e

- Supersolution = later
- <u>Some classical properties of viscosity solutions</u>

$$(HJ) \quad ru + H(x, Du, D^2 u) = 0 \quad \text{in } \mathbb{R}^d$$

- <u>Equivalent definition:</u>

Let $u : \mathbb{R}^d \to \mathbb{R}$ be continuous. Then u is a subsolution of (HJ)

$\Leftrightarrow$ for any test function $\varphi \in C^2(\mathbb{R}^d)$ s.t $u - \varphi$ has local strict maximum at $x \in \mathbb{R}^d$ with $u(x) = \varphi(x)$, then

$$ru(x) + H(x, D\varphi(x), D^2\varphi(x)) \leq 0.$$

(idem for supersolution)

<u>Proof</u>: $\boxed{\Leftarrow}$

Let $\varphi \in C^2(\mathbb{R}^d)$ s.t $u - \varphi$ has a max at $x \in \mathbb{R}^d$.

Set $\tilde{\varphi}(y) = \varphi(y) + \frac{1}{4}|x-y|^4 + u(x) - \varphi(x)$.

Then $\tilde{\varphi} \in C^2(\mathbb{R}^d)$, $\tilde{\varphi}(x) = u(x)$. and

$$u(y) - \tilde{\varphi}(y) = \underbrace{u(y) - \varphi(y)} - \frac{1}{4}|x-y|^4 - u(x) + \varphi(x)$$

$$\leq u(x) - \varphi(x) - \frac{1}{4}|x-y|^4 - u(x) + \varphi(x)$$

$$< 0 \text{ if } y \neq x, \quad 0 \text{ if } y = x$$

$$\Rightarrow \quad r u(x) + H\big(x, \underset{\substack{\| \\ D\varphi(x)}}{D\tilde{\varphi}(x)}, \underset{\substack{\| \\ D^2\varphi(x)}}{D^2\tilde{\varphi}(x)}\big) \leq 0 \qquad \square$$

$\boxed{\Leftarrow}$ exo.

Prop (Stability)

- Assume $H_n : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}^d \to \mathbb{R}$ is a sequence of continuous maps, non increasing with respect to the last variable and converges locally uniformly to H.
- Assume $u_n : \mathbb{R}^d \to \mathbb{R}$ is a seq. of viscosity subsol of $r u_n + H_n(x, Du_n, D^2 u_n) = 0$ and that $u_n \to u$ locally unif'ly.

Then u subsol of the limit eq

$$r u + H(x, Du, D^2 u) = 0.$$

OK also for supersol

__Sketch of proof:__ Assume that $u - \varphi$ has a strict local maximum at some $x \in \mathbb{R}^d$:

$$\exists r > 0, \quad \forall y \in B_r(x) \setminus \{x\}, \quad u(y) - \varphi(y) < u(x) - \varphi(x).$$

Note that $\exists \eta > 0$ s.t.

$$\forall y \in \partial B_r(x)$$

$$u(y) - \varphi(y) \leq -\eta + u(x) - \varphi(x).$$

Then $\exists n_0$ large enough s.t. $\forall n \geq n_0$,

$$u_n(y) - \varphi(y) \leq -\eta/2 + u_n(x) - \varphi(x)$$

$\forall y \in \partial B_r(x)$.

Rk: $\sup_{B_r(x)} u_n - \varphi = \max_{\mathring{B}_r(x)} u_n - \varphi$

let x_n point of max. $u_n - \varphi$. ($x_n \in \mathring{B}_r(x)$)

Then as u_n visco. subsol

$$r\, u_n(x_n) + H_n(x_n, D\varphi(x_n), D^2\varphi(x_n)) \leq 0.$$

(Exo: $x_n \longrightarrow x$)

$$\Rightarrow \quad r\, u(x) + H(x, D\varphi(x), D^2\varphi(x)) \leq 0.$$

$\square$

Exo: u is a subsol of

$$r\, w + H(x, Dw, D^2 w) = 0$$

$\Leftrightarrow$ $-u$ is a supersol of

$$r\, v + \tilde{H}(x, Dv, D^2 v) = 0$$

where $\tilde{H}(x, p, A) = -H(x, -p, -A)$.

