I 3) [Show $\theta = +\infty$ a.s.

We know (I 2) that $\mathbb{P}[\tau_M \le t] \le \frac{v(x)e^t}{M}$ $\forall t>0, \forall M>0$.

and that $\theta \geq \tau_M$.

So $\{\theta \le t\} \subset \{\tau_M \le t\}$.

$\Rightarrow \mathbb{P}[\theta \le t] \le \frac{v(x)e^t}{M}$ $\forall M>0$

As $M \to +\infty$, $\mathbb{P}[\theta \le t] = 0$

As $t \to +\infty \Rightarrow \mathbb{P}[\theta < +\infty] = 0$

So $\theta = +\infty$ a.s.

I.4) [Show that
$$\mathbb{E}\left[\int_0^{+\infty} e^{-t} |Dv(X_t)|^2 dt\right] < +\infty$$

By I.1) we have

$$\underbrace{\mathbb{E}\left[v(X_{t\wedge\tau_M}) e^{-t\wedge\tau_M}\right]}_{\geq 0} \le v(x) - \frac{1}{2}\mathbb{E}\left[\int_0^{t\wedge\tau_M} e^{-s}|Dv(X_s)|^2 ds\right]$$

$$\Rightarrow \mathbb{E}\left[\int_0^{t\wedge\tau_M} e^{-s}|Dv(X_s)|^2 ds\right] \le 2v(x)$$

Let $t \to +\infty$ and $M \to +\infty$.

Then $t\wedge\tau_M \xrightarrow[t\to+\infty]{} \tau_M$

and $\tau_M \xrightarrow[M\to+\infty]{} +\infty$ because $v(X_t) < +\infty$ $\forall t$

By monotone cV, we get
$$\mathbb{E}\left[\int_0^{+\infty} e^{-t}|Dv(X_t)|^2 dt\right] \leq 2v(x) < +\infty.$$
So $\bar{\alpha}_t := -Dv(X_t)$ satisfies

- $\mathbb{E}\left[\int_0^{+\infty} e^{-t}|\bar{\alpha}_t|^2 ds\right] < +\infty$
- $X_t^{x,\bar{\alpha}} \in G \quad \forall t \geq 0$ a.s.

$\Rightarrow \bar{\alpha} \in \mathcal{A}(x)$.

I.5) Let $x \in G$ and $a \in \mathbb{R}^N$.

Set $r = \frac{1-|x|}{2}$

Let $\tau = \mathrm{Inf}\{t \geq 0, |X_t^{x,a} - x| > r\}$

Finally ~~let~~ $(X_t)_{t \geq \tau}$ the sol. to
$$\begin{cases} dX_t = -Dv(X_t)\,dt + \sqrt{2}dB_t \\ X_\tau = X_\tau^{x,a}. \end{cases}$$
By I.3, $X_t$ exists on $[\tau, +\infty)$

and by I.4 (By Markov)
$$\mathbb{E}\left[\int_\tau^{+\infty} e^{-t}|Dv(X_t)|^2 dt \,\middle|\, \mathcal{F}_\tau\right] \leq 2e^{-\tau} v(X_\tau)$$
Let us set $\alpha_t = a$ on $[0, \tau]$ and

$\alpha_t = -Dv(X_t)$ on $(\tau, +\infty)$.

• Then $X_t^{x,\alpha} = \begin{cases} X_t^{x,a} \in G & \text{for } t \in [0,\tau] \\ X_t \in G & (\tau, +\infty) \end{cases}$

$\qquad \| $

$X_t^{X_\tau^{x,a}, -Dv, \tau}$

$\Rightarrow X_t^{x,\alpha} \in G \quad \forall t \geq 0$

• Moreover

$$\mathbb{E}\left[\int_0^{+\infty} e^{-t} |\alpha_t|^2 dt\right] =$$

$$\mathbb{E}\left[\int_0^{\tau} e^{-t} |a|^2 dt\right] + \mathbb{E}\left[\int_\tau^{+\infty} e^{-t} |Dv(X_t)|^2 dt\right]$$

$$\leq |a|^2 \int_0^{+\infty} e^{-t} dt + \mathbb{E}\left[\mathbb{E}\left[\int_\tau^{+\infty} e^{-t} |Dv(X_t)|^2 dt \,\Big|\, \mathcal{F}_\tau\right]\right]$$

$$\underset{\text{see above}}{\leq} |a|^2 + \mathbb{E}\Big[2 \underbrace{e^{-\tau}}_{\leq 1} v(\underbrace{X_\tau^{x,a}}_{\in B(x,r)})\Big]$$

$$\leq |a|^2 + 2\|v\|_{L^\infty(B(x,r))} < +\infty.$$

So $\begin{cases} \alpha \in \mathcal{A}(x). \\ \text{and } \alpha = a \text{ on } (0,\tau). \end{cases}$

**Part II** : Assume that $v$ is continuous in $G$. $\left(v(x) = \inf_{\alpha \in \mathcal{A}(x)} J(x,\alpha)\right)$

**II.1) DPP**

Explain heuristically why, $\forall \tau > 0$ stopping time

$$v(x) = \inf_{\alpha \in \mathcal{A}(x)} \mathbb{E}\left[ \int_0^{\tau} e^{-t} \left( h(X_t^{x,\alpha}) + \frac{1}{2}|\alpha_t|^2 \right) dt + e^{-\tau} v(X_\tau^{x,\alpha}) \right].$$

