Mean Field Games problems for linear control system and ergodic behavior of Mean Field Games problems depending on acceleration

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"Two Days Online Workshop on Mean Field Games"

Outline of the talk

1. MFG with linear control.
2. Example of MFG with control on the acceleration.
3. Asymptotic behavior of MFG with acceleration.

MFG with linear control

MFG system

Given an Hamiltonian function H defined as

$$H(x,p,m) = -\langle p, Ax\rangle + |B^*p|^2 - L(x,-B^*p,m)$$

for any $(x,p,m) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_\alpha(\mathbb{R}^d)$, where L is a Tonelli Lagrangian for any fixed m.
The MFG system we want to study in the following

$$\begin{cases} -\partial_t v(t,x) + H(x, D_x v(t,x)) = F(x, m_t), & (t,x) \in [0,T] \times \mathbb{R}^d \\ \partial_t m_t - \mathrm{div}(m_t D_p H(x, D_x v(t,x))) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d \\ m_0 = \bar{m}, \ \ v(T,x) = G(x, m_T) & x \in \mathbb{R}^d. \end{cases}$$

Main issues: H is not strictly convex and not coercive in p.

Presentation of the model

Control problem

This PDE system is associated with the following control problem.
Fix $T > 0$ and let A, B be $d \times d$ and $d \times k$ real matrices, respectively.
Consider the control system defined by

$$\dot{\gamma}(t) = A\gamma(t) + B\gamma(t), \quad t \in [0, T]$$

where $u : [0, T] \to \mathbb{R}^k$ is a summable function. For $x \in \mathbb{R}^d$, $u \in L^1(0, T; \mathbb{R}^k)$ and $m \in C([0, T]; \mathcal{P}_1(\mathbb{R}^d))$ set

$$\inf_{u \in L^1(0,T;\mathbb{R}^k)} \left\{ \int_0^T L(\gamma(s, x, u), u(s), m_s)\, ds + G(\gamma(T), m_T) \right\}.$$

Notation and hypothesis

Let the Lagrangian $L : \mathbb{R}^d \times \mathbb{R}^k \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ and the terminal costs $G : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ be:

1. $(x, v) \to L(x, v, m)$ Tonelli;
2. $m \to L(x, v, m)$ continuous w.r.t. the d_1 distance;
3. $G \in C_b(\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d))$

Set

$$\Gamma_T = \{\gamma(\cdot\ , x, u) : x \in \mathbb{R}^d, u \in L^1([0, T]; \mathbb{R}^k)\}$$

Let $m_0 \in \mathcal{P}_\alpha(\mathbb{R}^d)$, let $R \geq [m_0]_\alpha$ and set

$$\mathcal{P}_{m_0}(\Gamma_T, R) = \left\{ \eta \in \mathcal{P}(\Gamma_T) : \int_{\Gamma_T} \|\dot\gamma\|_2^\alpha \, \eta(d\gamma) \leq R, e_0\sharp\eta = m_0 \right\}.$$

We take a particular form of m in the above functional. i.e. we consider $m_t = e_t\sharp\eta$ for $\eta \in \mathcal{P}_{m_0}(\Gamma_T, R)$.

Equilibrium and mild solutions

Definition

We follow the Lagrangian approach to MFG system (see, for instance, Cannarsa–Capuani (2018) and Mazanti-Santabrogio (2018)).

Definition

Given $m_0 \in \mathcal{P}_\alpha(\mathbb{R}^d)$, we say that $\eta \in \mathcal{P}_{m_0}(\Gamma_T, R)$ is a **MFG equilibrium** for m_0 if

$$\mathrm{supp}(\eta) \subset \bigcup_{x \in \mathbb{R}^d} \Gamma^*_\eta(x).$$

Notation: $\Gamma^*_\eta(x)$ denotes the set of minimizing trajectories of the control problem associated with the measure $e_t \sharp \eta$.

