# Correction of the exam of June 2013

$\underline{\varepsilon > 0}$ $\left|\begin{array}{l} dX_s = \sqrt{2\varepsilon}\ \sigma(X_s)\, dB_s \\ X_0 = x_0 \in G \end{array}\right.$

$G = \{ x \in \mathbb{R}^d \,/\, \varphi(x) < 0 \}$.

$\forall x \in \mathbb{R}^d, \quad \varphi(x) = 0 \Rightarrow D\varphi(x) \neq 0$

$\tau_G(x_0) = \{ t \geq 0, \quad X_s^{x_0} \notin G \}$

and $\quad V_\varepsilon(x_0) = \mathbb{E}\left[ e^{-\tau_G(x_0)} \right]$.

$\underline{\text{Hyp}}$ $\left[\begin{array}{l} V_\varepsilon \text{ continuous on } \bar{G}, \\ V_\varepsilon = 1 \text{ on } \partial G. \end{array}\right.$

1) To show: $\quad V_\varepsilon(x_0) \to 0 \quad$ as $\varepsilon \to 0$.

We know that : $\forall t > 0$.

$$\mathbb{E}\left[ \sup_{s \in [0,t]} \left| X_s^{x_0} - x_0 \right|^2 \right] = \mathbb{E}\left( \sup_{s \in [0,t]} \left| \int_0^t \sqrt{2\varepsilon}\, \sigma(X_s)\, dB_s \right|^2 \right]$$

$$\leq 2C_d\varepsilon \quad \mathbb{E}\left[ \int_0^t \mathrm{Tr}(\sigma\sigma^*(X_s))\, ds \right]$$

$$\leq 2C_d\varepsilon \, \|\sigma\|_\infty^2 \, t \xrightarrow[\varepsilon \to 0]{} 0$$

let $r > 0$ s.t.

$$\mathring{B}(x_0, r) \subset G$$

If $\sup_{s\in[0,t]} |X_s^{x_0} - x_0| \leq r$

$\Rightarrow \quad \tau_G(x_0) \geq t.$

$$\mathbb{P}[\tau_G(x_0) < t] \leq$$

$$\mathbb{P}[\underbrace{\sup_{s\in[0,t]} |X_s^{x_0} - x_0|}_{\to 0 \text{ in } L^2} \geq r] \longrightarrow 0.$$

$\Rightarrow$ $\forall \delta > 0$ $\quad \mathbb{P}[e^{-\tau_G(x_0)} > \delta] \longrightarrow 0$

So $\mathbb{E}[e^{-\tau_G(x_0)}] = V_\varepsilon(x_0) \to 0.$

2) To prove: $\forall \theta > 0$ stopping time, $\forall x_0 \in G$,

$$V_\varepsilon(x_0) = \mathbb{E}\left[e^{-\theta \wedge \tau_G(x_0)} V_\varepsilon(X^{x_0}_{\theta\wedge\tau_G(x_0)})\right]$$

<u>Strong Markov property</u>:

$$\begin{cases} dY_s = b_s(Y_s)\,ds + \sigma_s(Y_s)\,dB_s \\ Y_{t_0} = y_0 \end{cases} \quad \text{where } b, \sigma \text{ deterministic}$$

$\to Y^{t_0, y_0}$

then $\forall g \in C_b^0(\mathbb{R}^d)$, $\forall \theta > 0$ stopping time

$$\forall t > \theta \quad \mathbb{E}[g(Y_t^{t_0,x_0}) \mid \mathcal{F}_\theta] = \mathbb{E}[g(Y_t^{\theta,z})]\Big|_{z = Y_\theta^{t_0,x_0}}$$

$$I = \mathbb{E}\left[ e^{-\tau_G(x_0)} \,\middle|\, \mathcal{F}_{\tau_G(x_0)\wedge\theta} \right] =$$

$$\mathbb{E}\left[ e^{-(\tau_G(x_0)\wedge\theta) - (\tau_G(x_0) - \tau_G(x_0)\wedge\theta)} \,\middle|\, \mathcal{F}_{\tau_G(x_0)\wedge\theta} \right]$$

$$= e^{-(\tau_G(x_0)\wedge\theta)}\, \mathbb{E}\left( e^{-(\tau_G(x_0) - \tau_G(x_0)\wedge\theta)} \,\middle|\, \mathcal{F}_{\tau_G(x_0)\wedge\theta} \right]$$

where $\tau_G(x_0) - (\tau_G(x_0)\wedge\theta)$

$$= \inf \left\{ t \geq 0 \;/\; X_t^{X^{x_0}_{\tau_G(x_0)\wedge\theta}} \notin G \right\}$$