Idea of proof for $\tau = \underbrace{t_1}_{\geq 0}$ deterministic

$$v(x) = \inf_{\alpha \in \mathcal{A}(x)} \mathbb{E}\left[ \int_0^{t_1} e^{-t} \left( h(X_t^{x,\alpha}) + \frac{1}{2}|\alpha_t|^2 \right) dt + \underbrace{\int_{t_1}^{+\infty} e^{-t} \left( h(X_t^{x,\alpha}) + \frac{1}{2}|\alpha_t|^2 \right) dt}_{*} \right]$$

$$(*) = \int_0^{+\infty} e^{-(t+t_1)} \left( h(X_{t_1+t}^{x,\alpha}) + \frac{1}{2}|\alpha_{t+t_1}|^2 \right) dt$$

$$= e^{-t_1} \int_0^{+\infty} e^{-t} \left( h(X_{t_1+t}^{x,\alpha}) + \frac{1}{2}|\alpha'_t|^2 \right) dt$$

where $\alpha'_t = \alpha_{t+t_1}$ $\quad t \geq 0$

and where

$$\mathbb{E}\left[ \int_0^{+\infty} e^{-t} h(X_{t_1+t}^{x,\alpha}) dt \,\Big|\, \mathcal{F}_{t_1} \right]$$

$$\text{"="} \quad \mathbb{E}\left[ \int_0^{+\infty} e^{-t} h(X_t^{y,\alpha'}) dt \right] \quad y = X_{t_1}^{x,\alpha}$$

(Markov, if we argue as if $\alpha$ is in a feedback form), because the eq does not depend on time.

Note that $\alpha' \in L^2_{adm}$ and $X_t^{y,\alpha'} \in \mathcal{A}(y)$

because $X_t^{y,\alpha'} \underset{law}{=} X_{t_1+t}^{x,\alpha} \in G$.

$\forall t \geq 0$

So

$$v(x) = \operatorname{Inf}_{\alpha \in \mathcal{A}(x)} \mathbb{E}\Big[ \int_0^{t_1} (-)\,dt + e^{-t_1} \mathbb{E}\Big[ \int_0^{+\infty} e^{-t} \big( h(X_t^{y,\alpha'}) + \tfrac{1}{2}|\alpha_t'|^2 \big) dt \,\Big|\, \mathcal{F}_{t_1} \Big] \Big]_{y = X_{t_1}^{x,\alpha}}$$

$$\text{"="} \operatorname{Inf}_{\alpha \in \mathcal{A}(x)} \mathbb{E}\Big[ \int_0^{t_1} (\ )\,dt + e^{-t_1} v(X_{t_1}^{x,\alpha}) \Big].$$

Parenthesis:

$$\mathbb{E}\Big[ \int_0^{+\infty} e^{-t} h(X_{t+t_1}^{0,x})\,dt \Big] =$$

$$\mathbb{E}\Big[ \mathbb{E}\Big[ \int_0^{+\infty} e^{-t} h(X_t^{t_1,y}) \Big]_{y = X_{t_1}^{0,x}} \Big]$$

$$\mathbb{E}\Big[ \int_0^{+\infty} e^{-t} h(X_t^{t_1, X_{t_1}^{0,x}})\,dt \Big]$$

$$\varphi(y) = \mathbb{E}\Big[ \int_0^{+\infty} e^{-t} h(X_t^{t_1,y})\,dt \Big] \qquad \overset{?}{=} \varphi(X_{t_1}^{0,x})$$

II 2) Prove that $v$ is a subsolution

to $\quad -\Delta v + \frac{1}{2}|Dv|^2 + v = h$ in $G$.

- $v$ is continuous
- Assume that $v - \varphi$ has a local max at $x_0 \in G$, where $\varphi$ is $C^2$.

We want to check that

$$-\Delta \varphi + \frac{1}{2}|D\varphi|^2 + u \leq h \quad \text{at } x_0 .$$

Fix $a \in \mathbb{R}^N$. We want to consider $X^{x_0, a}$. pb $a \notin \mathcal{A}(x)$.

By (I.5) $\exists\, \alpha \in \mathcal{A}(x)$ s.t.

$\alpha = a$ in $[0, \tau]$ where

$$\tau = \inf \{ t > 0 ,\ X_t^{x,a} \notin \overbrace{\bar{B}(x, r)}^{\subset G} \}$$

So by DPP

$$u(x_0) \leq \mathbb{E}\left[ \int_0^{\tau \wedge h} e^{-t} \left( h(X_t^{x_0,\alpha}) + \frac{1}{2}|\alpha_t|^2 \right) dt + e^{-\tau\wedge h} u(X_{\tau\wedge h}^{x_0,\alpha}) \right]$$

Without loss of generality we assume

$u(x_0) = \varphi(x_0)$. $\Rightarrow$ $u \leq \varphi$ in $G$.

$$\Rightarrow \varphi(x_0) \leq \mathbb{E}\left[ \int_0^{\tau\wedge h} e^{-t} (\quad) dt + e^{-\tau \wedge h} \varphi(X_{\tau\wedge h}^{x_0, a}) \right]$$

$$\Rightarrow \quad 0 \leq \mathbb{E}\left[ \int_0^{\tau \wedge h} e^{-t} \left( -\varphi(X_t) + a \cdot D\varphi + \Delta \varphi \right)(X_t^{x_0,a})\, dt \right]$$