For the exam (Dec. 15)

- Feynman-Kac (derive the DPP, and the HJ eq)
- For control pbs:
  - "guess" the DPP
  - write and prove a verification theorem.
  - Prove from DPP that the value f° is a viscosity sol of the HJ eq.
  - Regularity of the value f°.

From DPP to HJ eq (supersolution)

Control pb (infinity horizon)

$$X^{x,\alpha} \leftarrow \begin{cases} dX_s = b(X_s,\alpha_s)\,ds + \sigma(X_s,\alpha_s)\,dW_s \\ X_0 = x \end{cases}$$

$$V(x) = \inf_{\alpha} \mathbb{E}\left[\int_0^{+\infty} e^{-rs} f(X_s^{x,\alpha},\alpha_s)\,ds\right].$$

$(r>0,\ f\ \text{bd}).$

DPP: $\forall \tau \geq 0$ stopping time

$$V(x) = \inf_{\alpha} \mathbb{E}\left[\int_0^{\tau} e^{-rs} f(X_s^{x,\alpha},\alpha_s)\,ds + e^{-r\tau} V(X_\tau^{x,\alpha})\right].$$

Proved: • V is continuous in $\mathbb{R}^d$ and bd.

• V is a viscosity subsolution to the HJ eq:

$$(HJ) \quad rV(x) + H(x, DV(x), D^2V(x)) = 0 \quad \text{in } \mathbb{R}^d$$

where

$$H(x, p, A) = \sup_{a \in A} \{ -p \cdot b(x,a) - \tfrac{1}{2} Tr(\sigma\sigma^t(x,a)A) - f(x,a) \}.$$

[Claim: V is a viscosity supersolution of (HJ)]

Proof: Let $\varphi \in C_b^2(\mathbb{R}^d)$ such that $V - \varphi$ has a minimum at some point $x_0 \in \mathbb{R}^d$. Without loss of generality we assume $V(x_0) = \varphi(x_0)$.

Then $V(x) \geq \varphi(x)$ on $\mathbb{R}^d$ with $V(x_0) = \varphi(x_0)$.

• Argument by contradiction: assume that $rV(x_0) + H(x_0, D\varphi(x_0), D^2\varphi(x_0)) > 0$.

By continuity $\exists\, r > 0$ and $\eta > 0$ s.t

[$r\varphi(x) + H(x, D\varphi(x), D^2\varphi(x)) \leq -\eta$ in $B(x_0, r)$.]

We consider the stopping time

$$\tau := \tau(x_0, \alpha) = \operatorname{Inf}\{t > 0,\ X_t^{x_0,\alpha} \notin B(x_0, 1)\}$$

By the DPP, we have, $\forall h > 0$

$$(*) \quad V(x_0) = \operatorname{Inf}_{\alpha} \mathbb{E}\left[\int_0^{\tau_h} e^{-\lambda t} f(X_t^{x_0,\alpha}, \alpha_t)\, dt + e^{-\lambda \tau_h} V(X_{\tau_h}^{x_0,\alpha})\right]$$

where $\tau_h = \operatorname{Inf}\{\tau, h\}$

Let α^h be h^2-optimal in $(*)$:

$$h^2 + V(x_0) \geqslant \mathbb{E}\left[\int_0^{\tau_h} e^{-\lambda t} f(X_t^{x_0,\alpha^h}, \alpha_t^h)\, dt + e^{-\lambda \tau_h} V(X_{\tau_h}^{x_0,\alpha^h})\right]$$

As $V \geqslant \varphi$, and $V = \varphi$ at x_0,

we have:

$$h^2 + \varphi(x_0) \geqslant \mathbb{E}\left[\int_0^{\tau_h} e^{-\lambda t} f\, dt + e^{-\lambda \tau_h} \varphi(X_{\tau_h}^{x_0,\alpha^h})\right].$$

By Itô' formula for $e^{-\lambda t}\varphi(X_t^{x_0,\alpha^h})$:

$$h^2 \geqslant \mathbb{E}\Big[\int_0^{\tau_h} e^{-\lambda t}\Big(-\lambda\varphi(X_t) + D\varphi(X_t)\cdot b(X_t, \alpha_t^h) + \tfrac{1}{2}\operatorname{Tr}(\sigma\sigma^*(X_t, \alpha_t^h) D^2\varphi(X_t)) + f(X_t, \alpha_t^h)\Big)\, dt\Big]$$

