An Optimal Visiting Mean Field Game

Fabio Bagagiolo
Università degli Studi di Trento, Italy

Ongoing research project with Adriano Festa (Torino) and Luciano Marzufero (Trento)

Two-days online workshop on Mean Field Games - Les Andelys (France)
June 2020

An Optimal Visiting Mean Field Game

Fabio Bagagiolo
Università degli Studi di Trento, Italy

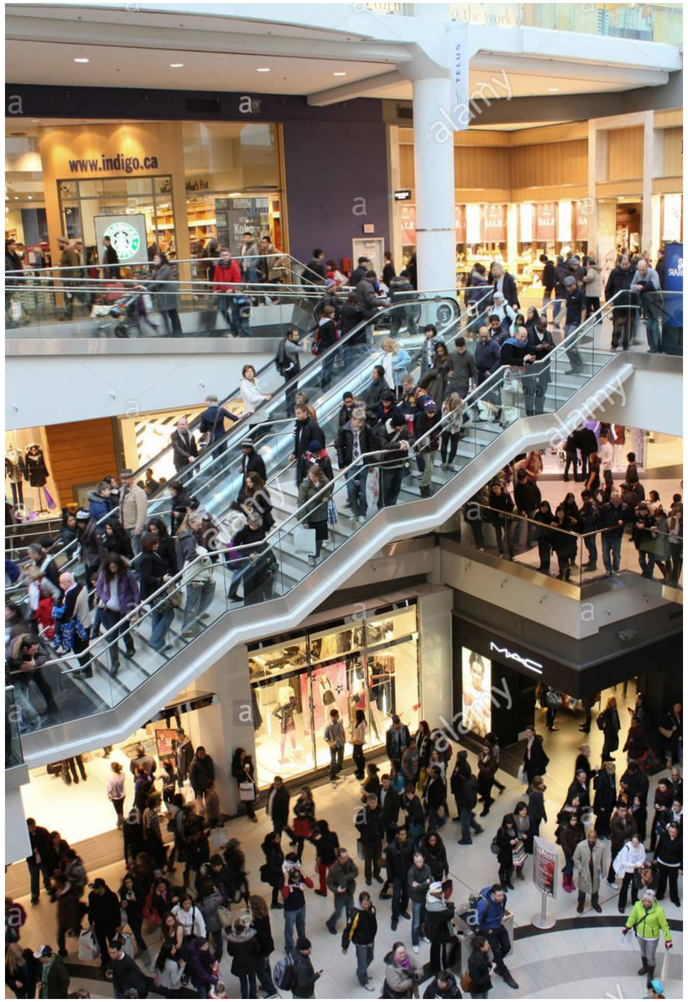
Ongoing research project with Adriano Festa (Torino) and Luciano Marzufero (Trento)

Two-days online workshop on Mean Field Games - Les Andelys (France)
June 2020

From the Abstract

In this talk I am going to present an n-dimensional deterministic mean field game, where the agents have to minimize a cost which also depends on more than one targets to be "visited".

Motivations can be found in models of traffic/pedestrian flows and congestion.


www.indigo.ca




Bagagiolo-Pesenti 2017, *Annals of ISDG*, 2017

Bagagiolo-Faggian-Maggistro-Pesenti, *Networks and Spatial Economics*, 2019 online

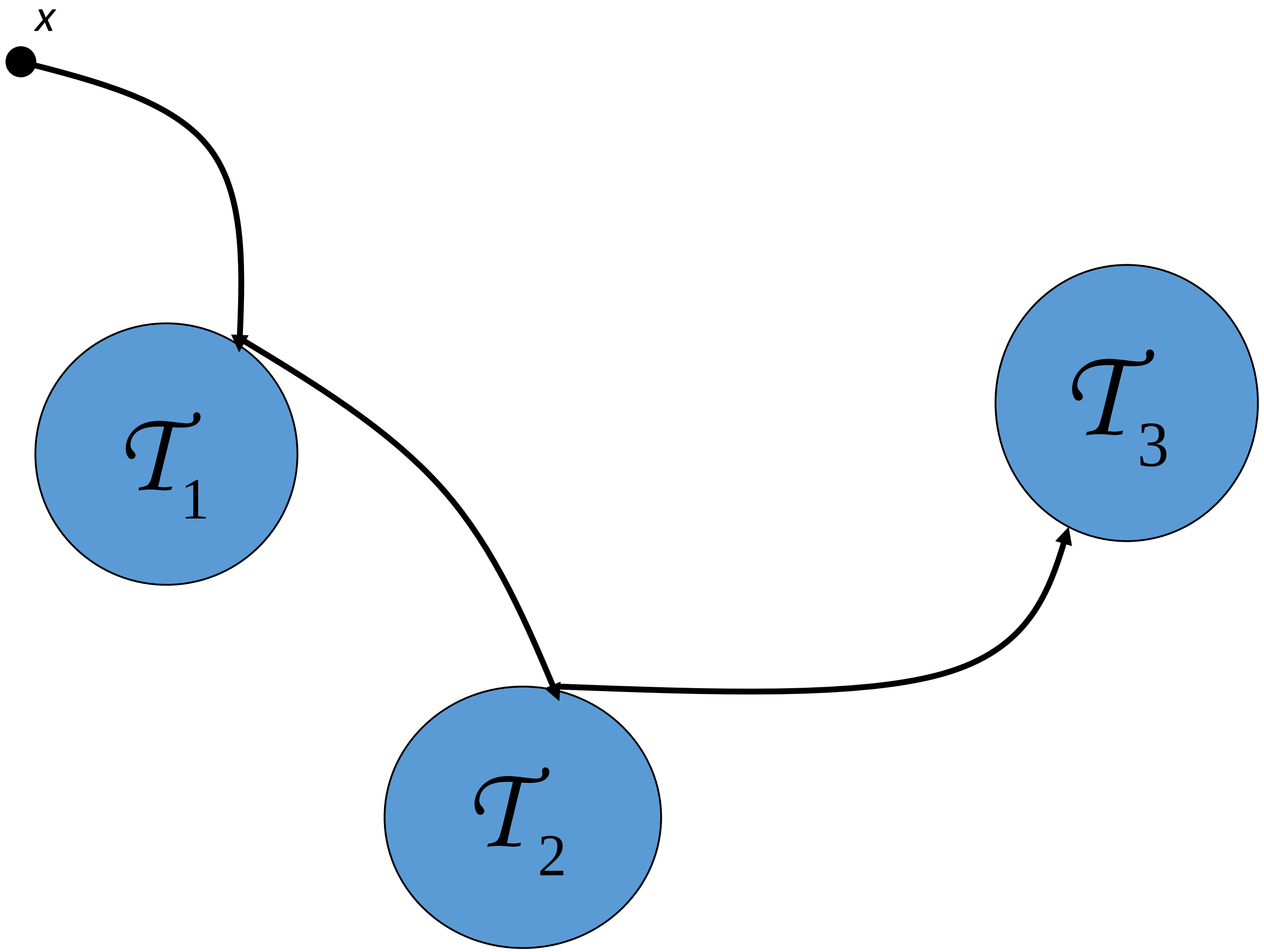
From the Abstract

In this talk I am going to present an n-dimensional deterministic mean field game, where the agents have to minimize a cost which also depends on more than one targets to be "visited".

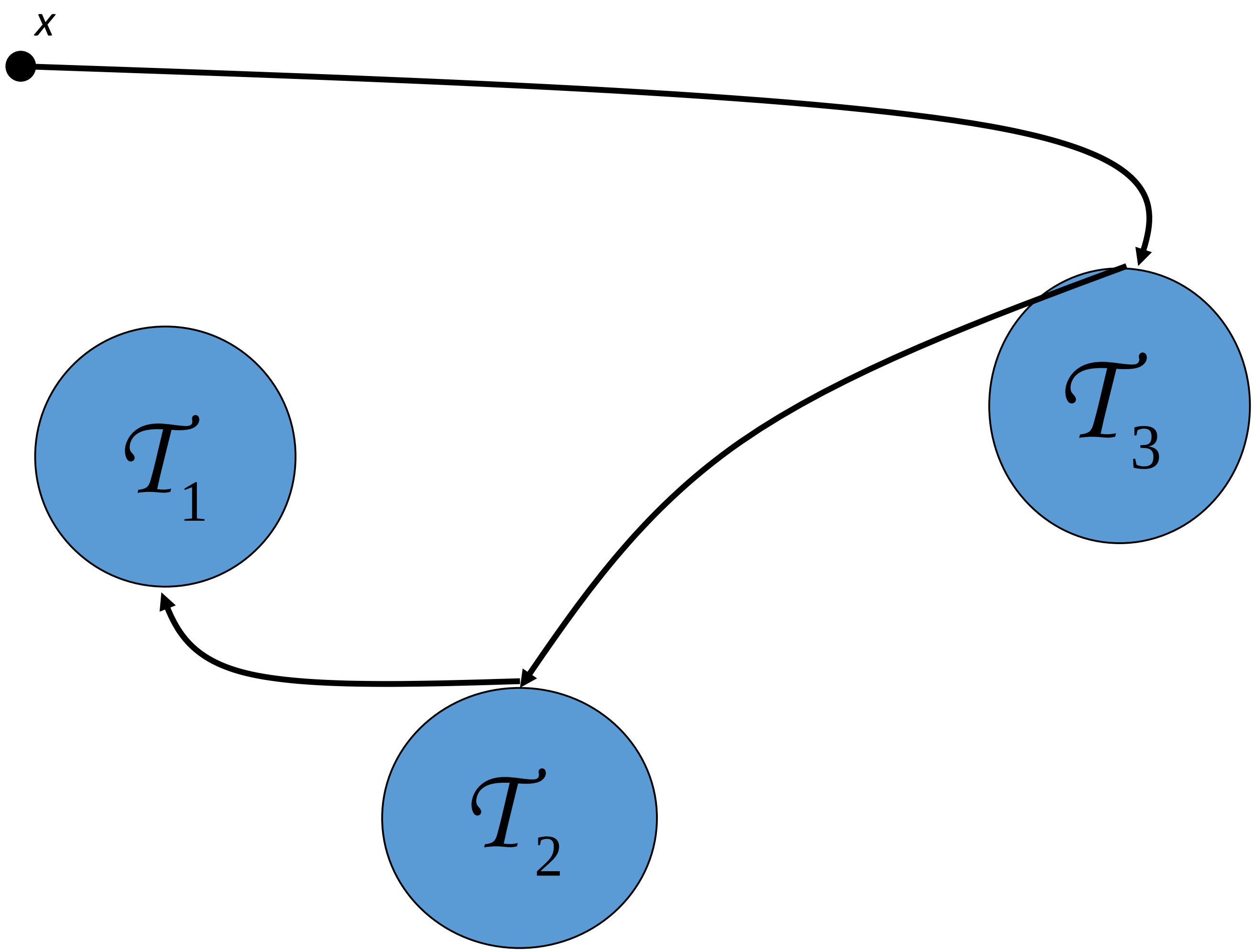
Motivations can be found in models of traffic/pedestrian flows and congestion.

