Classical and Weak Solutions to Local First Order Mean Field Games through Elliptic Regularity

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General Setting

- We study well-posedness of the local first-order MFG system:

$$\begin{cases} -u_t + H(x, D_x u) = f(x, m(x,t)) & \text{in } \mathbb{T}^d \times (0,T), \\ m_t - \operatorname{div}(m D_p H(x, D_x u)) = 0 & \text{in } \mathbb{T}^d \times (0,T), \\ m(0,x) = m_0(x),\ u(x,T) = g(x, m(x,T)) & \text{in } \mathbb{T}^d, \end{cases}$$

- $m_0 > 0$ is a probability density, H is a strictly convex Hamiltonian of quadratic growth, f and g are strictly increasing in m, f grows polynomially as $m \to \infty$.
- This system has been studied in the case where $g(x,m) = u_T(x)$ is independent of m, in the variational theory of weak solutions of P. Cardaliaguet and P.J. Graber.
- When $\lim_{m \to 0} f(\cdot, m) = -\infty$, classical solutions are obtained.
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Main Tools

- Classical maximum principle techniques and the Bernstein method for gradient estimates.
- Regularity theory for quasilinear elliptic problems with non-linear oblique derivative boundary conditions.
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First Order MFG System as an Elliptic Problem

- To make the presentation simpler, we will assume from now on that $H = H(p), f = f(m), g = g(m)$ are independent of x,

$$\begin{cases} -u_t + H(D_x u) = f(m) & \text{in } \mathbb{T}^d \times (0,T), \\ m_t - \operatorname{div}(mDH) = 0 & \text{in } \mathbb{T}^d \times (0,T), \\ m(0,x) = m_0(x),\ u(x,T) = g(m(\cdot,T)) & \text{in } \mathbb{T}^d, \end{cases}$$

- The strategy of proof follows the ideas of P. L. Lions from his work on the planning problem: setting $m = f^{-1}(-u_t + H)$ we can eliminate m from the system and rewrite it as a first order quasilinear elliptic problem:

$$\begin{cases} Qu = -\mathrm{Tr}(A(Du)D^2u) = 0 & \text{in } \mathbb{T}^d \times (0,T), \\ Nu = B(x,t,u,Du) = 0 & \text{on } \mathbb{T}^d \times \{t = 0, T\}. \end{cases}$$

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Ellipticity Condition

The matrix A, given by

$$A = \begin{pmatrix} DH \otimes DH + mf'(m)D^2H & -DH^T \\ -DH & 1 \end{pmatrix},$$

is strictly positive, except when $mf'(m) = 0$. In particular, when the players have a strong incentive to navigate areas of low density, which precludes m from vanishing, we expect regularity. This motivates the following definition:

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The MFG system is said to be ***strictly elliptic*** if $\lim_{m\to 0^+} f(m) = -\infty$. Otherwise, it is said to be ***degenerate elliptic***.

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Degenerate Elliptic Problem

Definition

A pair $(u, m) \in BV(Q_T) \times L^\infty_+(Q_T)$ is called a ***weak solution*** if:

(i) $D_x u \in L^2(Q_T)$, $u \in L^\infty(Q_T)$, $m \in C^0([0,T]; H^{-1}(\mathbb{T}^d))$, $m(\cdot, T) \in L^\infty(\mathbb{T}^d)$.

(ii) $-u_t + H(\cdot, D_x u) \leq f(\cdot, m)$ and $m_t - \operatorname{div}(m D_p H(\cdot, D_x u)) = 0$ hold in the distributional sense. Moreover, $u(\cdot, T) = g(\cdot, m(\cdot, T))$ in the sense of traces, and $m = m_0$ in $H^{-1}(\mathbb{T}^d)$.

(iii) The following identity holds:

$$\int\int_{Q_T} m(x,t)(H(x, D_x u) - D_p H(x, D_x u) \cdot D_x u - f(x,m))dxdt$$
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Degenerate Elliptic Problem

The following is our main result for the degenerate elliptic problem:

Theorem

Assume that the MFG system is degenerate elliptic. Then:

- *If (u, m), (u', m') are two weak solutions, then $m = m'$ a.e. in $\mathbb{T}^d \times [0, T]$, and $u = u'$ a.e. in $\{m > 0\}$. Moreover, $m(\cdot, T) = m'(\cdot, T)$ and $u(\cdot, T) = u'(\cdot, T)$ a.e. in $\mathbb{T}^d$.*
- *There exists a weak solution (u, m). Furthermore (u, m) is obtained as the "viscous limit" of classical solutions (u^ϵ, m^ϵ) to strictly elliptic first order MFG systems.*

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- Since u satisfies the elliptic equation $Qu = 0$, it satisfies the maximum principle. The boundary condition for the elliptic problem is given explicitly by

$$B(x, 0, u, Du) = -u_t + H(D_x u) - f(m_0(x)),$$

$$B(x, T, u, Du) = u - g(f^{-1}(-u_t + H(D_x u))).$$

- Thanks to the strict monotonicity of f, g, this is an oblique boundary condition. In fact, the linearization of this condition has the form

$$\alpha(x,t) \cdot Dw + \beta(x,t)w = \gamma(x,t),$$

where $\alpha \cdot \nu > 0$, $\beta(\cdot, 0) \equiv 0$ and $\beta(\cdot, T) \equiv 1$. In this sense, B is of "Neumann type" in the lower half of the boundary and of "Robin type" in the upper half.

