# Mean Field Games with controlled volatility and McKean-Vlasov second order backward SDEs

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# Plan

## The Mean Field Game (MFG)

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## Strong solution of an MFG with common noise

Let $M$ be a random probability measure and, $Q$ a set of control processes with values in $A$. We consider the following.

**Controlled dynamic in the environment $M$ :**

$$\begin{aligned} dX_t^{q,M} &= b_t\left(X_\cdot^{q,M}, q_t, M\right) dt + \sigma_t\left(X_\cdot^{q,M}, q_t, M\right) dW_t \\ &\quad + \sigma_t^0\left(X_\cdot^{q,M}, q_t, M\right) dW_t^0. \end{aligned}$$

**Optimization in the environment $M$ :**

$$\max_{q \in Q} \ \mathbb{E}\left[\xi\left(X_\cdot^{q,M}\right) + \int_0^T f_r\left(X_\cdot^{q,M}, q_r, M\right) dr\right],$$

# Strong solution of an MFG with common noise

## Strong solution

A strong solution of this MFG is a solution of the following fix point problem : find $M$ such that

$$M = \mathcal{L}(X^{\star,M}|W^0), \ a.s.$$

where $X^{\star,M}$ is an optimal diffusion in the environment $M$ and $\mathcal{L}(X^{\star,M}|W^0)$ is its conditional law with respect to $\sigma(W^0)$.

# Weak relaxed solutions

Weak relaxed solutions

# Relaxed controls

## Relaxed controls

We allow relaxed controls (or *mixed strategies*), so that control processes will take values $Prob(A)$ rather that $A$.

## Why relaxed controls ?

Because they will help us to get compactness and convexity, hence to apply the Kakutani's fixed point theorem.

The cost functional then becomes

## Cost functional in the environment $M$ :

$$\mathbb{E}\left[\xi(X) + \int_0^T \int_A f_r\left(X_\cdot, a, M\right) q_r(da)dr\right].$$

# Weak relaxed formulation of the MFG

We work "in law", meaning that the controlled SDE is replaced with a controlled martingale problem and that we will work with the possible joint laws of the quintuplet $(X, W, W^0, Q, M)$.

## Canonical space

$$\Omega := \mathcal{X} \times \mathcal{W} \times \mathcal{W}^0 \times \mathcal{Q} \times Prob(\mathcal{X}),$$

where $\mathcal{X}, \mathcal{W}, \mathcal{W}^0, \mathcal{Q}, Prob(\mathcal{X})$ are the canonical spaces for :

- $X$ the state space of the *typical player* ;
- $W$ its *individual noise* ;
- $W^0$ the *common noise* ;
- $Q$ the control process of the typical player ;
- $M$ the distribution of the other players.

All these spaces are equipped with natural filtrations.

# Admissible controls

Let $\pi^0 \in Prob(\mathcal{W}^0 \times Prob(\mathcal{X}))$ be a possible law for $(W^0, M)$.

**We say that $\mathbb{P} \in Prob(\Omega)$ is $\pi^0$-admissible if**

- $\mathbb{P} \circ (W^0, M)^{-1} = \pi^0$
- $W$ is a $\mathbb{P}$-Brownian motion independent from $(M, W^0)$
- for all $t$, $\mathcal{F}_t^Q$ is independent from $\mathcal{F}_T^{W,M}$ conditionally to $\mathcal{F}_t^{W,M}$
- for all $\phi \in \mathcal{C}_b^2(\mathbb{R}^d \times \mathbb{R}^p \times \mathbb{R}^{p_0})$,

$$\phi(\bar{X}_t) - \int_0^t \int_A \left( \bar{b}_r(X, a, M) \cdot D\phi(\bar{X}_r) + \frac{1}{2} \bar{\sigma}\bar{\sigma}_r^\intercal(X, a, M) : D^2\phi(\bar{X}_r) \right) Q_r(da)dr$$

is a $\mathbb{P}$-martingale.

Where $\bar{X} := (X, W, W^0)$ and $\bar{b}, \bar{\sigma}$ are its drift and volatility.

# Optimal controls and weak relaxed solution

## Optimal controls

We say that $\mathbb{P} \in Prob(\Omega)$ is $\pi^0$-optimal if it maximizes

$$\mathbb{E}^{\mathbb{P}}\left[\xi(X) + \int_0^T \int_A f_r\left(X_\cdot, a, M\right) Q_r(da)dr\right],$$

within the set of $\pi^0$-admissible controls.

## Solution of the MFG

$\mathbb{P} \in Prob(\Omega)$ is a weak relaxed solution of the MFG if :

- $\mathbb{P}$ is $\pi^0$-optimal for some $\pi^0$ ;
- $M = \mathbb{P}(X|M, W^0)$, $\mathbb{P}$ a.s.

## Existence of a weak relaxed solution

**B-Touzi 20'**

**There exists a weak relaxed solution** if :

- $f, \xi, b, \sigma, \sigma^0$ are **bounded continuous** ;
- $b, \sigma, \sigma^0$ are **locally Lipschitz** in $x$, uniformly in $(t, a, m)$.

To be compared to :

**Carmona-Delarue-Lacker 16'**

Same result but when $\sigma, \sigma^0$ do not depend on the control parameter a .

## Second order backward SDEs (2BSDEs)

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# Second order stochastic control and 2BSDEs

2BSDEs are called of *order 2* because they are related to volatility control problems, hence HJB PDEs with Hamiltonian of order 2.