The targets are points of the state-space

More than one target.

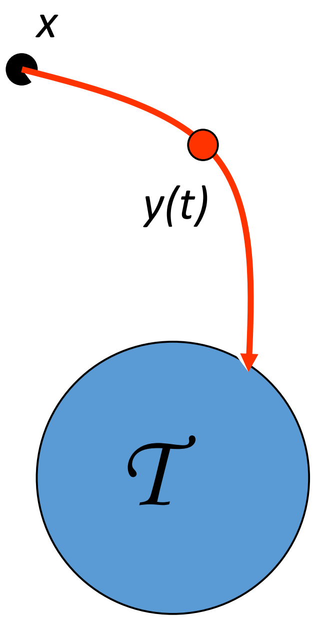


More than one target.



A first problem with Dynamic Programming

- The Dynamic Programming Principle “by words”:
- “Pieces of optimal trajectories are optimal”!

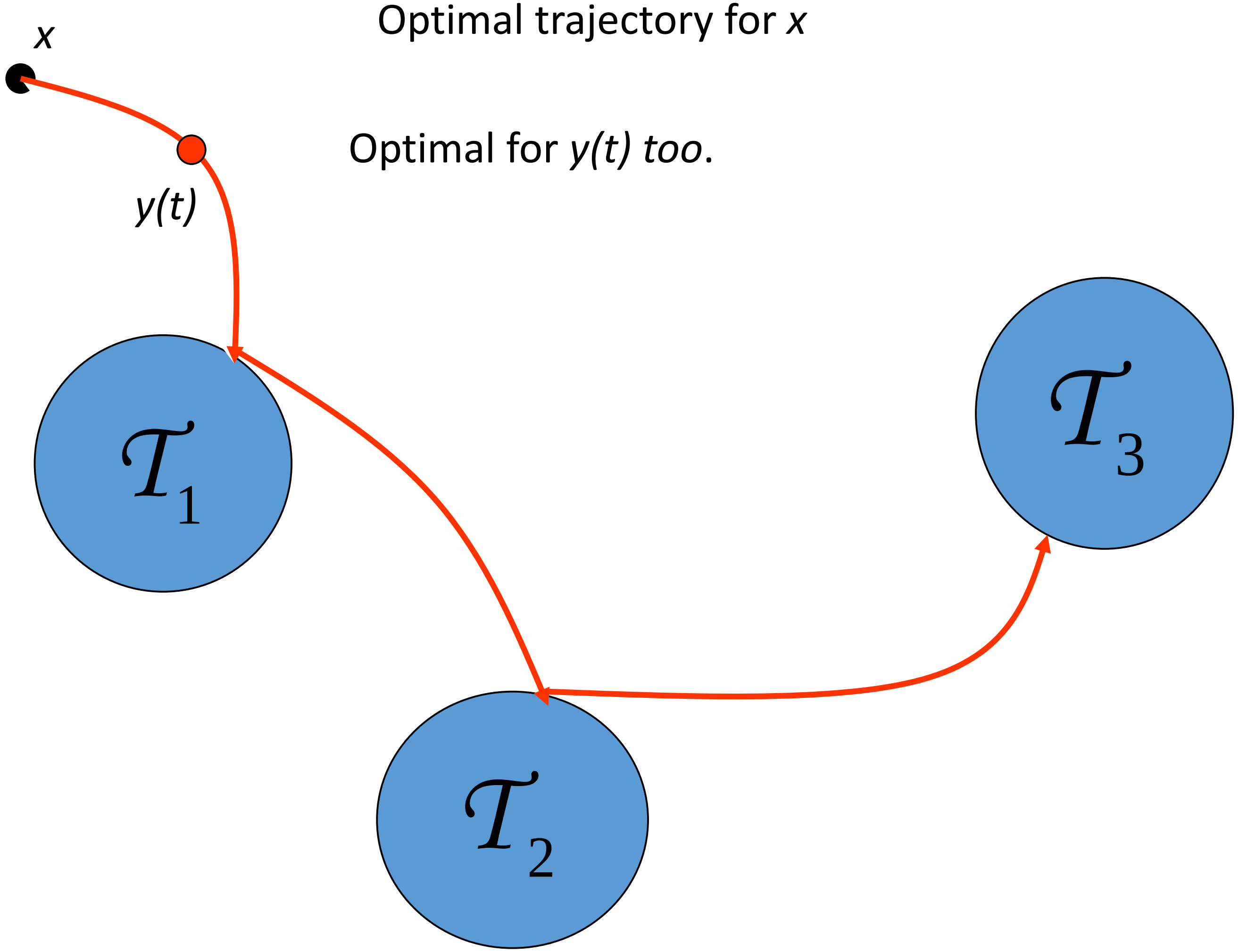


Optimal trajectory for *x*

Optimal for *y(t) too*.

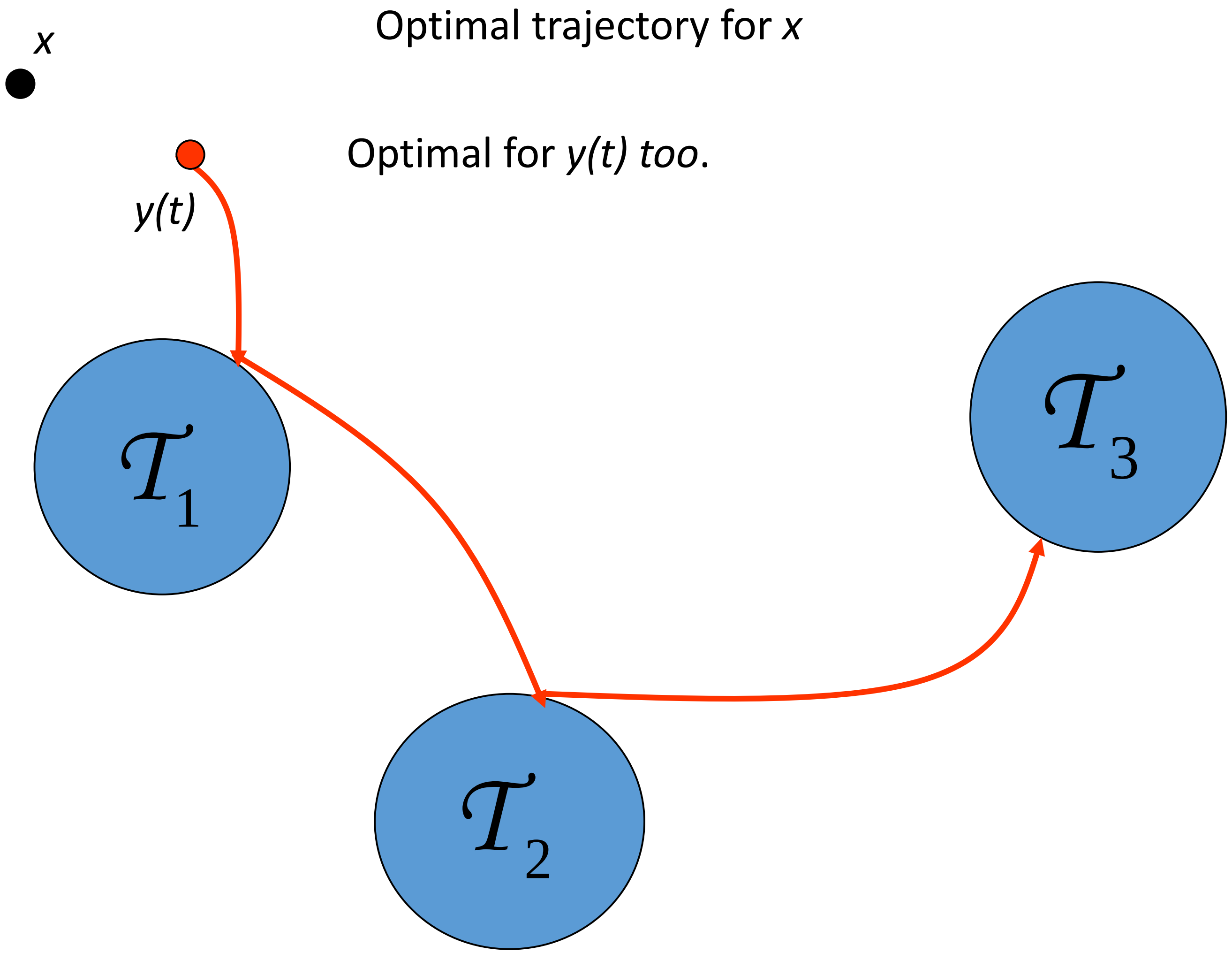
Problem with DPP

- The Dynamic Programming Principle does not hold.
- “Pieces of optimal trajectories are not optimal”!

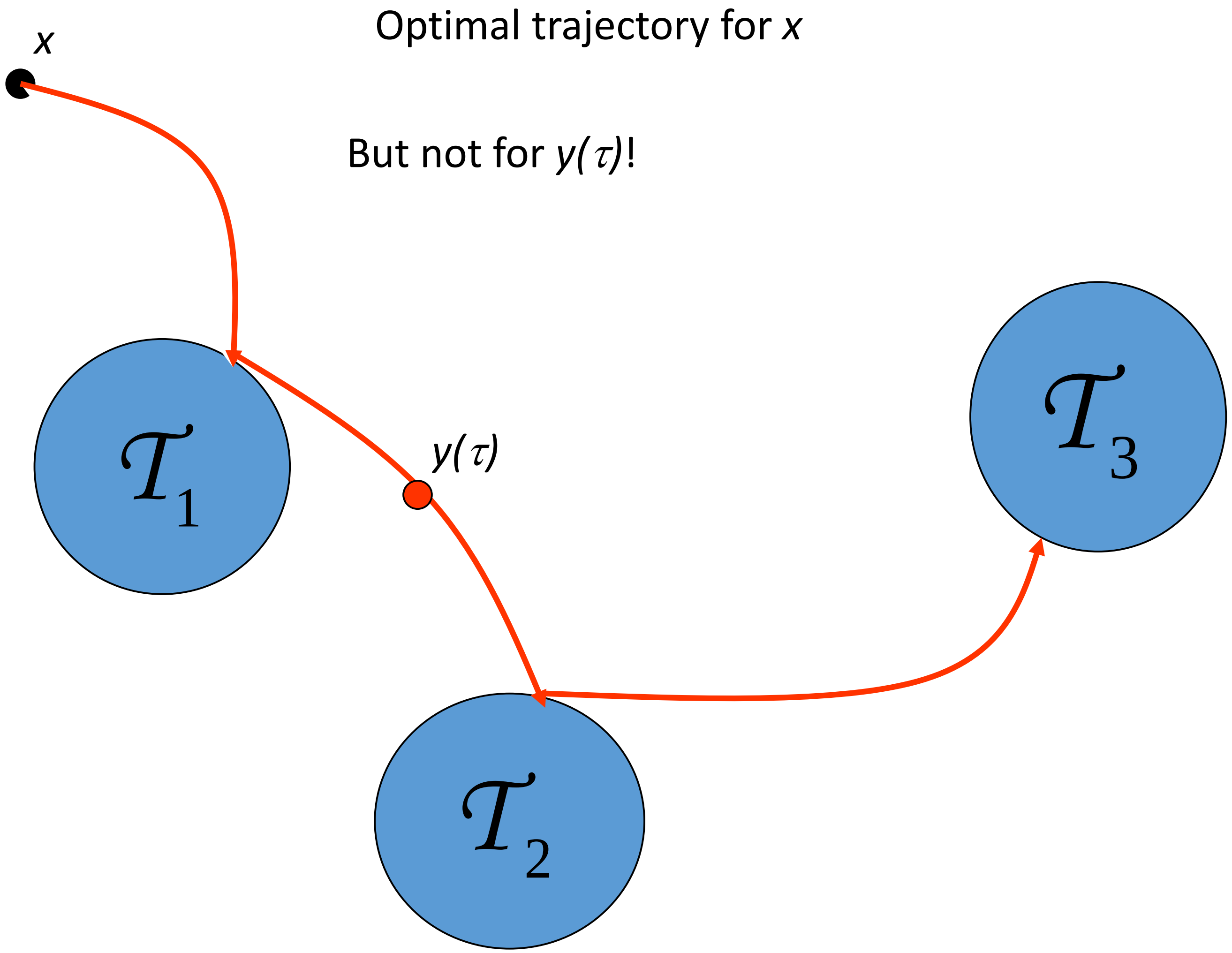


Optimal trajectory for *x*

Optimal for *y(t) too*.