$$= \tau_G\left( X^{x_0}_{\tau_G(x_0)\wedge\theta} \right).$$

So

$$I = e^{-\tau_G(x_0)\wedge\theta}\, \mathbb{E}\left( e^{-\tau_G\left(X^{x_0}_{\tau_G(x_0)\wedge\theta}\right)} \,\middle|\, \mathcal{F}_{\tau_G(x_0)\wedge\theta} \right]$$

$$\underset{\text{Markov}}{=} e^{-\tau_G(x_0)\wedge\theta}\, \mathbb{E}\left[ e^{-\tau_G(y)} \right]\Big|_{y = X^{x_0}_{\tau_G(x_0)\wedge\theta}}$$

$$= e^{-\tau_G(x_0)\wedge\theta}\, V_\varepsilon\left( X^{x_0}_{\tau_G(x_0)\wedge\theta} \right).$$

$$\Rightarrow \quad V_\varepsilon(x_0) = \mathbb{E}(I) = \mathbb{E}\left( e^{-\tau_G(x_0)\wedge\theta}\, V_\varepsilon\left( X^{x_0}_{\tau_G(x_0)\wedge\theta} \right) \right].$$

3) Show that $V_\varepsilon$ solves

$$(*) \quad \begin{cases} V_\varepsilon - \varepsilon \, \mathrm{Tr}(a D^2 V_\varepsilon) = 0 & \text{in } G \\ V_\varepsilon = 1 & \text{on } \partial G. \end{cases} \Big) \text{ assumption}$$

- $V_\varepsilon$ subsolution : ~~/~~ supersol

let $\varphi$ be $C^2$ and such that

$V_\varepsilon - \varphi$ has a minimum / maximum at some point $x_0 \in G$. Without loss of generality, we assume that $\varphi(x_0) = V_\varepsilon(x_0)$.

$\Rightarrow \quad V_\varepsilon(x) \underset{\geq}{\leq} \varphi(x)$ in $G$.

For $\theta = h > 0$, we have by (2) that

$$\cancel{V_\varepsilon(x_0)} = \mathbb{E}\Big[ \underbrace{e^{-h \wedge \tau_G(x_0)}}_{\geq 0} \underbrace{V_\varepsilon\big(X^{x_0}_{h\wedge \tau_G(x_0)}\big)}_{\leq \varphi} \Big]$$

$$\underset{\geq}{\leq} \mathbb{E}\Big[ e^{-h\wedge \tau_G(x_0)} \varphi\big(X^{x_0}_{h \wedge \tau_G(x_0)}\big)\Big]$$

$$\underset{\text{Itô formula}}{=} \mathbb{E}\Big[ \cancel{\varphi(x_0)} + \int_0^{h\wedge \tau_G(x_0)} \Big(-e^{-s}\varphi(X_s^x)$$

for $\psi(t,x) = e^{-t}\varphi(x)$

$$+ \frac{e^{-s}}{2} 2\varepsilon \, \mathrm{Tr}\big(\sigma\sigma^*(X_s^x) D^2\varphi(X_s^x)\big)\Big) ds$$

$$+ \int_0^{h\wedge \tau_G(x_0)} D\varphi(X_s^x) \sqrt{2\varepsilon}\, \sigma(X_s^x) dB_s \Big]$$

$\bar{G}$ compact $\Rightarrow$ $|D\varphi(X_s^x)|$ bd on $[0, \tau_G(x_0)]$

So $\mathbb{E}_L\left[\int_0^{h\wedge\tau_G(x_0)} \nabla\varphi \dots dB_s\right] = 0.$

So $0 \leq\!\!\!\!\!\geq \mathbb{E}_L\left[\int_0^{h\wedge\tau_G(x_0)} e^{-s}\left(-\varphi(X_s^{x_0}) + \varepsilon \operatorname{Tr}(a\nabla^2\varphi(X_s^{x_0}))\right)\right]$

$0 \leq\!\!\!\!\!\geq \frac{1}{h}\mathbb{E}\left[\int_0^h e^{-s}\left(-\varphi(X_s^{x_0}) + \varepsilon \operatorname{Tr}(aD^2\varphi(X_s^{x_0}))\right)\right]$

$+ \frac{1}{h} o(h)$

because $\tau_G(x_0) > 0$ a.s.

