The Planning Problem with common noise in finite state space

Charles Bertucci, JM Lasry, PL Lions

Ecole Polytechnique.

Two days online workshop on MFG, Les Andelys

Introduction : The Mean Field Planning Problem

A formal definition :

- A time dependent ($[0, t_f]$) Mean Field Game (MFG).
- In $(0, t_f)$ the non atomic players interact through mean field terms in costs, dynamics...
- The game is such that for any initial distribution of players m_0, the final distribution is m_f at time t_f.

Objective of the talk :

- Give a mathematical framework to study such games, even in the presence of common noise
- Focus on structural aspects of the problem more than on a particular instance

Games vs optimization

In an optimization problem :

- A constraint on the terminal state is well understood
- Solution via penalization for instance

In the MFG setting :

- no constraint (non atomic players cannot affect the distribution)
- it's all about the incentives !

In the potential case (when MFG reduces to an optimization problem) :

- the social planner problem is an **optimal transport** one.
- the final distribution is constrained

Common noise and the master equation

Common noise : Noise / randomness which is not i.i.d. between the players

- When there is common noise : the forward-backward structure fails, we are forced to work with the master equation (the pde satisfied by the value function when the density of other players is seen as a state variable)
- In the planning problem, a singularity is expected as $t \to t_f$:

$$U(t, m) \to_{t \to t_f} ??$$

Applications

- Delivery / transport problems with **competitive** agents delivering (MFG setting), price is **infinitely elastic** due to stock constraints... (planning problem)
- Common noise is more than plausible
- "Real life" example : delivery of oil from the americas to Europe

Bibliographical comments

Literature on the planning problem :

- General results on the forward-backward system : Lions ; Porretta
- Numerical methods on the FB system : Achdou-Camilli-Capuzzo-Dolcetta
- Variational approach on the FB system : Graber-Meszaros-Silva-Tonon ; Orrieri-Porretta-Savare
- Master equation in finite state space : BLL
- Master equation in continuous space (including optimal transport) : BLL (ongoing work)

The master equation in finite state space

Notations

- There are d states
- The time is reversed : t is the time remaining in the game (it ends at $t = 0$)
- $$U^i(t, x)$$
 denotes the value in the state i when it remains t time in the game and that the repartition of the other players is $x \in \mathbb{R}^d$
- The operators $F, G : \mathbb{R}^{2d} \to \mathbb{R}^d$ describe respectively the evolution of the density and the value function
- Monotonicity in $\mathbb{R}^d$:

$$\forall x, y \in \mathbb{R}^d, \langle A(x) - A(y), x - y \rangle \geq 0$$

The form of the master equation in a MFG

- In this context, without common noise, the typical form of the master equation is

$$\partial_t U + (F(x,U)\cdot\nabla_x)U = G(x,U) \text{ in } (0,\infty)\times\mathbb{R}^d;$$

$$U(0,x) = U_0(x) \text{ in } \mathbb{R}^d \text{ terminal cost.}$$

- The analogue of the forward-backward system is

$$\begin{cases} \frac{d}{dt}V(t) = G(x(t),V(t)); \\ \frac{d}{dt}x(t) = F(x(t),V(t)); \\ x(t_0) = x_0;\ V(0) = U_0(x(0)). \end{cases}$$

The following holds

$$U(t_0,x_0) = V(t_0).$$

Common noise in discrete state space

- We have to choose a certain type of noise, other noises are possible (see also Bayraktar, Cecchin, Cohen and Delarue)
- We look at the case in which the master equation is of the form

$$\partial_t U + (F(x, U) \cdot \nabla_x) U + \lambda (U - (DT)^* U(Tx)) = G(x, U) \text{ in } \mathbb{R}_+^* \times \mathbb{R}^d;$$

  where $T : \mathbb{R}^d \to \mathbb{R}^d$, $\lambda > 0$.
- At random times given by a Poisson process of intensity λ, all the players are affected by the map T $(x \to T(x))$.
- Fairly general type of noise if we consider limits of this class (see BLL19 for a discussion on this)

Existing results for those master equations

- "Good" class of monotonicity : $(G, F) : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ monotone **and** U_0 is monotone **and** T is linear
- Uniqueness of solutions in the monotone regime
- A priori estimates on $\|D_x U\|_\infty$ (which yields existence) in the monotone regime $(+\epsilon)$ if F, G and U_0 are Lipschitz

Penalized Planning Problem

A penalized initial condition

- We want to create incentives to the players which will induce a final density x_0.
- The following is well suited

$$U_0(x) = \frac{1}{\epsilon}(x - x_0)$$

- Already used in the literature
- Enjoys lipschitz and monotone properties
- We approximate the planning problem with a sequence of classical MFG
- In the potential case, it is associated with a quadratic penalization

The penalized master equation

- How does the solution U_ϵ of

$$\partial_t U_\epsilon + (F(x, U_\epsilon)\cdot\nabla_x)U_\epsilon + \lambda(U_\epsilon - (DT)^* U_\epsilon(Tx)) = G(x, U_\epsilon) \text{ in } \mathbb{R}^*_+ \times \mathbb{R}$$

$$U_\epsilon(0, x) = \frac{1}{\epsilon}(x - x_0);$$

  behaves as $\epsilon \to 0$.
- For $\epsilon > 0$, the problem falls in the known MFG class.