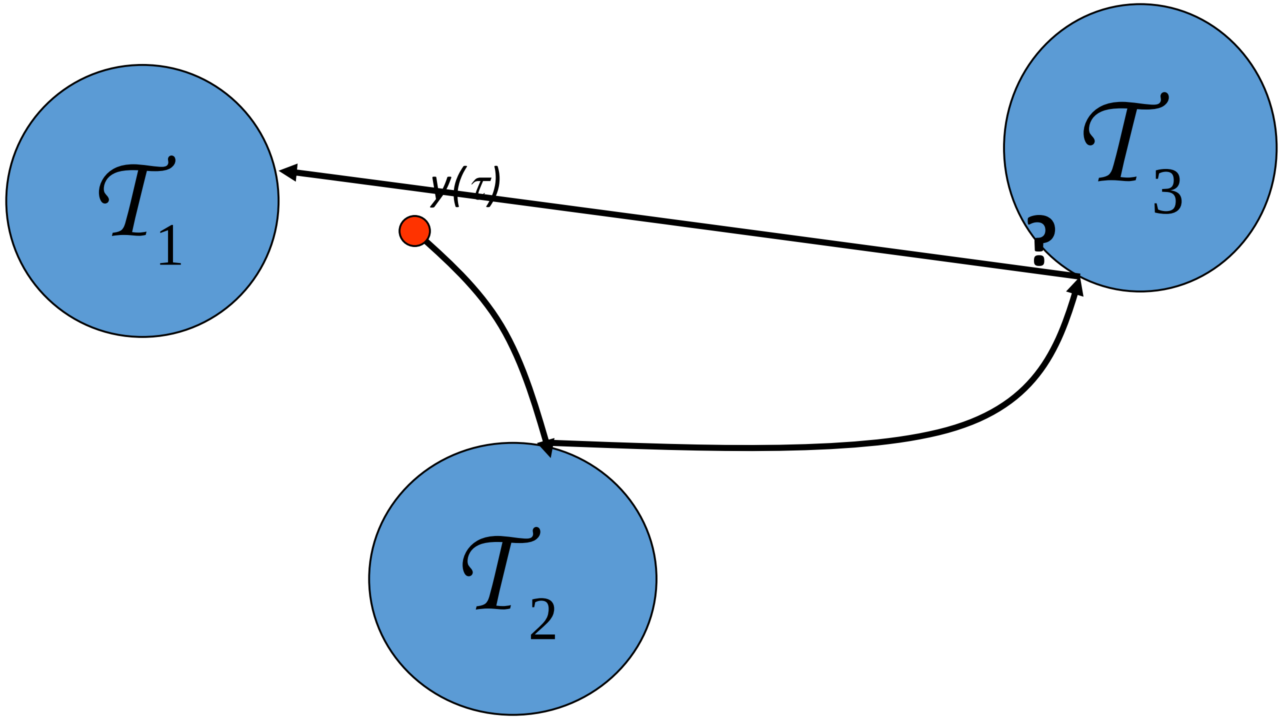


x
Optimal trajectory for x
Optimal for $y(t)$ *too*.
$y(t)$
$\mathcal{T}_1$
$\mathcal{T}_2$
$\mathcal{T}_3$


Optimal trajectory for x
But not for y(τ)!
x
y(τ)
T1
T2
T3

Optimal trajectory for *x*

But not for *y(τ)*!



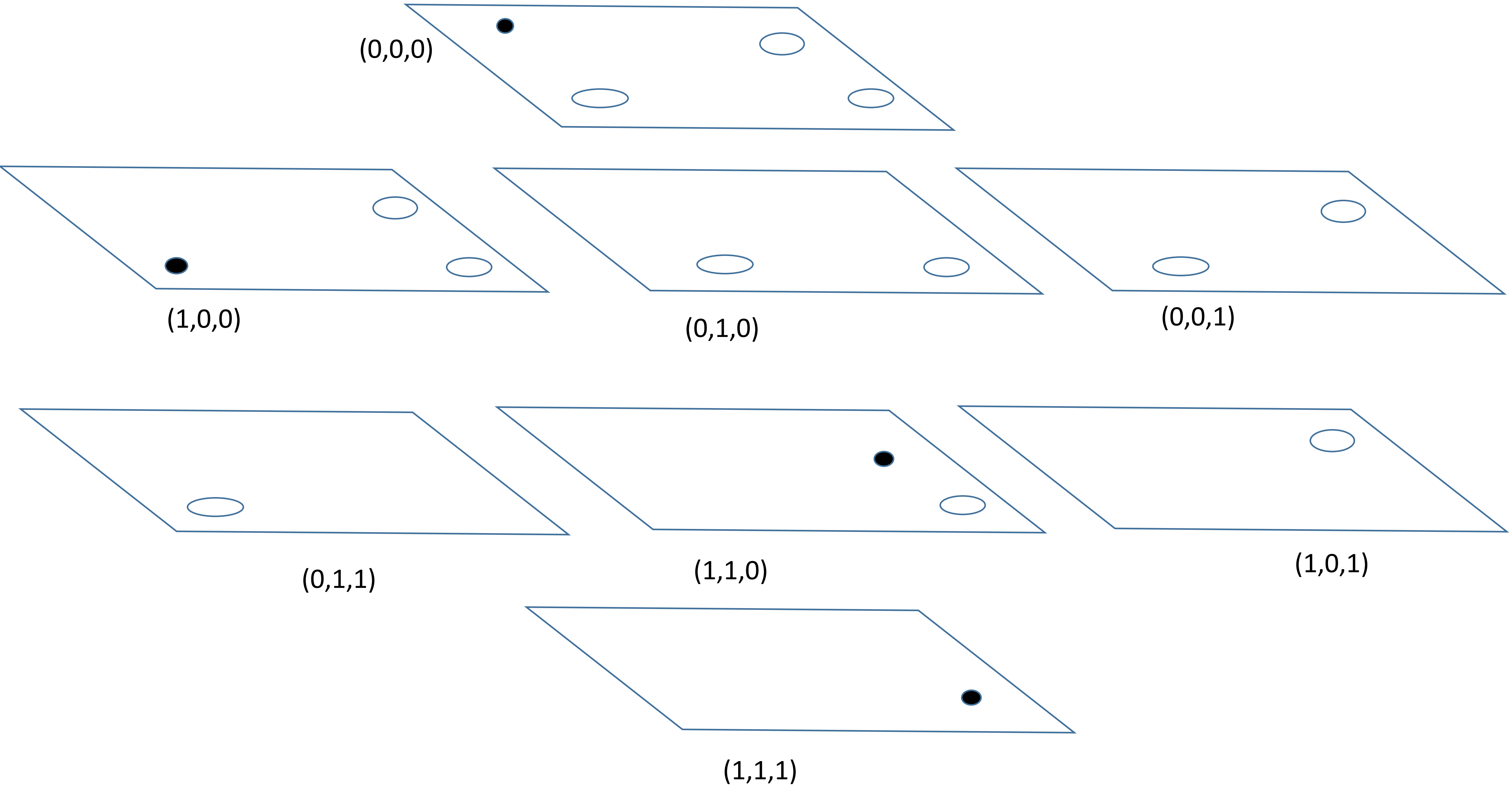
Problem with DPP

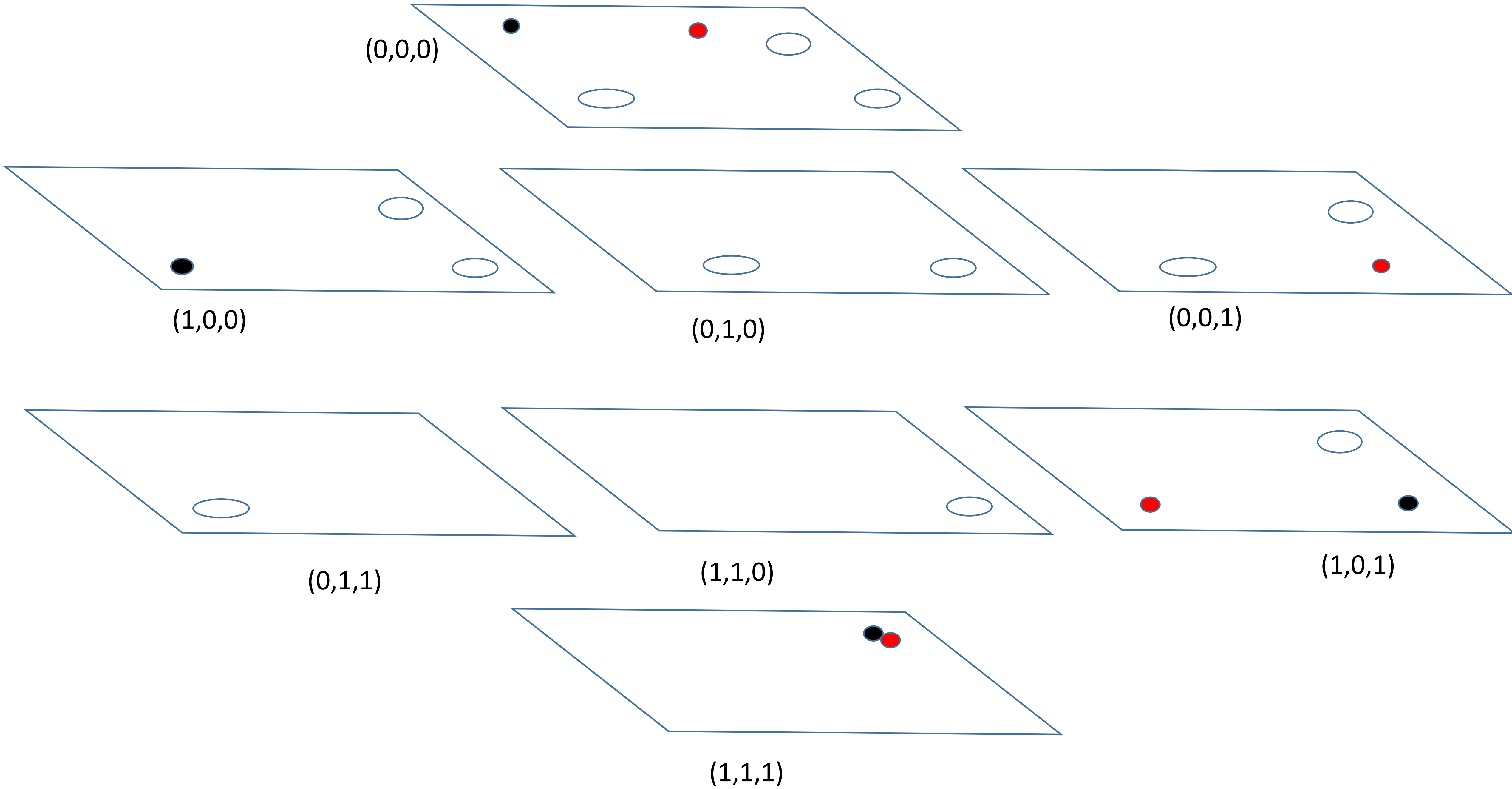
- The Dynamic Programming Principle does not hold.
- “Pieces of optimal trajectories are not optimal”!
- And, if we do not have DPP then we do not have HJB