Equilibrium and mild solutions

We prove that there exists at least one equilibrium Define the set-valued map

$$E : \left(\mathcal{P}_{m_0}(\Gamma_T, R), d_1\right) \rightrightarrows \left(\mathcal{P}_{m_0}(\Gamma_T, R), d_1\right)$$

such that

$$E(\eta) = \{\mu \in \mathcal{P}_{m_0}(\Gamma_T, R) : \mathrm{supp}(\mu_x) \subset \Gamma^*_\eta(x), m_0 - \text{a.e.}\}.$$

1. For $R \geq [m_0]_\alpha$ the set-valued map has closed graph;
2. there exists a constant $R(\alpha, [m_0]_\alpha) \geq 0$ such that $E(\eta)$ is non-empty, convex and compact and moreover, E has closed graph.

Theorem (P. Cannarsa, M.C.)

There exists at least one MFG equilibrium.

Definition

We say that $(V, m) \in C([0, T] \times \mathbb{R}^d) \times C([0, T]; \mathcal{P}_\alpha(\mathbb{R}^d))$ is a **mild solution** if there exists a MFG equilibrium η such that

1. $m_t = m_t^\eta$ for any $t \in [0, T]$
2. V is the value function associated with the underlying control problem.

Weak solution

Equivalence between weak solutions and mild solutions

Definition

$(v, m) \in W^{1,+\infty}([0, T] \times \mathbb{R}^d) \times C([0, T]; \mathcal{P}_\alpha(\mathbb{R}^d))$ is a **weak solution** if: v is a continuous viscosity solution of the HJ-eq., $D_x v$ exists m-a.e. and m is a solution in the sense of distributions of the continuity equation.

Theorem (P. Cannarsa, M.C.)

Let m_0 be absolutely continuous w.r.t. the Lebesgue measure with compactly supported density. Then, (V, m) is a mild solutions if and only if it is a weak solution.

Example: MFG with acceleration

Consider the control dynamics

$$\begin{cases} \dot{x}(t) & = v(t), \\ \dot{v}(t) & = u(t). \end{cases}$$

and consider the Lagrangian of the form
$L(x, v, w, m) = \frac{1}{2}|w|^2 + F(x, v, m)$. Then, we obtain the following MFG system

$$\begin{cases} -\partial_t u^T + \frac{1}{2}|D_v u^T|^2 - \langle D_x u^T, v\rangle = F(x, v, m_t^T) \\ \partial_t m_t^T - \langle v, D_x m_t^T\rangle - \mathrm{div}(m_t^T D_v u^T)) = 0 \\ u^T(T, x, v) = g(x, v, m_T^T), m_0^T = m_0 \end{cases}$$

for $(x, v, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, T]$.

Example: MFG with acceleration

As a corollary of the above existence result we know that there exists at least one mild solution (u, m^{η}) where η is a MFG equilibrium.
Moreover, as proved in Y. Achdou, P. Mannucci, C. Marchi and N. Tchou (2020) the unique solution m is absolutely continuous w.r.t. the Lebesgue measure with bounded density and it is the image of m_0 by the flow

$$\begin{cases} \dot{x}(t) & = v(t) \\ \dot{v}(t) & = -D_v u(t, x(t), v(t)). \end{cases}$$

Asymptotic behavior of MFG with acceleration

The problem

We recall that the time-dependent MFG reads as

$$\begin{cases} -\partial_t u^T + \frac{1}{2}|D_v u^T|^2 - \langle D_x u^T, v\rangle = F(x, v, m_t^T) \\ \partial_t m_t^T - \langle v, D_x m_t^T\rangle - \operatorname{div}(m_t^T D_v u^T)) = 0 \\ u^T(T, x, v) = g(x, v, m_T^T), m_0^T = m_0 \end{cases}$$

where $m^T : [0, T] \to \mathcal{P}_1(\mathbb{T}^d \times \mathbb{R}^d)$ and the terminal costs g belongs to $C_b^1(\mathbb{T}^d \times \mathbb{R}^d)$ for any $m \in \mathcal{P}_1(\mathbb{T}^d \times \mathbb{R}^d)$ and $m \mapsto g(x, v, m)$ is Lipschitz continuous w.r.t. the d_1-distance, uniformly in (x, v).

It should be...