A regularizing effect

- We want an argument of compactness to pass to the limit $\epsilon \to 0$.

Proposition (BLL)

Assume U_0 and (G, F) are monotone, T is linear, G, F lipschitz, $F(x,\cdot)$ α monotone uniformly in x, U is a classical solution of the master equation, then there exists $C > 0$ independent of U_0 such that for $0 < t \leq 1$

$$\|D_x U(t)\|_\infty \leq \frac{C}{t}.$$

- Remark : α monotone means $A - \alpha Id$ is monotone
- Proof : Auxiliary function :

$$(t, x, \xi) \to \xi D_x U \xi - \beta(t)|D_x U\xi|^2 + \gamma(t)|\xi|^2$$

The planning problem

A starter : the initial condition

- What is the limit of $\epsilon^{-1}(Id - x_0)$ as ϵ tends to 0 ?
- The answer is
$$\frac{1}{\epsilon}(Id - x_0) \underset{\epsilon\to 0}{\overset{G}{\to}} A_{x_0}$$
- A_{x_0} is the maximal monotone operator defined by $D(A) = \{x_0\}$ and $A(x_0) = \mathbb{R}^d$.
- $$A_n \underset{n\to\infty}{\overset{G}{\to}} A$$
if for all $(x_n, y_n)_{n\geq 0}$ which converges in $\mathbb{R}^{2d}$ toward (x, y) such that $y_n \in A(x_n)$, then $y \in A(x)$.

Definition of a solution

We call $U : (0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ a solution of the problem if U satisfies

$$\partial_t U + (F(x, U) \cdot \nabla_x) U + \lambda (U - (DT)^* U(Tx)) = G(x, U) \text{ in } (0, \infty) \times \mathbb{R}^d;$$

$$U(t) \underset{t \to 0}{\overset{G}{\to}} A_{x_0}$$

Theorem (BLL)

Under the assumptions of the proposition, there exists a unique solution U of the problem which is lipschitz in space for all $t > 0$.

Main ideas of the proof

- Existence : We use the Yosida approximation $V_{\delta,\epsilon} = U_\epsilon \circ (Id + \delta U_\epsilon)^{-1}$ of U_ϵ to analyse precisely the convergence of the penalized problem and to use properly the lipschitz estimate. Formally, $\epsilon \to 0$ and then $\delta \to 0$.
- Uniqueness : Monotonicity as usual...

Other remarks

Convergence of the induced trajectories

Assume $\lambda = 0$ to simplify a little

Proposition

Under the assumption of the theorem, the trajectories converge toward x_0 as $t \to 0$.

- Take a trajectory which is at x_1 at t_1, it then evolves according to

$$\frac{d}{dt}x(t) = F(x(t), U(t, x(t))).$$

- Remark that

$$\frac{d}{dt}U(t, x(t)) = G(x(t), U(t, x(t))).$$

- Thus $(x(t), U(t, x(t)))_{0<t\leq t_1}$ is bounded from the finitude of $U(t_1, x_1)$.
- From the convergence in the sense of graphs, we deduce that

$$x(t) \underset{t\to 0}{\to} x_0.$$

The case of a moving target

- Take a permutation σ of $\{1; ...; d\}$, and T_σ the associated application on $\mathbb{R}^d$.
- Assume that at random times given by a Poisson process, the "target" x_0 is affected by $T_{\sigma^{-1}}$, i.e. $x_0 \to T_{\sigma^{-1}} x_0$.
- Assume F and G satisfies

$$T_{\sigma^{-1}} G(T_\sigma x, T_\sigma p) = G(x, p)$$

- Using this invariance, we can model the change of the final "target" as a change in the current density
- We can associate to this problem the master equation

$$\partial_t U + (F(x, U) \cdot \nabla_x) U + \lambda (U - (T_\sigma)^* U(T_\sigma x)) = G(x, U) \text{ in } (0, T) \times \mathbb{R}^d;$$

$$U(t) \underset{t \to 0}{\overset{G}{\to}} A_{x_0}$$

given that at $t = T$, the target is x_0.

The case with boundaries

- Model case : the half space $\{x_1 \geq 0\}$.
- Natural condition for well-posedness :

$$F^1(x,p) \leq 0 \text{ on } \{x_1 = 0\}.$$

- The uniform α monotonicity of F is no longer possible.
- First case : Work by hand the same type of regularizing results and obtain the same type of solutions
- Second case : No regularizing effect. For instance $F^1(x,p) = x_1 F(x,p)$. Then, the value function explodes for all time $t > 0$ near $x_1 = 0$.

Thank you !