We know that $\forall x \in B(x_0, r)$, $\forall a \in A$,

$$r\varphi(x) - b(x,a)\cdot D\varphi(x) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^t(x,a)D^2\varphi(x)) - f(x,a) \leq -\eta.$$

So

$$h^2 \geq \mathbb{E}\left[\int_0^{\tau_h} e^{-rt}\cdot\eta\, dt\right] = \eta\cdot\mathbb{E}\left[\frac{1-e^{-r\tau_h}}{r}\right]$$

We know,

$$(**) \quad \mathbb{P}[\tau_h \neq h] \to 0$$

Divide by h and let $h \to 0^+$:

$$0 \geq \eta\cdot 1 > 0 \quad \text{imp.}$$

<u>Proof of $(**)$</u> as $\tau_h = \tau \wedge h$

$$\{\tau_h \neq h\} = \{\tau < h\} = \{\sup_{0\leq t\leq h} |X_t^{x_0,\alpha} - x_0| \geq r\}$$

where $\mathbb{E}\left[\sup_{0\leq t\leq h} |X_t^{x_0,\alpha} - x_0|^2\right] \leq C h$

where $C = C(\|b\|_\infty, \|\sigma\|_\infty)$.

Exam of May 5, 2014

- We work in $\mathbb{R}^N$, $N \geq 2$.
- (B_t) N-dim Brownian Motion.
- $(\mathcal{F}_t)$ associated filtration (completed)
- $L^2_{ad} = \{ \alpha : [0,+\infty[\times \Omega \to \mathbb{R}^N$, adapted $\mathbb{E}\left[\int_0^{+\infty} e^{-t} |\alpha_t|^2 dt\right] < +\infty \}$.
- <u>SDE</u>: For $\alpha \in L^2_{ad}$,

$X^{x,\alpha}$ sol to $\begin{cases} dX_t = \alpha_t dt + \sqrt{2}\, dB_t \\ X_0 = x \in \mathbb{R}^N \end{cases}$

$\mathcal{A}(x) = \{ \alpha \in L^2_{ad}$, s.t. $\mathbb{P}[X^{x,\alpha}_t \in G \ \forall t \geq 0] = 1 \}$

where $G = \{ x \in \mathbb{R}^N / |x| < 1 \}$.

<u>Cost</u> $J(x, \alpha) =$

$$\mathbb{E}\left[\int_0^{+\infty} e^{-t} \left(h(X^{x,\alpha}_t) + \frac{1}{2} |\alpha_t|^2 \right) dt \right]$$

where $h : G \to \mathbb{R}$ is bd.

<u>Value function</u>:

$$v(x) = \inf_{\alpha \in \mathcal{A}(x)} J(x, \alpha).$$

Part I

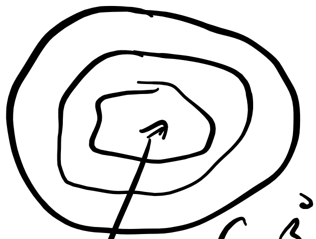
We assume that there exists $v : G \to (0, +\infty)$, C^∞,

s.t. $\quad \| \; v(x) \underset{|x| \to 1^-}{\longrightarrow} +\infty$

and $\quad -\Delta v(x) + \frac{1}{2}\| Dv(x)\|^2 + v(x) \geq 0$ in G.

- Let (X_t) sol to

$$\begin{cases} dX_t = -Dv(X_t)\,dt + \sqrt{2}\,dB_t \\ X_0 = x \in G \end{cases}$$



$G = \overset{\circ}{B}(0,1)$

$\{v \leq m\}$

defined on $[0, \theta)$ where

$$\theta = \operatorname{Inf}\{ t > 0 \;/\; X_t \notin G \}$$

$(+\infty$ if $X_t \in G \;\; \forall t \geq 0)$.