To recover DPP we need a sort of memory

- We need a sort of memory!
- We have to keep in mind whether the target is already visited or not.
- For every target, we need a positive scalar *w*, evolving in time, which is zero if and only if we have already visited the target.
- Bagagiolo-Benetton, *Applied Mathematics and Optimization*, 2012 (continuous memory (hysteresis));
- Here we adopt a “switching” memory, as in

  Bagagiolo-Pesenti 2017, *Annals of ISDG*, 2017
  Bagagiolo-Faggian-Maggistro-Pesenti, *Networks and Spatial Economics*, 2019 online


(0,0,0)
(1,0,0)
(0,1,0)
(0,0,1)
(0,1,1)
(1,1,0)
(1,0,1)
(1,1,1)


(0,0,0)
(1,0,0)
(0,1,0)
(0,0,1)
(0,1,1)
(1,1,0)
(1,0,1)
(1,1,1)

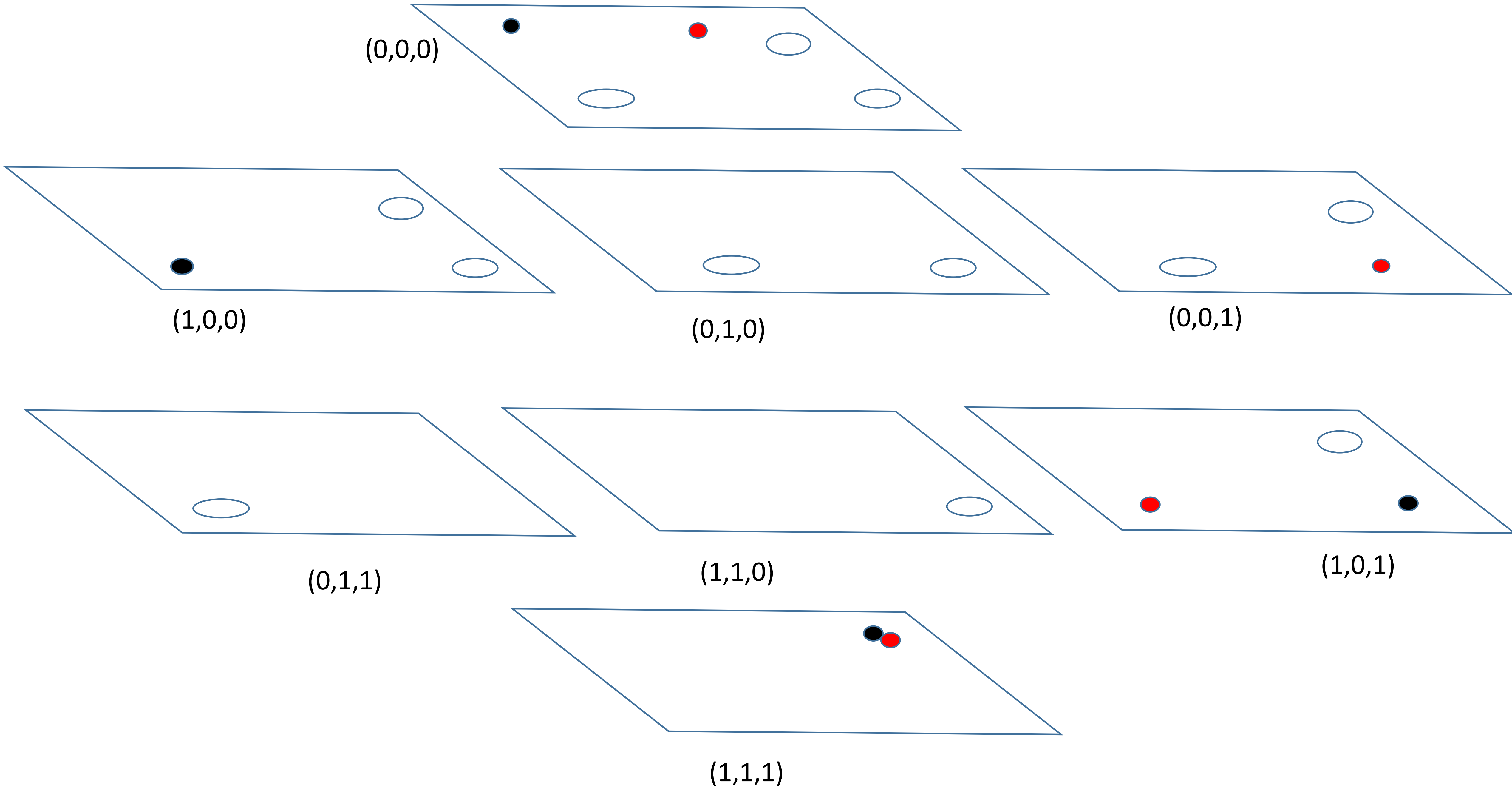
From the Abstract

In this talk I am going to present an n-dimensional deterministic mean field game, where the agents have to minimize a cost which also depends on more than one targets to be "visited".

Motivations can be found in models of traffic/pedestrian flows and congestion.

The targets are points of the state-space

Depending on the targets already visited, the population is then split into several populations, with a transfer between them.


(0,0,0)
(1,0,0)
(0,1,0)
(0,0,1)
(0,1,1)
(1,1,0)
(1,0,1)
(1,1,1)

From the Abstract

In this talk I am going to present an n-dimensional deterministic mean field game, where the agents have to minimize a cost which also depends on more than one targets to be "visited".

Motivations can be found in models of traffic/pedestrian flows and congestion.

The targets are points of the state-space

Depending on the targets already visited, the population is then split into several populations, with a transfer between them.

"to visit" can be relaxed in "to pass as close as possible". The agents can also give up some targets, paying a suitable instantaneous cost.

From the Abstract

In this talk I am going to present an n-dimensional deterministic mean field game, where the agents have to minimize a cost which also depends on more than one targets to be "visited".

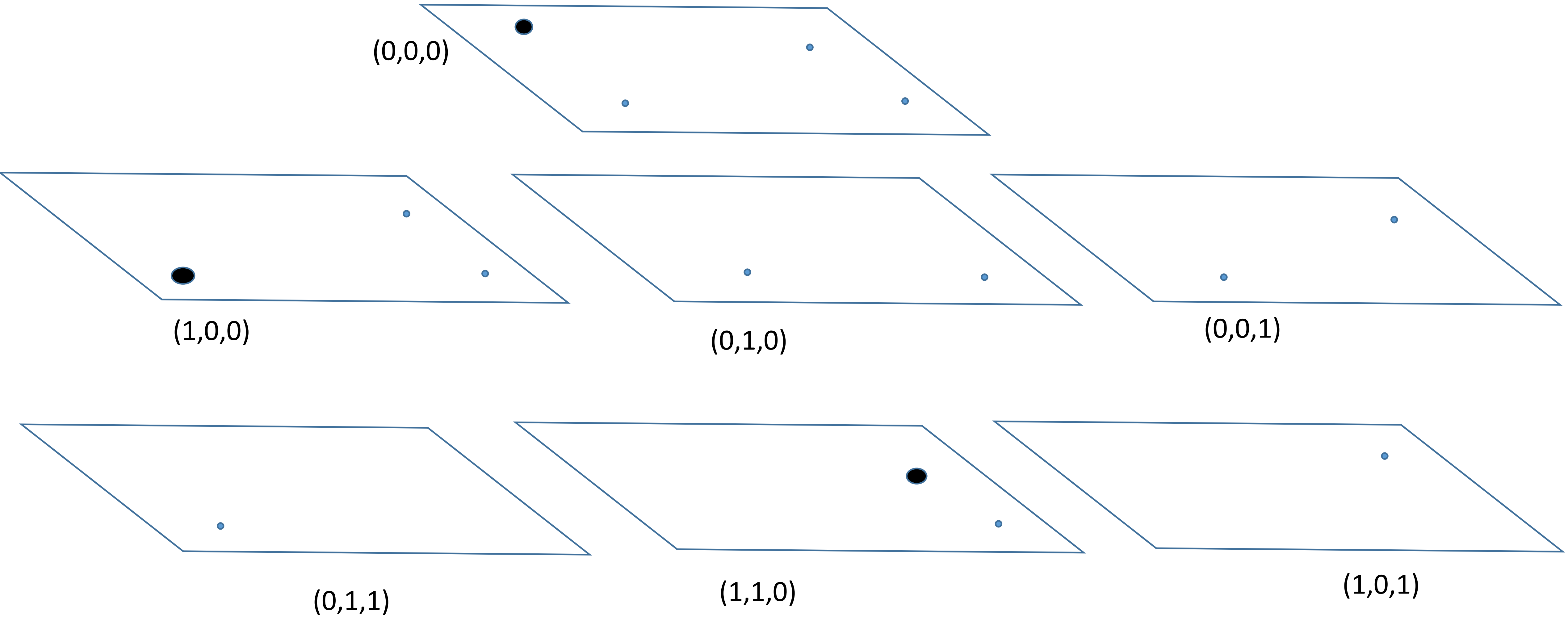
Motivations can be found in models of traffic/pedestrian flows and congestion.

The targets are points of the state-space

Depending on the targets already visited, the population is then split into several populations, with a transfer between them.

"to visit" can be relaxed in "to pass as close as possible". The agents can also give up some targets, paying a suitable instantaneous cost.

