\left((X_t^{X_{t_1}^{x_0,\alpha}, \tilde{\alpha}}, \tilde{\alpha}_t)_{t \geq 0} \right)$$

because $Y_t = X_{t+t_1}$ solves

$$dY_t = b(Y_t, \alpha_{t+t_1})\,dt + \sigma(Y_t, \alpha_{t+t_1})\,dW_t$$
$$= b(Y_t, \tilde{\alpha}_t)\,dt + \sigma(Y_t, \tilde{\alpha}_t)\,dW_t$$
$$Y_0 = X_{t_1}^{x_0,\alpha}.$$

Guess: DPP:

$$v(x_0) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^{t_1} e^{-\lambda t} f(X_t^{x_0,\alpha}, \alpha_t)\,dt + e^{-\lambda t_1} v(X_{t_1}^{x_0,\alpha})\right]$$

Theo: For any stopping time $\tau \geq 0$ we have:

$$v(x_0) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^{\tau} e^{-\lambda t} f(X_t^{x_0,\alpha}, \alpha_t)\,dt + e^{-\lambda \tau} v(X_\tau^{x_0,\alpha})\right].$$

Derive the HJ eq:

Assume v is C^2 with ∇v, $D^2 v$ bd: then, for $\tau = h > 0$ small

$$\cancel{v(x_0)} \underset{\substack{\uparrow \\ \text{Ito}}}{=} \inf_{\alpha} \mathbb{E}\left[\cancel{v(x_0)} + \int_0^h e^{-\lambda t}\left(f - \lambda v(X_t) + \right.\right.$$

$$\nabla v(X_t)\cdot b(X_t,\alpha_t) + \frac{1}{2}\operatorname{Tr}\left(\sigma\sigma^T(X_t,\alpha_t)\, D^2 v(X_t)\right)dt + \int_0^h \nabla v(\cancel{X_t})\,\sigma(X_t,\alpha_t)\,dW_t\Big].$$

martingale

Heuristically: divide by $h>0$, let $h\to 0^+$

$$0 = \inf_{a\in A}\Big[f(x_0,a) + b(x_0,a)\cdot\nabla v(x_0) - r v(x_0) + \frac{1}{2}\operatorname{Tr}\left(\sigma\sigma^T(x_0,a)\,D^2 v(x_0)\right)\Big]$$

Set $H(x,p,A) =$

$$\sup_{a\in A}\Big\{-f(x,a) - b(x,a)\cdot p - \frac{1}{2}\operatorname{Tr}\left(\sigma\sigma^T(x,a)\,A\right)\Big\}$$

HJ eq $\quad r v(x) + H(x, Dv(x), D^2 v(x)) = 0.$

Theo (Verification theorem)

1. Assume that $v:\mathbb{R}^d\to\mathbb{R}$ is bd and of C^2 in $\mathbb{R}^d$ and satisfies
$$r v + H(x, Dv, D^2 v) = 0 \quad \text{in } \mathbb{R}^d$$
then $v \le u$ in $\mathbb{R}^d$

2. Inf there exist $\alpha^*: \mathbb{R}^d \to A$

s.t. $\forall x$
$$-f(x,\alpha^*(x)) - \nabla v(x)\cdot b(x,\alpha^*(x)) - \tfrac{1}{2}\operatorname{Tr}\left(\sigma\sigma^T(x,\alpha^*(x))\, D^2 v(x)\right) = H(x, Dv(x), D^2 v(x))$$
and α^* is Lipschitz, then $v = V$ and α^* is an optimal feedback.

