The Master Equation in a Bounded Domain with Neumann Conditions

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Introduction

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Dynamic of the player i**,** $1 \le i \le N$:

$$\begin{cases} dX_t^i = b(X_t^i, \alpha_t^i)\, dt + \sqrt{2}\sigma(X_t^i) dB_t^i\,, \\ X_{t_0}^i = x_0^i\,. \end{cases}$$

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Here, $x_0^i \in \mathbb{R}^d$, α_t^i is the control, b and σ are the *drift* term and the *diffusion* matrix and $(B_t)^i$ are independent d-dimensional Brownian motions.

Introduction

Cost for the player i:

$$J_i^N(t_0, \boldsymbol{x}_0, \boldsymbol{\alpha}.) = \mathbb{E}\left[\int_{t_0}^{T} \left(L(s, X_s^i, \alpha_s^i) + F_i^N(s, \boldsymbol{X}_s)\right) ds + G_i^N(\boldsymbol{X}_T)\right] ,$$

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where F_i^N and G_i^N are the *cost functions* of the player i and L is the *Langrangian* cost for the control.

We say that a control $\boldsymbol{\alpha}_\cdot^*$ provides a *Nash equilibrium* if, for all controls $\boldsymbol{\alpha}_\cdot$ and for all i we have

$$J_i^N(t_0, \boldsymbol{x}_0, \boldsymbol{\alpha}_\cdot^*) \leq J_i^N(t_0, \boldsymbol{x}_0, \alpha_i, (\alpha_j^*)_{j \neq i}),$$

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Value function:

$$v_i^N(t_0, \boldsymbol{x}_0) = J_i^N(t_0, \boldsymbol{x}_0, \boldsymbol{\alpha}^*).$$

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Using Ito's formula and the dynamic programming principle, one can prove that v_i^N solves the so-called Nash system:

$$\begin{cases} -\partial_t v_i^N(t,\boldsymbol{x}) - \sum_j \mathrm{tr}(a(x_j)D^2_{x_jx_j}v_i^N(t,\boldsymbol{x})) \quad + H(x_i, D_{x_i}v_i^N(t,\boldsymbol{x})) \\ \qquad + \sum_{j\neq i} H_p(x_j, D_{x_j}v_j^N(\boldsymbol{x}))\cdot D_{x_j}v_i^N(t,\boldsymbol{x}) \quad = F_i^N(\boldsymbol{x})\,, \\ v_i^N(T,\boldsymbol{x}) = G_i^N(\boldsymbol{x})\,, \end{cases} \tag{1}$$

for $(t,\boldsymbol{x}) \in [0,T]\times\mathbb{R}^{Nd}$. Here H is the *Hamiltonian* of the system and $a = \sigma\sigma^*$.

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The idea of Lasry and Lions is to simplify the Nash system, with suitable symmetry conditions for the agents and their dynamics, for $N \gg 1$. This leads us to the study of the so-called ***Mean Field Games System***.

Introduction

We suppose that F_i^N and G_i^N are of this form:

$$F_i^N(\boldsymbol{x}) = F(x_i, m_{\boldsymbol{x}}^{N,i}), \qquad G_i^N(t, \boldsymbol{x}) = G(x_i, m_{\boldsymbol{x}}^{N,i}),$$

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Heuristically, when $N \to +\infty$, the Mean Field Games system takes the following form:

$$\begin{cases} -\partial_t u - \mathrm{tr}(a(x)D^2u) + H(x,Du) = F(x,m), \\ \partial_t m - \sum_{i,j}\partial^2_{ij}(a_{ij}(x)m) - \mathrm{div}(mH_p(x,Du)) = 0, \\ m(0) = m_0, \qquad u(T) = G(x, m(T)), \end{cases} \tag{2}$$

with a Hamilton-Jacobi-Bellman equation for u coupled with a Fokker-Planck equation for the law of the population m.

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In order to describe this limit problem, Lasry and Lions introduced the ***Master Equation***, which summarizes the MFG system in a unique equation.