The problem is recast in the framework of optimal stopping/optimal switching problems, with the add of a continuum of agents.



Whenever the agent switches from one level to the other it pays a swithcing cost given by the distance from the target it is giving up.

We can also see such a process as an optimal stopping control problem in anyone of the levels, where the stopping cost is given by the distance from the target plus the value function (the best the agent can do) on the new level.

Another little problem with DPP

- In infinite horizon optimal control problems

$$J(x,\alpha)=\int_0^{+\infty} e^{-\lambda t}\ell\big(y(t),\alpha(t)\big)dt$$

- an explicitly time dependent running cost

$$J(x,\alpha)=\int_0^{+\infty} e^{-\lambda t}\ell\big(y(t),\alpha(t),t\big)dt$$

- is a problem for DPP and HJB, because usually you cannot glue time inside the non linear running cost.
- In that case, you need an explicitly time dependent cost

$$J(x,\alpha,t)=\int_t^{T} e^{-\lambda(s-t)}\ell\big(y(s),\alpha(s),s\big)ds+G\big(y(T)\big)$$

Another little problem with DPP

- Deterministic optimal stopping control problems are very often written in an infinite horizon feature

$$J(x,\alpha,\tau)=\int_0^\tau e^{-\lambda t}\ell\big(y(t),\alpha(t)\big)dt+e^{-\lambda\tau}\psi\big(y(\tau)\big)$$

$$V(x)=\inf_{\alpha;\tau\geq 0} J(x,\alpha,\tau),\quad \begin{cases} y'(t)=f(y(t),\alpha(t)) \\ y(0)=x \end{cases}$$

$$m(\cdot)\to J\big(x,\alpha,m(\cdot),\tau\big)=\int_0^\tau e^{-\lambda t}\ell\big(y(t),\alpha(t),m(t)\big)dt+e^{-\lambda t}\psi\big(y(\tau),m(\tau)\big)$$

$$=\int_0^\tau e^{-\lambda t}\tilde{\ell}\big(y(t),\alpha(t),t\big)dt+e^{-\lambda t}\tilde{\psi}\big(y(\tau),\tau\big)$$

Time dependent optimal stopping

$$J(x,\alpha,t,\tau) = \int_t^\tau e^{-\lambda(s-t)} \ell\big(y(s),\alpha(s),s\big)dt + e^{-\lambda(\tau-t)} \psi\big(y(\tau),\tau\big)$$

$$\begin{cases} y'(s) = f\big(y(s),\alpha(s)\big), & s > t \\ y(t) = x \end{cases}$$

$$V(x,t) = \inf_{\alpha;\tau\geq t} J(x,\alpha,t,\tau)$$

$q=(q_1,q_2,\ldots,q_N)$ dependence

$$\begin{cases} y'(s) = f\big(y(s),\alpha(s),q\big), & s>t \\ y(t)=x \end{cases}$$

$$J_q(x,t,\alpha,\tau,q') = \int_t^\tau e^{-\lambda(s-t)}\ell\big(y(s),\alpha(s),s,q\big)dt + e^{-\lambda(\tau-t)}\Big(C\big(y(\tau),q,q'\big)+V_{q'}\big(y(\tau),\tau\big)\Big)$$

$$J_q(x,t,\alpha,\tau) = \int_t^\tau e^{-\lambda(s-t)}\ell\big(y(s),\alpha(s),s,q\big)dt + e^{-\lambda(\tau-t)}\psi_q\big(y(\tau),\tau\big)$$

$$\psi_q(x,\tau) = \inf_{q'\in I_q}\big(C(x,q,q')+V_{q'}(x,\tau)\big)$$

$$q'=(1,1,\ldots,1) \Rightarrow V_{q'}=0$$

$$I_q = \{q' \mid q\to q' \text{ admissible}\}$$

$$C(x,q,q') = \sum_{q_j'\neq q_j}\left\|x-X_j\right\|^2$$

On the dependence on $q=(q_1,q_2,\ldots,q_N)$

m_q portion of population labelled by q

$$f(x,\alpha,m^q),\quad \ell(x,\alpha,m^q)$$

m^q suitable weighted sum of mass densities of populations q'

with similar goals: $q_i=0 \Rightarrow q_i^{'}=0;\quad \exists i \,|\, q_i=q_i^{'}=0$

a Bellman equation in a Variational Inequality of Obstacle Type form for every $q \in \mathcal{I}$. In other words, defining for $x, p \in \mathbb{R}^d$, $t \in [0, T[$ and $q \in \mathcal{I}$ the Hamiltonian function by

$$H^q(x,t,p) = \sup_{\alpha \in A}\{-f(x,\alpha,q)\cdot p - \ell(x,a,q,t)\}, \tag{9}$$

we have the following result:

Proposition 3. *Under the hypotheses of Proposition 2, for every $q \in \mathcal{I}$ the value function V_q is the unique viscosity solution of*

$$\begin{cases} \max\{V_q(x,t) - \psi_q(x,t), -V_{q_t}(x,t) + \lambda V_q(x,t) + H^q(x,t,D_x V_q(x,t))\} = 0, & (x,t) \in \mathbb{R}^d \times [0,T[\\ V_q(x,T) = \psi_q(x,T), & x \in \mathbb{R}^d \end{cases}. \tag{10}$$

Proposition 3. *Under the hypotheses of Proposition 2, for every $q \in \mathcal{I}$ the value function V_q is the unique viscosity solution of*

$$\begin{cases} \max\{V_q(x,t) - \psi_q(x,t), -V_{q_t}(x,t) + \lambda V_q(x,t) + H^q(x,t,D_x V_q(x,t))\} = 0, & (x,t) \in \mathbb{R}^d \times [0,T[\\ V_q(x,T) = \psi_q(x,T), & x \in \mathbb{R}^d \end{cases} . \tag{10}$$

Proof. See Theorem A.1 in §A.1. □

Proposition 4. *the family of functions $\{V_q : q \in \mathcal{I}\}$ is the unique solution of the problem*

$$\{\textit{for every } q \in \mathcal{I},\ V_q \textit{ is the unique viscosity solution of } (10)\}. \tag{11}$$

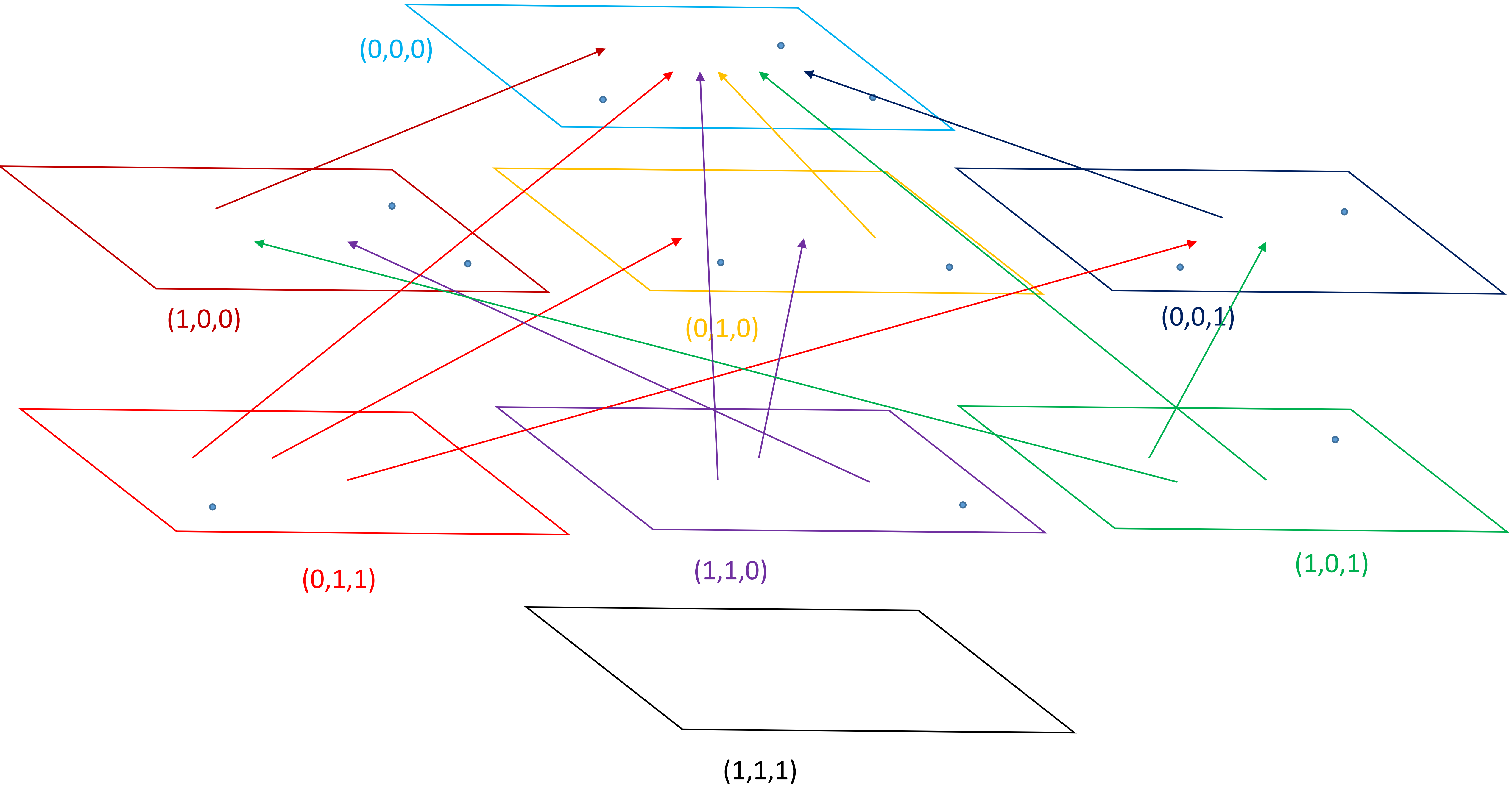
Proof. Let $\{U_q : q \in \mathcal{I}\}$, where $U_q \in \mathrm{BUC}(\mathbb{R}^d \times [0,T[)$ for every $q \in \mathcal{I}$, be a solution of (11) with stopping cost