Set $\quad \tau_M = \operatorname{Inf}\{ t > 0 \;/\; v(X_t) > M \}$

$\forall M > 0$

I.1) Show that $\forall t \geq 0$,

$$\mathbb{E}\left[v(X_{t \wedge \tau_M})\, e^{-(t \wedge \tau_M)} \right] \leq v(x) - \mathbb{E}\left[\int_0^{\tau_M \wedge t} \frac{e^{-s}}{2} |Dv(X_s)|^2 \, ds \right].$$

Itô's formula for $t \mapsto e^{-t} v(X_t)$

$$e^{-(t \wedge \tau_M)}\, v(X_{t \wedge \tau_M}) =$$

$$v(x) + \int_0^{t \wedge \tau_M} e^{-s} \Big(\underbrace{- v(X_s) - \|Dv(X_s)\|^2}_{(*)} +$$

$$+ \overbrace{\cancel{\tfrac{1}{2}}\, \mathrm{Tr}^*\left(\cancel{\sqrt{2}}^2 D^2 v(X_s)\right)} \, ds$$

$$+ \sqrt{2} \int_0^{t \wedge T_m} e^{-s} Dv(X_s) \cdot dB_s$$

$(*)$ $\begin{cases} \text{As } \{v \le M\} \subset\subset G, \\ Dv \text{ is bd in } \{v \le M\}. \end{cases}$

$\Rightarrow \sqrt{2}\,\mathbb{E}\left[\int_0^{t \wedge T_m} e^{-s} Dv(X_s)\, ds\right] = 0$ because the intégral is a martingale.

For $(*)$ let us check that $\{v \le M\}$ compact.

Let (x_n) a sequence in $\{v \le M\} \subset G = \mathring{B}(0,1)$.

$\exists\, (x_{n_k})$ a subseq which converges to some $x \in \overline{G} = \overline{B}(0,1)$.

if $x \in \partial G$, then $|x| = 1 \Rightarrow |x_{n_k}| \xrightarrow[k]{} 1$.

then $\underset{\underset{M}{\wedge|}}{v(x_{n_k})} \to +\infty$. contradiction

So $x \in G$ & As v continuous in G

$\Rightarrow \underset{\underset{M}{\wedge|}}{v(x_{n_k})} \to v(x) \le M$

$$(*) = \left(-v - |Dv|^2 + \Delta v\right)(X_s)$$

but v solves $-\Delta v + \frac{1}{2}|Dv|^2 + v \geq 0$

$$\le -\tfrac{1}{2} |Dv|^2 (X_s).$$

So $\mathbb{E}\left[e^{-\tau_M \wedge t} v(X_{t \wedge \tau_M})\right] \leq$

$$v(x) - \frac{1}{2}\mathbb{E}\left[\int_0^{t \wedge \tau_M} e^{-s} |Dv(X_s)|^2 ds\right].$$

I 2) Check that

$$v(X_{\tau_M \wedge t}) \mathbb{1}_{\{\tau_M \leq t\}} = M \mathbb{1}_{\{\tau_M \leq t\}}$$

and infer that $\mathbb{P}[\tau_M \leq t] \leq \dfrac{v(x) e^t}{M}$.

We know by definition of τ_M that

$v(X_s) < M$ on $[0, \tau_M)$ and

$\exists h_n \to 0$ s.t. $v(X_{\tau_M + h_n}) \geq M$.

$h_n \geq 0$

But $s \mapsto v(X_s)$ is continuous a.s

$$\Rightarrow v(X_{\tau_M}) = M.$$

So that $v(X_{\tau_M \wedge t}) \mathbb{1}_{\{\tau_M \leq t\}} =$

$$v(X_{\tau_M}) \mathbb{1}_{\{\tau_M \leq t\}} = M \mathbb{1}_{\{\tau_M \leq t\}}.$$

Recall

$$\mathbb{E}\Big[\underbrace{v(X_{t \wedge \tau_M})}_{\geq 0} \underbrace{e^{-\tau_M \wedge t}}_{\geq 0}\Big] \leq v(x)$$

$$\geq \mathbb{E}\left[\mathbb{1}_{\{\tau_M \leq t\}} v(X_{t \wedge \tau_M}) e^{-\tau_M \wedge t}\right]$$

$$\geq \mathbb{E}\left[\mathbb{1}_{\{\tau_M \leq t\}} M e^{-t}\right]$$

$\mathbb{P}[\tau_M \leq t] \leq v(x) e^t / M$. $\Leftarrow$