Introduction

We consider the solution (u, m) of (2) with $m(t_0) = m_0 \in \mathcal{P}(\mathbb{R}^d)$, where $\mathcal{P}(\mathbb{R}^d)$ is the set of Borel probability measures, and we define

$$U : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}\,, \qquad U(t_0, x, m_0) = u(t_0, x)\,, \tag{3}$$

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Formulation of the Master Equation

$$\begin{cases} -\partial_t U(t,x,m) - \mathrm{tr}\left(a(x)D_x^2 U(t,x,m)\right) + H\left(x, D_x U(t,x,m)\right) \\ -\displaystyle\int_\Omega \mathrm{tr}\left(a(y) D_y D_m U(t,x,m,y)\right) dm(y) \\ +\displaystyle\int_\Omega D_m U(t,x,m,y)\cdot H_p(y, D_x U(t,y,m)) dm(y) = F(x,m)\,, \\ U(T,x,m) = G(x,m)\,. \end{cases} \tag{4}$$

Here, $D_m U$ is a suitable derivative of U with respect to the measure m.

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In this talk I will be focused on the first aspect.

Introduction - In the Literature

- **Mean Field Games system**: Lasry, Lions; Huang, Caines, Malhamé, . . . ;
- **Neumann boundary conditions**:
  - Achdou, Bardi-Cirant (2018), Porretta (2015), Gomes, . . . ;
  - Achdou-Dao-Ley-Tchou (2019), Camilli-Carlini-Marchi (2015) *(Mean Field Games on Networks)*;
- **The Master Equation**
  - Lions *(Derivation, Finite state space, Short time existence)*;
  - Gangbo-Swiech (2015) *(First order and no Diffusion)*;
  - Chassagneux-Crisan-Delarue (2014, 2015) *(First order)*;
  - Cardaliaguet-Delarue-Lasry-Lions (2015) *(Second order in the Torus)*;
  - Carmona-Delarue (2014) *(Second order, in the whole Space)*; . . .
- **The convergence problem**:
  - Lasry-Lions, Fischer (2017), Lacker (2016),. . . *(Open loop strategies)*
  - Cardaliaguet-Delarue-Lasry-Lions, Lacker (2018), . . . *(Closed-loop)*

The Master Equation in a Bounded Domain with Neumann Conditions

Outline

Stochastic interpretation and Equations involved

We analyze the asymptotic behaviour of an N-players differential game, where each player chooses his own control and plays his dynamic in a closed bounded domain $\Omega \subseteq \mathbb{R}^d$.

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As already said, here the invariance of the domain is obtained by adding a reflecting process on the boundary $\partial\Omega$.

Hence, the dynamic of the single player i becomes

$$\begin{cases} dX_t^i = b(X_t^i, \alpha_t^i)\, dt + \sqrt{2}\sigma(X_t^i) dB_t^i - dk_t^i\,, \\ X_{t_0}^i = x_0^i\,, \end{cases}$$

where k_t^i is a *reflected process along the co-normal*.

This reflected process is defined in the following way (see Lions, Snitzman, 1984).

$$k_t^i = \int_0^t a(X_s^i)\nu(X_s^i)\, d|k^i|_s\,, \qquad |k^i|_t = \int_0^t \mathbb{1}_{\{X_s^i\in\partial\Omega\}}\, d|k^i|_s\,,$$

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The Nash system (1) for the value function becomes in this case

$$\begin{cases} -\partial_t v_i^N - \displaystyle\sum_j \mathrm{tr}(a(x_j)D^2_{x_j x_j} v_i^N) + H(x_i, D_{x_i} v_i^N) \\ \qquad\qquad\qquad\qquad + \displaystyle\sum_{j \neq i} H_p(x_j, D_{x_j} v_j^N) \cdot D_{x_j} v_i^N = F(x_i, m_{\boldsymbol{x}}^{N,i})\,, \\ v_i^N(T, \boldsymbol{x}) = G(x_i, m_{\boldsymbol{x}}^{N,i})\,, \\ a(x_j) D_{x_j} v_i^N \cdot \nu(x_j)_{|x_j \in \partial\Omega} = 0\,, \qquad j = 1, \cdots, N\,, \end{cases} \tag{5}$$

with Neumann boundary conditions for the functions $(v_i^N)_i$.

We recall that the solution of the Master Equation is defined from its trajectories, which are the solutions of the MFG system (2).