$$\psi_q^U(x,t) := \inf_{q' \in \mathcal{I}_q} (C(x,q,q') + U_{q'}(x,t)),$$

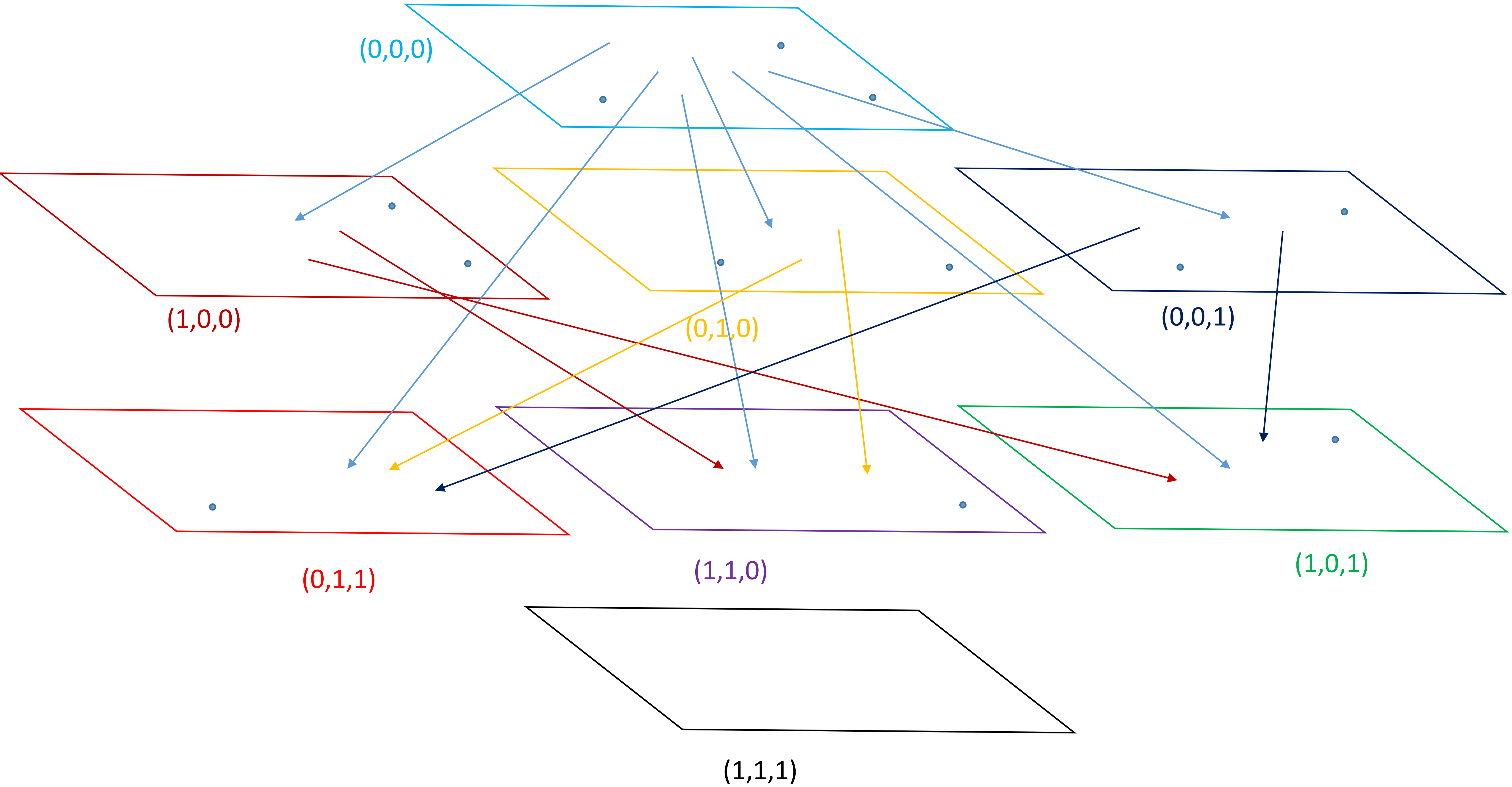
$$U_{\bar{q}} \equiv 0, \quad \bar{q} = (1,1,\ldots,1)$$

that is for every $q \in \mathcal{I}$, U_q is the unique viscosity solution of

$$\begin{cases} \max\{U_q(x,t) - \psi_q^U(x,t), -U_{q_t}(x,t) + \lambda U_q(x,t) + H^q(x,t,D_x U_q(x,t))\} = 0, & (x,t) \in \mathbb{R}^d \times [0,T[\\ U_q(x.T) = \psi_q^U(x,T), & x \in \mathbb{R}^d \end{cases} .$$


(0,0,0)
(1,0,0)
(0,1,0)
(0,0,1)
(0,1,1)
(1,1,0)
(1,0,1)
(1,1,1)

On the continuity equation with a sink in $\mathbf{R}^d$


(0,0,0)
(1,0,0)
(0,1,0)
(0,0,1)
(0,1,1)
(1,1,0)
(1,0,1)
(1,1,1)

On the continuity equation with a sink in $\mathbf{R}^d$

$$\mu_t(x,t) + div\big(b(x,t)\mu(x,t)\big) + \text{"sink term"} = 0$$

$\Phi : (x,t) \mapsto \Phi(x,t)$ the flow given by the field b

$$\begin{cases} y'(t) = b(y(t),t) \\ y(0) = x \end{cases} ; \quad y(t) = \Phi(x,t)$$

$$\boldsymbol{A} \subset \mathbf{R}^d \quad \textbf{the sink}$$

m_0 initial distribution

m_0 "flows" with Φ, but when an agent touches A

it falls in the sink and it is no longer present

Candidate for the evolution

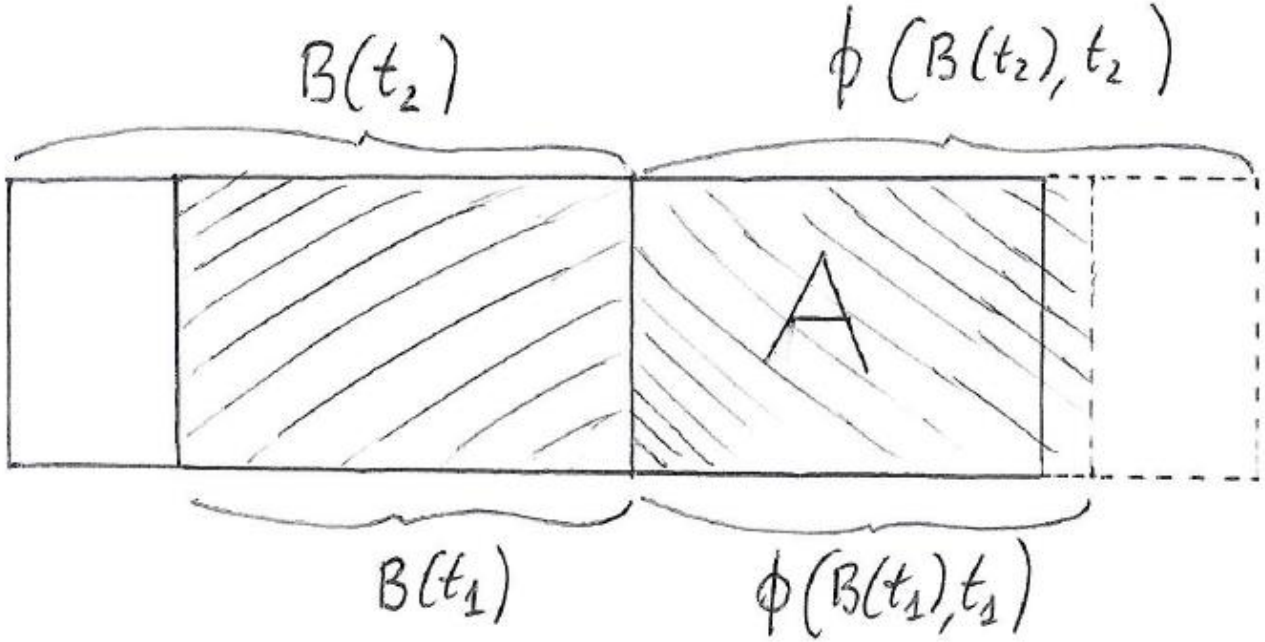
$$\forall x,\ t_x \ \text{ is the first arrival time in } A$$

$$\forall t \geq 0, \quad B(t) = \left\{ x \notin A \middle| 0 \leq t_x \leq t \right\}$$

$$\text{At any time } t,\ \text{"no one is around" the region } \Phi\left(B(t), t\right)$$

$\mathbb{R}^2$

$\longrightarrow$ constant flow, $0 \leq t_1 < t_2$



Candidate for the evolution

$$\forall x,\ t_x \ \text{ is the first arrival time in } A$$

$$\forall t \geq 0,\quad B(t) = \left\{ x \notin A \middle| 0 \leq t_x \leq t \right\}$$

$$\text{At any time } t,\ \text{"no one is around" the region } \Phi\big(B(t),t\big)$$

$$\mu(t) = \begin{cases} \Phi(\cdot,t)\# m_0 & \text{in } \mathbf{R}^d \setminus \Phi\big(B(t),t\big) \\ 0 & \text{in } \Phi(B(t),t) \end{cases}$$

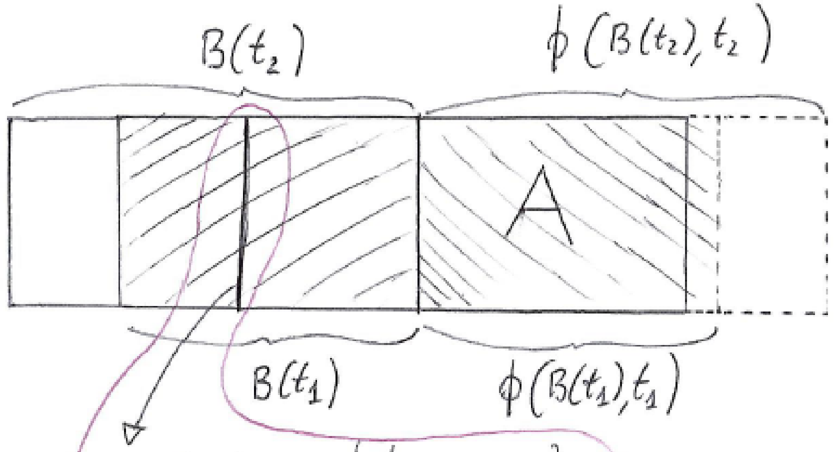
Weak formulation of the continuity equation

For every test function φ,

$$\int_{\mathbf{R}^d} \varphi(x,0)\,dm_0 + \int_0^T \int_{\mathbf{R}^d \setminus \Phi(B(t),t)} \left(\varphi_t(x,t) + \langle D_x\varphi(x,t), b(x,t)\rangle\right) d\mu(t)\,dt$$

$$-\int_0^T \int_{S^t} \varphi\left(\Phi(x,t)\right) d\mu^t\,dt = 0$$

$\longrightarrow$ constant flow, $0 \le t_1 < t_2$ $\mathbb{R}^2$



$S^t = \{x \in \mathbb{R}^d \mid t_x = t\}$

$0 < t < t_1$

Weak formulation of the continuity equation

For every test function φ,

$$\int_{\mathbf{R}^d} \varphi(x,0)dm_0 + \int_0^T \int_{\mathbf{R}^d \setminus \Phi(B(t),t)} \left(\varphi_t(x,t) + \langle D_x\varphi(x,t), b(x,t)\rangle\right) d\mu(t)dt$$

$$-\int_0^T \int_{S^t} \varphi(\Phi(x,t))d\mu^t dt = 0$$

$$\mu^t = g(t)\mu(0)^t$$

$\mu(0)^t$ is the "disintegration" of $\mu(0)$ on the fibers S^t that compose $B(t)$;