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- Since Robin boundary conditions provide L^∞ estimates, there would be nothing to prove if u achieved its maximum value at $t = T$.
- The proof thus consists of adequately choosing a function $\psi(t)$ such that $v = u + \psi(t)$ still satisfies the maximum principle, but forcefully achieves its maximum value at $t = T$. This yields an estimate of the form

$$g(\min m_0) - C(T - t) \leq u(x,t) \leq g(\max m_0) + C(T - t).$$

- As a Corollary, since $u(x, T) = g(m(x, T))$, we also obtain two-sided bounds for the terminal density:

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A Priori Gradient Estimate

- The L^∞ bound for the space-time gradient is obtained in two steps.
- First, differentiating the equation $Qu = 0$ one sees that the time derivative u_t satisfies the maximum principle.
- Thus, since $-u_t + H = f(m)$, and $m(0), m(T)$ are a priori bounded above and below, it follows that $||u_t||_{L^\infty} \leq ||H(D_x u)||_{L^\infty} + C$, and in particular reduces the problem to estimating only the space gradient.

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- The second step is to bound the space gradient using Bernstein's method.
- The bound $||u_t||_{L^\infty} \leq ||H(D_x u)||_{L^\infty} + C$ obtained in the first step comes into play here as well, because, at points (x_0, t_0) where $H(D_x u)$ is near its maximum value, it provides an a priori lower bound $f(m(x_0, t_0)) = -u_t + H \approx -u_t + ||H(D_x u)||_{L^\infty} \geq -C$.
- This amounts to a strictly positive lower bound $m(x_0, t_0) \geq f^{-1}(-M) > 0$, and thus a lower bound for the ellipticity of the problem, which is essential in the Bernstein estimate.

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Existence of Classical Solutions

- The classical a priori estimates for elliptic, quasilinear, oblique derivative problems yield a Hölder bound for Du.
- Smooth solutions are then obtained through the non-linear method of continuity. Namely, we consider, for $0 \leq \theta \leq 1$ the following homotopy of MFG systems:

$$\begin{cases} -u_t + H^\theta = f(m(x,t)) & \text{in } \mathbb{T}^d \times [0,T], \\ m_t - \operatorname{div}(mD_pH^\theta) = 0 & \text{in } \mathbb{T}^d \times [0,T], \\ m(0,x) = m_0^\theta(x), & x \in \mathbb{T}^d, \\ u(x,T) = g(m(x,T)) & x \in \mathbb{T}^d. \end{cases} \qquad (\text{MFG}_\theta)$$

where $H^\theta(p) = \theta H(p) + (1-\theta)(\frac{1}{2}|p|^2 + f(1))$, $m_0^\theta(x) = \theta m_0(x) + (1-\theta)$. When $\theta = 0$, this has the trivial solution $(u,m) \equiv (g(1), 1)$.

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Existence of Classical Solutions

- We fix $0 \leq \alpha < 1$. We consider the set

  $$S = \{\theta \in [0,1] : (\mathrm{MFG}_\theta) \text{ has a unique } C^{3,\alpha} \times C^{2,\alpha} \text{ solution}\}.$$

  We know that $0 \in S$, and S can be seen to be open by the Implicit Function Theorem and the classical theory of linear elliptic oblique problems. We also know that $0 \in S$.

- The a priori estimates obtained so far, together with a classical stability theorem for quasilinear oblique problems, can be shown to imply that S is closed as well. Thus $S = [0,1]$, which completes the proof.

Existence of Classical Solutions

- We fix $0 \leq \alpha < 1$. We consider the set

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Contents

Existence and Uniqueness of Weak Solutions

- Strategy: obtain the weak solution (u, m) as $\lim_{\epsilon\to 0^+}(u^\epsilon, m^\epsilon)$, where (u^ϵ, m^ϵ) is the classical solution to the strictly elliptic MFG system:

$$\begin{cases} -u_t^\epsilon + H(D_x u^\epsilon) = f + \epsilon \log m^\epsilon & \text{in } \mathbb{T}^d \times [0, T], \\ m_t^\epsilon - \operatorname{div}(m^\epsilon D_p H) = 0 & \text{in } \mathbb{T}^d \times [0, T], \\ m^\epsilon(0, x) = m_0(x),\ u^\epsilon(x, T) = g(m^\epsilon(x, T)) & x \in \mathbb{T}^d, \end{cases}$$

- The key point is that some of the a priori estimates obtained previously are independent of ϵ. We lose the a priori lower bound on m^ϵ. However, the bounds on $||u||_{L^\infty}$ are still available, and it is possible to modify the Bernstein argument to obtain an upper bound on $||m||_{L^\infty}$ and $||u_t^-||_{L^\infty}$.

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- One can then combine the a priori L^∞ estimates obtained here with some standard integral estimates that come from the Lasry-Lions monotonicity method.
- This yields enough compactness to guarantee that, up to a subsequence, (u^ϵ, m^ϵ) converges to the weak solution (u, m).
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