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The Master Equation, in this case, takes the following form

$$\begin{cases} -\partial_t U(t,x,m) - \mathrm{tr}\left(a(x) D_x^2 U(t,x,m)\right) + H\left(x, D_x U(t,x,m)\right) \\ -\displaystyle\int_\Omega \mathrm{tr}\left(a(y) D_y D_m U(t,x,m,y)\right) dm(y) + \\ \displaystyle\int_\Omega D_m U(t,x,m,y) \cdot H_p(y, D_x U(t,y,m)) dm(y) = F(x,m) \\ \text{in } (0,T) \times \Omega \times \mathcal{P}(\Omega)\,, \\ U(T,x,m) = G(x,m) \quad \text{in } \Omega \times \mathcal{P}(\Omega)\,, \\ a(x) D_x U(t,x,m) \cdot \nu(x) = 0 \qquad \text{for } (t,x,m) \in (0,T) \times \partial\Omega \times \mathcal{P}(\Omega)\,, \\ a(y) D_m U(t,x,m,y) \cdot \nu(y) = 0 \quad \text{for } (t,x,m,y) \in (0,T) \times \Omega \times \mathcal{P}(\Omega) \times \partial\Omega\,. \end{cases}$$

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- Boundary conditions of the Nash system and the Master Equation are closely connected. Roughly speaking, the symmetrical structure of the problem implies
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Hence,

$$a(x_i)D_{x_i}v_i^N\cdot\nu(x_i)=0\longrightarrow a(x)D_xU(t,x,m)\cdot\nu(x)=0\,,$$
$$a(x_j)D_{x_j}v_i^N\cdot\nu(x_j)_{j\neq i}=0\longrightarrow a(y)D_mU(t,x,m,y)\cdot\nu(y)=0\,.$$

Notations and Derivatives

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The topology on $\mathcal{P}(\Omega)$ is defined from the so-called *Wasserstein distance*. For $m_1, m_2 \in \mathcal{P}(\Omega)$, we set

$$\mathbf{d}_1(m_1, m_2) := \sup_{\phi \, 1-\mathrm{Lip.}} \int_\Omega \phi(x)(m_1(dx) - m_2(dx)) \, .$$

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$$\mathbf{d}_1(m_1, m_2) := \sup_{\phi \text{ 1-Lip.}} \int_\Omega \phi(x)(m_1(dx) - m_2(dx))\,.$$

This distance set a topology on $\mathcal{P}(\Omega)$ and allows us to talk about continuity of U with respect to m.

We introduce two notions of derivation with respect to the measure.

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Definition 1.1

Let $U : \mathcal{P}(\Omega) \to \mathbb{R}$. We say that U is of class $\mathcal{C}^1$ if there exists a continuous map $\frac{\delta U}{\delta m} : \mathcal{P}(\Omega) \times \Omega \to \mathbb{R}$ such that, for all m_1, $m_2 \in \mathcal{P}(\Omega)$ we have

$$\lim_{t \to 0} \frac{U(m_1 + s(m_2 - m_1)) - U(m_1)}{s} = \int_\Omega \frac{\delta U}{\delta m}(m_1, y)(m_2(dy) - m_1(dy)),$$

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with the normalization convention

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Then, if $\frac{\delta U}{\delta m}$ is of class $\mathcal{C}^1$ with respect to the space variable, we define the intrinsic derivative of U with respect to m as

$$D_m U(m, y) = D_y \frac{\delta U}{\delta m}(m, y).$$

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For $n \geq 0$ and $\alpha \in (0,1)$, we denote with $\mathcal{C}^{n+\alpha}$ the space of functions $\phi \in \mathcal{C}^n(\Omega)$ with bounded norm

$$\|\phi\|_{n+\alpha} := \sum_{|\ell| \leq n} \left\| D^\ell \phi \right\|_\infty + \sum_{|\ell| = n} \sup_{x \neq y} \frac{|D^\ell \phi(x) - D^\ell \phi(y)|}{|x-y|^\alpha} .$$