Proof: ① Fix $\alpha \in \mathcal{A}$ and $X_t := X_t^{x_0,\alpha}$

$$e^{-rt} v(X_t^*) \underset{(\text{Itô})}{=} v(x_0) + \int_0^t e^{-rs}\Big(-r v(X_s^*) + \nabla v(X_s^*)\cdot b(X_s^*, \alpha_s^*) + \tfrac{1}{2}\operatorname{Tr}\left(\sigma\sigma^T(X_s^*,\alpha_s^*) D^2 v(X_s^*)\right)\Big) ds$$
(Hyp: ∇v bd)
$$+ \int_0^t e^{-rs} \nabla v(X_s^*)\, \sigma(X_s^*, \alpha_s^*)\, dW_s .$$

As v solves the PDE:
$$r v(X_s^*) - \nabla v(X_s^*)\, b(X_s^*, \alpha_s^*) - \tfrac{1}{2}\operatorname{Tr}(\sigma\sigma^T D^2 v) - f(X_s^*, \alpha_s^*)$$
$$\underset{=}{\leq} r v(X_s^*) + H(X_s^*, \nabla v(X_s^*), D^2 v(X_s^*))$$
$$= 0$$
$$\Rightarrow \mathbb{E}\left[e^{-rt} v(X_t^*)\right] \underset{=}{\geq} v(x_0) + \mathbb{E}\left[\int_0^t e^{-rs} f(X_s^*, \alpha_s^*)\, ds\right]$$

$$\Rightarrow \quad v(x_0) \underset{=}{\leq} \mathbb{E}\left[\int_0^t e^{-rs} f(X_s^*, \alpha_s^*)\, ds + e^{-rt} v(X_t^*) \right]$$

$t \to +\infty \Rightarrow$

$$v(x_0) \underset{=}{\leq} \mathbb{E}\Big[\underbrace{\int_0^{+\infty} e^{-rs} f(X_s^*, \alpha_s^*)\, ds}_{J(x_0,\alpha)} \Big]$$

As $\alpha \in \mathcal{A}$ is arbitrary

$$v(x_0) \leq \inf_\alpha J(x_0,\alpha) = u(x_0).$$

• ② Let X^* the sol to

$$\begin{cases} dX_t^* = b(X_t^*, \alpha^*(X_t^*))\, dt \\ \qquad\quad + \sigma(X_t^*, \alpha^*(X_t^*))\, dW_t \\ X_0^* = x_0 \end{cases}$$

Let $\alpha_t^* = \alpha^*(X_t^*)$. $\Rightarrow \alpha^* \in \mathcal{A}$.

Then $$\underset{\substack{\wedge \\ u(x_0)}}{v(x_0)} = \mathbb{E}\Big[\underset{\substack{\vee \\ u(x_0)}}{\int_0^{+\infty} e^{-rt} f(X_t^*, \alpha_t^*)\, dt} \Big]$$

$\Rightarrow \quad u(x_0) = v(x_0)$ & α^* optimal for $u(x_0)$.

2) Exit-time pb

- Dynamic : $\begin{cases} dX_t = b(X_t, \alpha_t)\,dt \\ \qquad\quad + \sigma(X_t, \alpha_t)\,dW_t \\ X_0 = x_0 \end{cases}$

$\rightsquigarrow X^{x_0,\alpha}$ unique sol.

- Cost function : fix $G \subset \mathbb{R}^d$ open. (with $x_0 \in G$).

Stopping time of a trajectory X

$$\tau_G(X) = \operatorname{Inf}\{t \geq 0,\ X_t \notin G\}.$$

$$J(x_0, \alpha) = \mathbb{E}\left[\int_0^{\tau_G(X^{x_0,\alpha})} e^{-\lambda t} f(X_t, \alpha_t)\,dt\right]$$

- Value function

$$v(x_0) = \operatorname*{Inf}_{\alpha \in \mathcal{A}} J(x_0, \alpha).$$

- DPP $\forall t_1 \geq 0$,

$$v(x_0) = \operatorname*{Inf}_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^{t_1 \wedge \tau_G(X^{x_0,\alpha})} e^{-\lambda t} f(X_t, \alpha_t)\,dt + e^{-\lambda(t_1 \wedge \tau)} v\left(X_{t_1 \wedge \tau_G(X)}\right)\right]$$

$\forall x_0 \in G$

Assumption : $f \geq 0$, $\sigma\sigma^T > 0$

One can show that u is continuous on $\overline{G}$, with $u = 0$ on ∂G.

<u>HJ eq</u>:

$$\begin{cases} ru + H(x, Du, D^2u) = 0 & \text{in } G \\ u = 0 & \text{on } \partial G. \end{cases}$$

<u>Exo</u>: Write the verification theo. and prove it.