From the recent literature on MFG on this subject (Cardaliaguet (2013) and Cannarsa-Cheng-M.-Wang (2019-2020)), we would expect that the limit of u^T/T, as $T \to +\infty$, is described by the following ergodic system

$$\begin{cases} \frac{1}{2}|D_v u|^2 - \langle D_x u, v\rangle = F(x, v, m) \\ -\langle v, D_x m\rangle - \mathrm{div}(mD_v u)) = 0. \end{cases}$$

Even for problems without mean field interaction, we cannot expect to have a solution to the corresponding ergodic Hamilton-Jacobi equation **due to the lack of coercivity and due to the lack of small time controllability**. Moreover, **as the drift of the continuity equation is given in terms of solution to the ergodic Hamilton-Jacobi equation**, there is no hope to formulate the problem in this way.

References on the context

Even for first-order Hamilton-Jacobi equations without mean field interaction, i.e.

$$\partial_t u(t,x) + H(x, D_x u(t,x)) = 0$$

where H is not coercive in the gradient term, the long time behavior of u has been an open issue since several years. For the analysis of special cases we refer to:
G. Barles (2007); P. Cardaliaguet (2010); Y. Giga, Q. Liu, H. Mitake (2012); M. Arisawa, P.L. Lions (1998); O. Alvarez, M. Bardi (2010);Z. Artstein, V. Gaitsgory (2000).

Presentation of the model

We consider the Lagrangian $L : \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d) \to \mathbb{R}$ of the form

$$L(x, v, w, m) = \frac{1}{2}|w|^2 + F(x, v, m)$$

where F is such that

1. F is globally continuous;
2. there exists $\alpha > 1$ and there exists $c_F \geq 0$ such that

$$\frac{1}{c_F}|v|^\alpha - c_F \leq F(x, v, m) \leq c_F(1 + |v|^\alpha);$$

3. there exists $C_F \geq 0$ such that

$$|D_x F(x, v, m)| + |D_v F(x, v, m)| \leq C_F(1 + |v|^\alpha).$$

Let $\mathcal{P}_{\alpha,2}(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ be the set of Borel probability measures satisfying

$$\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(|w|^2 + |v|^\alpha\right) \, \mu(dx, dv, dw) < +\infty.$$

Definition

Let $\eta \in \mathcal{P}_{\alpha,2}(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d)$. We say that η is a **closed measure** if for any test function $\varphi \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ the following holds

$$\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(\langle D_x\varphi(x, v), v\rangle + \langle D_v\varphi(x, v), w\rangle\Big) \, d\eta(x, v, w) = 0.$$

We denote by $\mathcal{C}$ the set of closed measures.

New problem and new method

Ergodic MFG problem

Definition

We say that $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R} \times \mathcal{C}$ is a solution of the **ergodic MFG problem** if

$$\begin{aligned} \bar{\lambda} &= \inf_{\mu \in \mathcal{C}} \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x, v, \pi \sharp \bar{\mu}) \right) \, d\mu(x, v, w) \\ &= \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x, v, \pi \sharp \bar{\mu}) \right) \, d\bar{\mu}(x, v, w). \end{aligned} \tag{1}$$

Theorem (P. Cardaliaguet, M.C.)

There exists at least one solution $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R} \times \mathcal{C}$ of the ergodic MFG problem. Moreover, if F satisfies the monotonicity assumption then the ergodic constant is unique: If $(\bar{\lambda}_1, \bar{\mu}_1)$ and $(\bar{\lambda}_2, \bar{\mu}_2)$ are two solutions of the ergodic MFG problem, then $\bar{\lambda}_1 = \bar{\lambda}_2$.

Main result

Convergence

Theorem (P. Cardaliaguet, M.C.)

Assume that $\alpha = 2$ and that the initial distribution m_0 is in $\mathcal{P}_2(\mathbb{T}^d \times \mathbb{R}^d)$. Let (u^T, m^T) be a solution of the MFG system and let $(\bar{\lambda}, \bar{\mu})$ be a solution of the ergodic MFG problem. Then $T^{-1}u^T(0,\cdot,\cdot)$ converges locally uniformly to $\bar{\lambda}$.

References

1. P. Cannarsa, C. Mendico, Mild and weak solutions of Mean Field Games problem for linear control systems, Volume 5 (2020), No. 2, 1-xx;
2. P. Cardaliaguet, C. Mendico, Ergodic behavior of control and mean field games problems depending on acceleration, forthcoming.

Thank you for the attention!