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- a uniformly elliptic with $\|a(\cdot)\|_{1+\alpha} < \infty$;
- H smooth, Lipschitz and strictly convex with respect to the last variable;
- F and G smooth and non-decreasing in the last variable, with

$$\sup_{m \in \mathcal{P}(\Omega)} \left(\|F(\cdot, m)\|_\alpha + \left\| \frac{\delta F}{\delta m}(\cdot, m, \cdot) \right\|_{\alpha, 2+\alpha} \right) + \mathrm{Lip}\left(\frac{\delta F}{\delta m}\right) \le C_F \ ,$$

with

$$\mathrm{Lip}\left(\frac{\delta F}{\delta m}\right) := \sup_{m_1 \neq m_2} \left(\mathbf{d}_1(m_1, m_2)^{-1} \left\| \frac{\delta F}{\delta m}(\cdot, m_1, \cdot) - \frac{\delta F}{\delta m}(\cdot, m_2, \cdot) \right\|_{\alpha, 1+\alpha} \right) \ ,$$

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and G satisfies the same estimates with α and $1+\alpha$ replaced by $2+\alpha$;
- The following Neumann boundary conditions are satisfied:

$$\left\langle a(y) D_y \frac{\delta F}{\delta m}(x,m,y), \nu(y) \right\rangle_{|\partial\Omega} = 0\,, \quad \left\langle a(y) D_y \frac{\delta G}{\delta m}(x,m,y), \nu(y) \right\rangle_{|\partial\Omega} = 0\,,$$

$$\langle a(x) D_x G(x,m), \nu(x) \rangle_{|\partial\Omega} = 0\,,$$

for all $m \in \mathcal{P}(\Omega)$.

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\displaystyle\int_\Omega D_m U(t,x,m,y)\cdot H_p(y, D_x U(t,y,m)) dm(y) = F(x,m) \\
\text{in } (0,T)\times\Omega\times\mathcal{P}(\Omega)\,, \\
U(T,x,m) = G(x,m) \quad \text{in } \Omega\times\mathcal{P}(\Omega)\,, \\
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Theorem 1.2

Suppose main hypotheses are satisfied. Then there exists a unique classical solution U of the Master Equation.

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Preliminary results

- If (u, m) is a solution of the MFG system with Neumann conditions, then

$$\|u\|_{1+\frac{\alpha}{2}, 2+\alpha} \leq C\,, \qquad \sup_{t\neq s} \mathbf{d}_1(m(t), m(s)) \leq C|t-s|^{\frac{1}{2}}\,;$$

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- If (u,m) and $(\tilde{u},\tilde{m})$ are solutions of the MFG system with initial conditions m_0 and $\tilde{m}_0$, then

$$\|u-\tilde{u}\|_{1,2+\alpha} + \sup_{t\in[0,T]} \mathbf{d}_1(m(t),\tilde{m}(t)) \le C\mathbf{d}_1(m_0,\tilde{m}_0)$$

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We have to prove the existence of the derivative $\frac{\delta U}{\delta m}$.

Idea: For (u, m), $(\tilde{u}, \tilde{m})$ defined before, linearize the equation of $(\tilde{u} - u, \tilde{m} - m)$. We obtain the following linear system

Linearized system and C^1 character of U

General linearized system:

$$\begin{cases} -z_t - \mathrm{tr}(a(x)D^2 z) + H_p(x,Du)Dz = \dfrac{\delta F}{\delta m}(x,m(t))(\rho(t)) + h(t,x)\,, \\ \rho_t - \mathrm{div}(a(x)D\rho) - \mathrm{div}(\rho(H_p(x,Du)+\tilde{b})) - \mathrm{div}(mH_{pp}(x,Du)Dz + c) = 0\,, \\ z(T,x) = \dfrac{\delta G}{\delta m}(x,m(T))(\rho(T)) + z_T(x)\,, \qquad \rho(t_0) = \rho_0\,, \\ aDz\cdot\nu_{|\partial\Omega} = 0\,, \quad \big(aD\rho + \rho(H_p(x,Du)+\tilde{b}) + mH_{pp}(x,Du)Dz + c\big)\cdot\nu_{|\partial\Omega} = 0\,, \end{cases} \tag{6}$$

where z_T, ρ_0, h and c are small if m_0 and $\tilde{m}_0$ are "close".