As $X_s^{x_0} \to x_0$ as $s \to 0$, we have

$-\varphi(x_0) + \varepsilon \operatorname{Tr}(aD^2\varphi(x_0)) \geq\!\!\!\!\!\leq 0$

$\Rightarrow \quad V_\varepsilon(x_0) - \varepsilon \operatorname{Tr}(aD^2\varphi) \leq\!\!\!\!\!\geq 0$

So $V_\varepsilon$ is a subsolution.

The proof for super sol. is the same.

4) Set $W_\varepsilon(x) = -\sqrt{\varepsilon}\,\operatorname{Log}(V_\varepsilon(x))$, $x \in G$

let us check that $W_\varepsilon$ is a viscosity subsolution of (1).

Let $\varphi$ be $C^2$ st. $W_\varepsilon - \varphi$ has a max at $x_0 \in G$. and $W_\varepsilon(x_0) = \varphi(x_0)$.

$W_\varepsilon(x) = -\sqrt{\varepsilon}\,\operatorname{Log}(V_\varepsilon(x)) \leq \varphi(x)$

$$\Rightarrow \quad V_\varepsilon(x) \geq e^{-\frac{1}{\sqrt{\varepsilon}}\varphi(x)} := \psi(x)$$

with an equality at $x_0$.

As $V_\varepsilon$ is a viscosity supersolution of (*) and $\psi$ is $C^2$ and such that $V_\varepsilon - \psi$ has a minimum at $x_0$, we have:

$$V_\varepsilon(x_0) - \varepsilon \, Tr\left(a(x_0)\, D^2\psi(x_0)\right) \geq 0$$

where $D\psi(x) = -\frac{1}{\sqrt{\varepsilon}} e^{-\frac{1}{\sqrt{\varepsilon}}\varphi(x)} D\varphi(x)$

and $D^2\psi(x) = e^{-\frac{1}{\sqrt{\varepsilon}}\varphi(x)}\left(-\frac{1}{\sqrt{\varepsilon}} D^2\varphi(x) + \frac{1}{\varepsilon} D\varphi(x)^* D\varphi(x)\right)$

$$) \; Tr\big(\underbrace{a(x)}_{\sigma\sigma^*(x)} D^2\psi(x)\big) = e^{-\frac{1}{\sqrt{\varepsilon}}\varphi(x)} \cdot \times$$

$$\left[-\frac{1}{\sqrt{\varepsilon}} Tr\left(a(x) D^2\varphi(x)\right) + \frac{1}{\varepsilon}\underbrace{Tr\left(\sigma\sigma^* D\varphi^* D\varphi\right)}_{= |\sigma^*(x) D\varphi(x)|^2}\right]$$

$$\underbrace{e^{\frac{1}{\sqrt{\varepsilon}}W_\varepsilon(x_0)} V_\varepsilon(x_0)}_{\overset{\shortparallel}{1}} + \sqrt{\varepsilon}\, Tr\left(a(x_0) D^2\varphi(x_0)\right) - |\sigma^*(x_0) D\varphi(x_0)|^2 \geq 0$$

$$\Rightarrow -\sqrt{\varepsilon}\, Tr\left(a D^2\varphi\right) + |\sigma^* D\varphi|^2 \leq 1$$

So $W_\varepsilon$ is a subsol.

- The proof that $W_\varepsilon$ is a supersolution is symmetrical.
- On $\partial G$, $V_\varepsilon(x) = 1 \Rightarrow W_\varepsilon(x) = 0$.

5) Check that $W_\varepsilon \geq 0$ in $G$.

We know that $V_\varepsilon(x) = \mathbb{E}\left[\underbrace{e^{-\tau_G(x)}}_{\in (0,1)}\right] \in (0,1)$

$W_\varepsilon(x) = -\sqrt{\varepsilon} \log\left(V_\varepsilon(x)\right) > 0$.

6) a) let us check that $-\lambda\varphi$ is a viscosity super sol to (1).

Recall that $G = \{\varphi < 0\}$.

We know that $\varphi$ is $C^2$, so we only need to check that

(pb) $\begin{cases} -\sqrt{\varepsilon}\, \mathrm{Tr}\left(a(-\lambda D^2\varphi)\right) + \left|\sigma^*(-\lambda D\varphi)\right|^2 \geq 1. \end{cases}$

(note that $-\lambda\varphi(x) = 0 \quad x \in \partial G$)

We know that $D^2\varphi > 0$ in $G$.

b) By comparison (if (a) holds) we have $0 \leq W_\varepsilon \leq \underbrace{-\lambda\varphi}_{\text{bd indep of } \varepsilon}$.

$\Rightarrow W_\varepsilon$ is bd