• Comparison principle:

" subsolutions are below supersolutions "

Consequence: at most one visc. sol.

Theo: Let $H(x,p,A) =$

$$\sup_{a\in A} \Big\{ -f(x,a) - b(x,a)\cdot p - \tfrac{1}{2} \operatorname{Tr}(\sigma\sigma^T(x,a) A) \Big\}.$$

where f, b, σ satisfy our standing assumptions.

Let u be a viscosity sub of

(HJ) $\quad r w + H(x, Dw, D^2 w) = 0$

and v is a viscosity supersol of (HJ) and v and u are bounded.

Then $u \le v$.

(Proof: postponed - difficult)

Corollary: There exists exactly one bd viscosity sol to (HJ).

Proof: The value function u is a bd viscosity sol of (HJ).

- Assume v is another bd sol.

then ① u subsol & v supersol + u, v bd. By comparison $u \le v$.

② Exchanging the roles of u and v $\Rightarrow$ $v \leq u$

So $u = v$. $\square$

<u>Ex:</u> $d = 1$, $\lambda u + |u_x| = 0$. (HJ)

$u = 0$ is the bd sol to (HJ)

But $u(x) = -e^{\lambda x}$ is also a sol. but not bd..

<u>Uniqueness for Eikonal eq:</u>

Let $G \subset \mathbb{R}^d$ be open bounded

$$(HJ) \begin{cases} u(x) + \|Du(x)\| = 1 & \text{in } G \\ u(x) = 0 & \partial G. \end{cases}$$

Supersol

<u>Subsol:</u> ① if $u - \varphi$ has min max at $x \in G$ $\Rightarrow$ $u(x) + |D\varphi(x)| \leq 1$ $\geq$

② $u(x) \leq 0$ $\geq$ $\forall x \in \partial G$.

(u is continuous on $\overline{G}$)

<u>Theo:</u> (comparison principle)

If u is a subsol & v is a supersol of

(HJ) then $u \le v$ in $\bar{G}$.

<u>Sketch of proof</u>: By contradiction:

Assume that $\sup_G (u - v) =: M > 0$

Doubling of variation: for $\varepsilon > 0$ small

$$M_\varepsilon := \sup_{(x,y) \in \bar{\Omega} \times \bar{\Omega}} u(x) - v(y) - \frac{1}{2\varepsilon}|x-y|^2 .$$

Let $x_\varepsilon, y_\varepsilon$ be maximum point.

Exo: · check $x_\varepsilon - y_\varepsilon \xrightarrow[\varepsilon \to 0]{} 0$, $u(x_\varepsilon) - v(y_\varepsilon) \to M > 0$
and for $\varepsilon > 0$ small, $x_\varepsilon, y_\varepsilon \in G$.

· For $y = y_\varepsilon$, then

$$x \mapsto u(x) - v(y_\varepsilon) - \frac{1}{2\varepsilon}|x - y_\varepsilon|^2$$

has a max at $x_\varepsilon \in G$. (Test function $\varphi(x) = v(y_\varepsilon) + \frac{1}{2\varepsilon}|x - y_\varepsilon|^2$)

$D\varphi(x_\varepsilon) = \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}$.

So as u subsol:

$$(1) \quad u(x_\varepsilon) + \left| \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} \right| \le 0$$

In the same way, for $x = x_\varepsilon$

$$y \mapsto -u(x_\varepsilon) + v(y) + \frac{1}{2\varepsilon}|x_\varepsilon - y|^2$$

has a min at $y_\varepsilon \in G$

As v supersol

(test fn: $+u(x_\varepsilon) - \frac{1}{2\varepsilon}|x_\varepsilon - y|^2$)

(2) $\quad v(y_\varepsilon) + \left| - \frac{y_\varepsilon - x_\varepsilon}{\varepsilon} \right| \geq 0.$

(1) - (2) $\Rightarrow$

$$u(x_\varepsilon) - v(y_\varepsilon) \leq 0.$$

$M > 0$ imp.