$g(\cdot)$ is the density of the measure ν on the indices τ of the fibers S^τ such that

$$\nu = \pi \# \mu(0), \quad \pi : B(t) \to [0,t], x \mapsto t_x$$

$$E \subseteq B(t) \Rightarrow \mu(0)(E) = \int_0^t \mu(0)^\tau \left(S^\tau \cap E\right) d\nu = \int_0^t g(\tau)\mu(0)^\tau \left(S^\tau \cap E\right) d\tau$$

g is the "transformation parameter" between spatial-density (kg/m^d) mass and time-density $(Kg/\sec)$ mass and depends on Φ

Camilli-De Maio-Tosin, *Networks and Heterogeneous Media*, 2017
Bagagiolo-Faggian-Maggistro-Pesenti, *Networks and Spatial Economics*, 2019 online

Uniqueness

$$\mu(t)=\begin{cases}\Phi(\cdot,t)\#m_0 & \text{in } \mathbf{R}^d\setminus\Phi\big(B(t),t\big)\\ 0 & \text{in } \Phi(B(t),t)\end{cases}$$

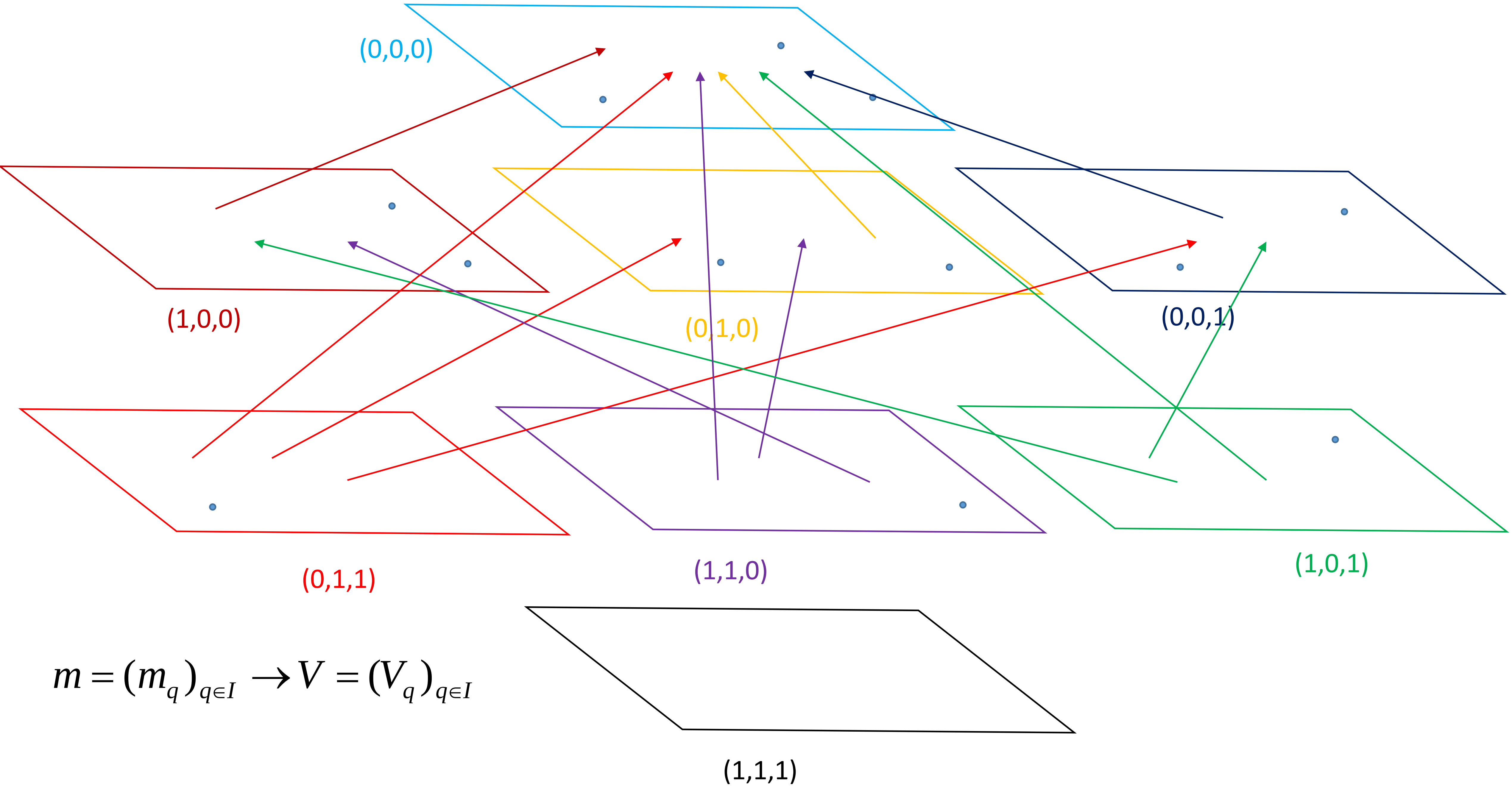
is the unique solution in $C^0\big(0,T;\mathcal{B}(\mathbf{R}^d)\big)$ of the weak formulation

$$\int_{\mathbf{R}^d}\varphi(x,0)\,dm_0+\int_0^T\int_{\mathbf{R}^d\setminus\Phi(B(t),t)}\Big(\varphi_t(x,t)+\big\langle D_x\varphi(x,t),b(x,t)\big\rangle\Big)d\mu(t)\,dt$$

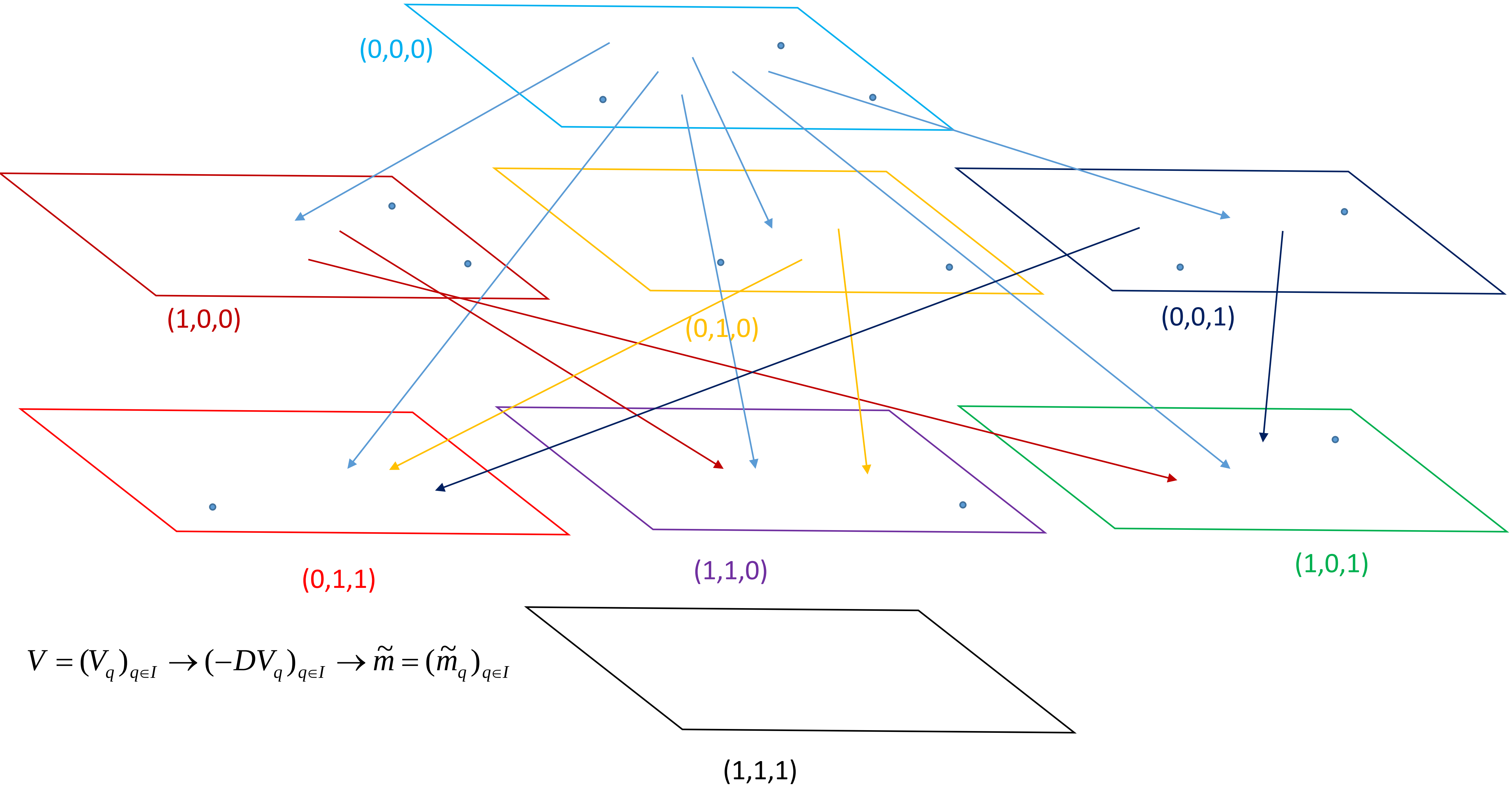
$$-\int_0^T\int_{S^t}\varphi\big(\Phi(x,t)\big)d\mu^t(t)\,dt=0$$

Work in progress still to do

- Is the measure ν really absolutely continuous, $d\nu=g(t)dt$?
- It depends on Φ, A (which can also vary in time: when coupling optimal control and continuity equation (MFG), the sink is the set where the value function is equal to the stopping cost).
- Beside sinks we also have sources.
- If the dependence of the costs on the mass is just via the total masses present at the time t, independently on the local state-position, then maybe the sources can be seen just as the sinks with opposite signs. Otherwise it also must flow…
- Coupling HJBVI-Continuity (MFG), fixed point, mean field equilibrium.