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Actually, we will have

$$\frac{\delta U}{\delta m}(t_0,x,m_0,y) = z(t_0,x)\,,$$

where z solves (6) with $\rho_0 = \delta_y$, $h = c = z_T = 0$ (Pure linearized system) .

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Suppose $z_T \in \mathcal{C}^{2+\alpha}$, $\rho_0 \in \mathcal{C}^{-(1+\alpha)}$, $h \in \mathcal{C}^{0,\alpha}([t_0,T]\times\Omega)$, , $c \in L^1([t_0,T]\times\Omega)$.
We have a regularity result for the system (6).

Proposition

If main hypotheses are satisfied, and then there exists a unique solution $(z, \rho) \in \mathcal{C}^{1,2+\alpha} \times \left(\mathcal{C}([0,T]; \mathcal{C}^{-(1+\alpha),N}) \cap L^1(Q_T)\right)$ *of system* (6). *This solution satisfies, for a certain* $p > 1$ *and* $C > 0$,

$$\|z\|_{1,2+\alpha} + \sup_t \|\rho(t)\|_{-(1+\alpha),N} + \|\rho\|_{L^p} \leq CM\,, \tag{7}$$

where $M := \|z_T\|_{2+\alpha} + \|\rho_0\|_{-(1+\alpha)} + \|h\|_{0,\alpha} + \|c\|_{L^1}$.

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These estimates allows us to prove the following Theorem.

Theorem 1.3

Suppose main hypotheses are satisfied. Then U *is* $\mathcal{C}^1$ *with respect to* m, *and the following boundary condition holds true:*

$$a(y)D_mU(t_0,x,m_0,y)\cdot\nu(y)=0\,,\qquad y\in\partial\Omega\,.$$

Sketch of the proof:

- Prove that the system (6) with $z_T = c = h = 0$ admits a fundamental solution: $\exists K$ such that, if (z, ρ) is the solution,

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- The couple $(\tilde{u} - u - z, \tilde{m} - m - \rho)$, with $\rho_0 = \tilde{m}_0 - m_0$, solves (6). Applying (7), we have

$$\left\| U(t, \cdot, \tilde{m}_0) - U(t, \cdot, m_0) - \int_\Omega K(t, \cdot, m_0, y)(\tilde{m}_0 - m_0)(dy) \right\|_{2+\alpha} \leq C \mathbf{d}_1(m_0, \tilde{m}_0)^2\,.$$

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- We prove that the solution (z, ρ) of (6) with $h = c = z_T = 0$ and $\rho_0 = -\partial_{a\nu} \delta_y$ satisfies $z = 0$;

Existence and uniqueness of solutions

- This reduces to prove, using boundary conditions on F and G, that

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Improving the estimates on the linear system from $\mathcal{C}^{-(1+\alpha)}$ to $\mathcal{C}^{-(2+\alpha)}$, we have

$$\left\| \frac{\delta U}{\delta m}(t, \cdot, m, \cdot) \right\|_{2+\alpha, 2+\alpha} \le C\,.$$

This allows us to prove the main theorem of this section.

The convergence problem

Now we are able to prove that the solution U of the Master Equation approximates the N-players differential game, readapting the ideas of Cardaliaguet, Delarue, Lasry, Lions.

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Now we are able to prove that the solution U of the Master Equation approximates the N-players differential game, readapting the ideas of Cardaliaguet, Delarue, Lasry, Lions.

To do that, we consider, for $1 \le i \le N$, the solutions v_i^N of the Nash system:

$$\begin{cases} -\partial_t v_i^N - \sum_j \mathrm{tr}(a(x_j) D^2_{x_j x_j} v_i^N) + H(x_i, D_{x_i} v_i^N) \\ \qquad\qquad\qquad + \sum_{j \neq i} H_p(x_j, D_{x_j} v_j^N) \cdot D_{x_j} v_i^N = F(x_i, m_{\boldsymbol{x}}^{N,i}) \,, \\ v_i^N(T, \boldsymbol{x}) = G(x_i, m_{\boldsymbol{x}}^{N,i}) \,, \\ a(x_j) D_{x_j} v_i^N \cdot \nu(x_j)_{|x_j \in \partial\Omega} = 0 \,, \qquad j = 1, \cdots, N \,, \end{cases}$$

The convergence problem

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and the auxiliary functions

$$u_i^N(t, \boldsymbol{x}) = U(t, x_i, m_{\boldsymbol{x}}^{N,i}) \,.$$

We want to prove that u_i^N and v_i^N are close if N is sufficiently large.