## Control problems of order 2

Assume that we want to maximize over $q$ the functional

$$\mathbb{E}\left[\xi + \int_0^T f_r(X, q_r)dr\right],$$

where $X$ has dynamic

$$dX_t = \sigma_t \lambda_t(X, q_t)dt + \sigma_t(X, q_t)dW_t.$$

## Value function of the problem

$$Y_t(x) := \sup_q \ \mathbb{E}^{\mathbb{P}^q}\left[\xi + \int_t^T f_r(X, q_r)dr \middle| X_{\wedge t} = x_{\wedge t}\right].$$

# Stochastic control of *order 1* and BSDEs

## BSDE and Hamiltonian without volatility control

**Without volatility control**, the value function of the control problem satisfies the BSDE

$$Y_t = \xi + \int_t^T F_r(X, Z_r)dr - \int_s^T Z_r dX_r, \; s \in [t, T], \quad \mathbb{P} - \text{p.s.}$$

where $\mathbb{P}$ is the driftless law of $X$, and $F$ is the **Hamiltonian** :

$$F_t(x, z) := \sup_{a \in A} \quad f_t(x, a) + z \cdot \sigma_t \lambda_t(x, a).$$

# Stochastic control of *order 2* and 2BSDEs

## Formulation of the 2BSDE (Soner-Touzi-Zhang 11')

When $\sigma$ is controlled, there exists processes $Z, U$ such that

- $$Y_t = \xi + \int_t^T \hat{F}_r(X, Z_r)dr - \int_t^T Z_r dX_r - (U_T - U_t),\ t \in [0, T],\ \mathbb{P}^q - \text{p.s.}, \forall q;$$
- $U$ is a $\mathbb{P}^q$-**super**martingale for all $q$ ;
- $\inf_q \mathbb{E}^{\mathbb{P}^q}[U_T] = U_0 = 0.$

$\hat{F}$ is a type of Hamiltonian with **aggregated volatility**.

## Interpretation of $U$

$U$ measures the lack of optimality of a control $q$.

($q$ is optimal) $\Longleftrightarrow$ ($\mathbb{E}^{\mathbb{P}^q}[U_T] = 0$) $\Longleftrightarrow$ ($U$ is a $\mathbb{P}^q$-martingale).

# Representation of the MFG though a McKean-Vlasov 2BSDE

Linking an MFG with **controlled volatility** to a McKean-Vlasov BSDE of **order 2** :

A first step towards a generalization of Carmona-Delarue's theory to the order 2.

# 2BSDE McKean-Vlasov associated to the MFG

We consider the **NO common noise setup**.

For all, $m \in Prob(\mathcal{X})$ let $\mathcal{P}(m)$ be the set of all $X$-marginals of $m$-admissible probabilities.

**Here, $\mathbb{P}$ is $m$-admissible if :**

for all $\phi \in \mathcal{C}_b^2(\mathbb{R}^d)$,

$$\phi(X_t) - \int_0^t \int_A \left( b_r(X, a, m) \cdot D\phi(X_r) + \frac{1}{2}\sigma\sigma_r^\intercal(X, a, m) : D^2\phi(X_r) \right) Q_r(da)dr \quad (1)$$

is a $\mathbb{P}$-martingale.

# McKean-Vlasov 2BSDE

## Definition

We say that $\mathbb{P}_X^* \in Prob(\mathcal{X})$ and the processes $Y, Z, U$ solve the **McKean-Vlasov 2BSDE**

$$Y_\cdot = \xi + \int_\cdot^T \hat{F}_r(X, Z_r, \mathbb{P}_X^*)dr - \int_\cdot^T Z_r dX_r - (U_T - U_\cdot), \quad \mathcal{P}(\mathbb{P}_X^*) - \text{q.s.} \quad (2)$$

if :

- (2) holds $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}(\mathbb{P}_X^*)$ ;
- $U_0 = 0$ and $U$ is a $\mathbb{P}$-supermartingale for all $\mathbb{P} \in \mathcal{P}(\mathbb{P}_X^*)$ ;
- $\mathbb{P}_X^* \in \mathcal{P}(\mathbb{P}_X^*)$ and $\mathbb{E}^{\mathbb{P}_X^*}[U_T] = 0$ (i.e $U$ is a $\mathbb{P}_X^*$-martingale).

# Representation theorem

## Hypothesis

- Those for which the (no common noise) MFG has a solution ;
- the drift $b_t(x, m, a)$ is of type $\sigma_t(x, m, a)\lambda_t(x, m, b)$, $(a, b) \in A \times B$.

## B-Touzi 20'

- The McKean-Vlasov 2BSDE

$$Y_{\cdot} = \xi + \int_{\cdot}^{T} \hat{F}_r(X, Z_r, \mathbb{P}_X^*)dr - \int_{\cdot}^{T} Z_r dX_r - (U_T - U_{\cdot}), \ \mathcal{P}(\mathbb{P}_X^*) - \text{q.s.} \quad (3)$$

associated to the MFG **admits a solution** $(\mathbb{P}_X^*, Y, Z, U)$ ;
- $\mathbb{P}_X^*$ is the $X$-marginal of a solution of the MFG ;
- $Y$ is the **value function** of the corresponding control problem.

Here $\hat{F}$ is the "aggregated" Hamiltonian of the MFG.

# Questions

Thanks for your attention !

Questions ?