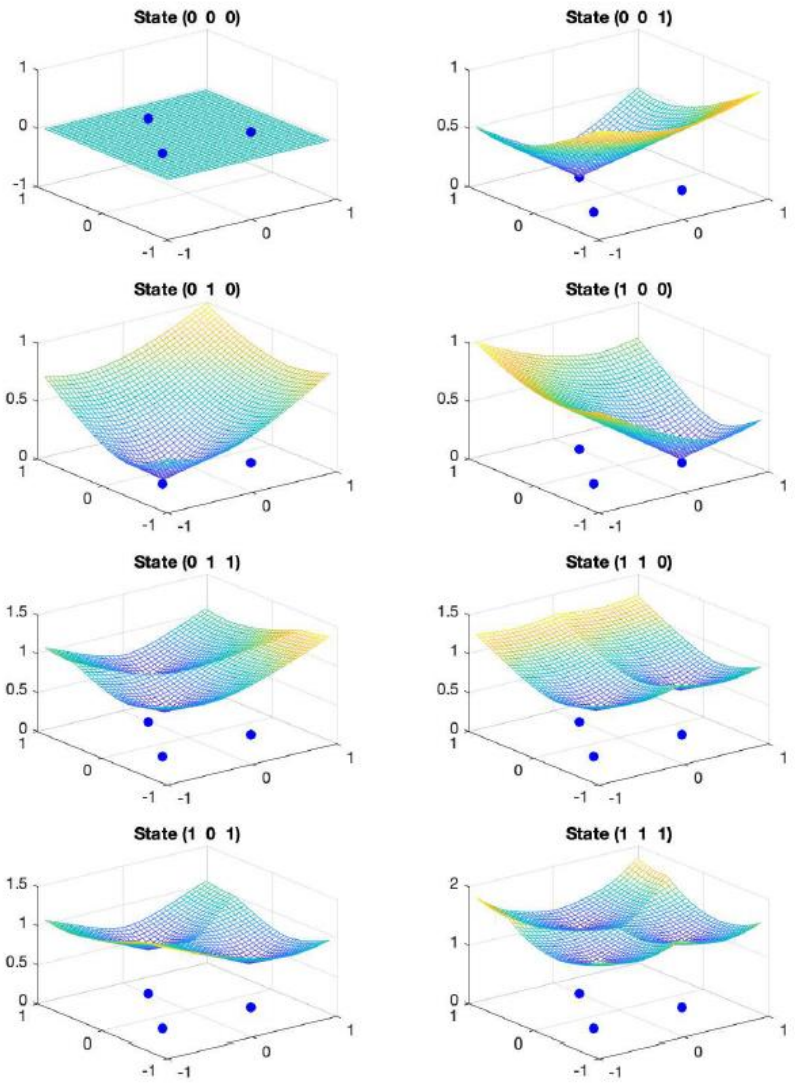
$$m=(m_q)_{q\in I}\to V=(V_q)_{q\in I}$$



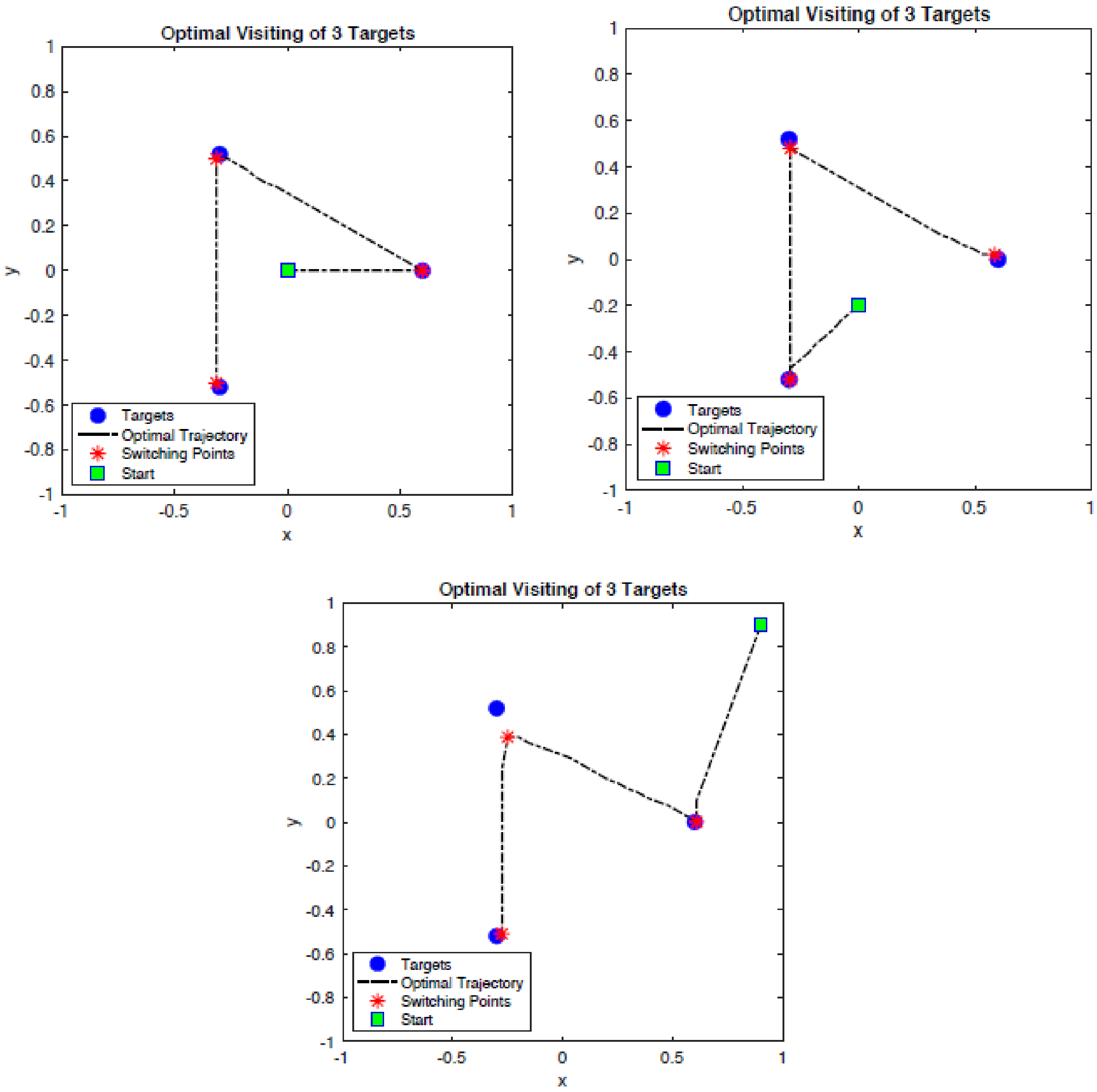
$$V=(V_q)_{q\in I}\rightarrow(-DV_q)_{q\in I}\rightarrow\tilde{m}=(\tilde{m}_q)_{q\in I}$$

Work in progress still to do

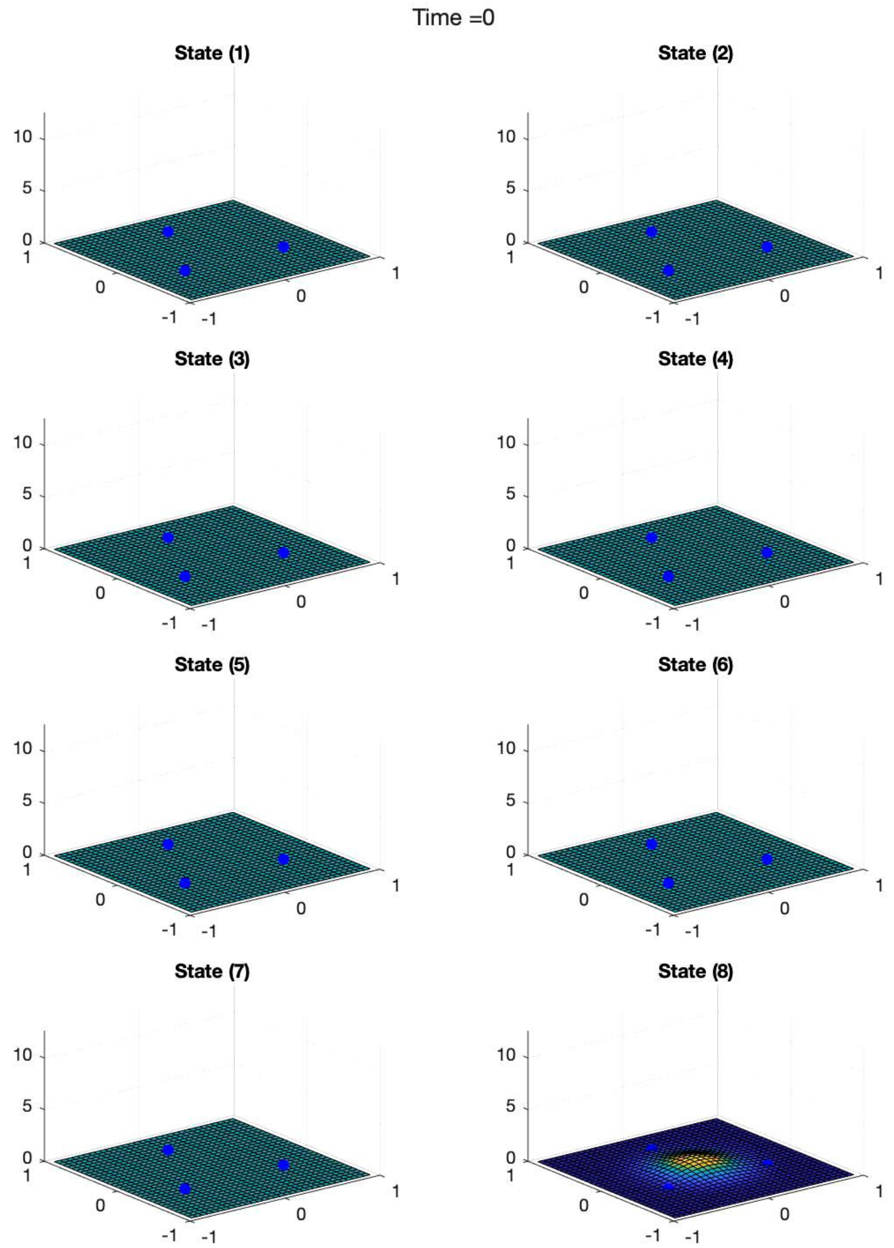
- Is the measure ν really absolutely continuous, $d\nu=g(t)dt$?
- It depends on Φ, A (which can also vary in time: when coupling optimal control and continuity equation (MFG), the sink is the set where the value function is equal to the stopping cost).
- Beside sinks we also have sources.
- If the dependence of the costs on the mass is just via the total masses present at the time t, independently on the local state-position, then maybe the sources can be seen just as the sinks with opposite signs. Otherwise it also must flow…
- Coupling HJBVI-Continuity (MFG), mean field equilibrium.
- We have already some numerical simulations, to be improved.



gure 1: Test 1. Approximated value functions in the various discrete states of the system



igure 2: Test 1. Optimal trajectories for various starting points: (top/left) $x_0 = (0, 0)$ top/right) $x_0 = (0, -0.2)$, (bottom) $x_0 = (0.9, 0.9)$.


Time =0
State (1)
State (2)
State (3)
State (4)
State (5)
State (6)
State (7)
State (8)
10
5
0
1
0
-1
-1
0
1

References and recent preprints on MFG and optimal stopping/switching

- Bertucci: *Optimal stopping in mean field games, an obstacle problem approach*, 2017.
- Nutz: *A Mean Field Games of optimal stopping*, 2017.
- Bertucci: *Fokker-Planck equations of jumping particles and mean field games of impulse control*, 2018.
- Bouveret, Dumitrescu, Tankov: *Mean-Field Games of optimal stopping: a relaxed solution approach*, 2018.
- Firoozi, Pakniyat, Caines: *A hybrid optimal control approach to LQG mean field games with switching and stopping strategies*, 2018.
- Aïd, Dumitrescu, Tankov: *The entry and the exit game in the electricity markets: a mean-field game approach*, 2020.