- Thanks to the regularity of U, we prove the following representation formulas for the derivatives of u_i^N: for all $j \neq i$,

$$
\begin{aligned}
D_{x_j} u_i^N(t,\boldsymbol{x}) &= \frac{1}{N-1} D_m U(t, x_i, m_{\boldsymbol{x}}^{N,i}, x_j)\,,\\
D^2_{x_i,x_j} u_i^N(t,\boldsymbol{x}) &= \frac{1}{N-1} D_x D_m U(t, x_i, m_{\boldsymbol{x}}^{N,i}, x_j)\,,\\
\left| D^2_{x_j,x_j} u_i^N(t,\boldsymbol{x}) - \frac{1}{N-1} D_y D_m U(t, x_i, m_{\boldsymbol{x}}^{N,i}, x_j)\right| &\leq \frac{C}{N^2}\,.
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where $r_i^N \in L^\infty$ with $\left\| r_i^N \right\|_\infty \leq \frac{C}{N}$.

We consider the optimal dynamic $\boldsymbol{Y}_t$ of the N-players system, starting from m_0:

$$\begin{cases} dY_t^i = -H_p(Y_t^i, D_{x_i} v_i^N(t, \boldsymbol{Y}_t))\, dt + \sqrt{2}\sigma(Y_t^i) dB_t^i - dk_t^i\,, \\ Y_{t_0}^i = Z^i\,, \end{cases}$$

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Theorem 1.4

Assume main hypotheses hold. Then, for any $1 \le i \le N$, we have

$$\mathbb{E}\left[\int_{t_0}^T \left|D_{x_i} v_i^N(t, \boldsymbol{Y}_t) - D_{x_i} u_i^N(t, \boldsymbol{Y}_t)\right|^2 dt\right] \le \frac{C}{N^2}\,. \tag{8}$$

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- **Idea of the proof**: Estimate the term $\left(u_i^N(t, \boldsymbol{Y}_t) - v_i^N(t, \boldsymbol{Y}_t)\right)^2$, using Ito's formula and the equations satisfied by u_i^N and v_i^N.

Finally, we can state the main convergence result of the Nash system towards the Master Equation.

Theorem 1.5

Suppose main hypotheses hold true. Then, if we define $m_{\boldsymbol{x}}^N := \frac{1}{N}\sum_i \delta_{x_i}$, *we have*

$$\sup_i \left| v_i^N(t_0, \boldsymbol{x}) - U(t_0, x_i, m_{\boldsymbol{x}}^N) \right| \le \frac{C}{N}\,. \tag{10}$$

Moreover, if we set

$$w_i^N(t_0, x_i, m_0) := \int_{\Omega^{N-1}} v_i^N(t_0, \boldsymbol{x}) \prod_{j \neq i} m_0(dx_j)\,,$$

then $$\left\| w_i^N(t_0, \cdot, m_0) - U(t_0, \cdot, m_0) \right\|_{L^1(m_0)} \le C\omega_N\,, \qquad \text{with } \omega_N \overset{N \to +\infty}{\to} 0 \tag{11}$$

Open problems

- Study the convergence of the trajectories: if X_t^i and Y_t^i are defined in this way:

$$\begin{cases} dX_t^i = -H_p(X_t^i, D_{x_i} u_i^N(t, \boldsymbol{X}_t))\, dt + \sqrt{2}\sigma(X_t^i) dB_t^i - dk_t^i\,, \\ X_{t_0}^i = Z^i\,, \end{cases}$$

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prove that

$$\mathbb{E}\left[\sup_{t\in[t_0,T]} \left|Y_t^i - X_t^i\right|\right] \le \frac{C}{N}\,.$$

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- Well-posedness of the Master Equation and Convergence problem in a framework of invariance condition.


THANK YOU!