Convergence for finite state mean field control problems

Alekos Cecchin

Laboratoire de Mathématiques “J.A. Dieudonné”
Université Côte d’Azur

Workshop on mean field games. 18-19 June 2020

Introduction

Mean field control problems as limits of N-agent optimization, when the number N of agents tends to infinity.

- We consider problems in continuous time where the position of each agent belongs to a finite state space $[\![d]\!] = \{1, \dots, d\}$.
- Agents are indistinguishable and control their transition rate from state to state in order to minimize a cost.
- Symmetric and mean field interaction: each agent knows its position and the number of other agents in any of the d states.
- Notion of optimality at prelimit level: Pareto equilibrium. Agents are cooperative and have a common cost to minimize.

Results for continuous state space

Mean field control problem analyzed in:

- [Carmona-Delarue '15]: open-loop controls, use stochastic maximum principle, get FBSDEs of McKean-Vlasov type;
- [Pham-Wei '18]: closed-loop controls, use dynamic programming, get HJB equation, viscosity solutions.
- [Cardaliaguet-Graber-Porretta-Tonon '15]: potential mean field game, MFG system as necessary conditions for optimality.

Results for continuous state space

Mean field control problem analyzed in:

- [Carmona-Delarue '15]: open-loop controls, use stochastic maximum principle, get FBSDEs of McKean-Vlasov type;
- [Pham-Wei '18]: closed-loop controls, use dynamic programming, get HJB equation, viscosity solutions.
- [Cardaliaguet-Graber-Porretta-Tonon '15]: potential mean field game, MFG system as necessary conditions for optimality.

Convergence of N-agent optimization studied in:

- [Lacker '17]: convergence of optimal controls via compactness arguments, no convergence rate;
- [Carmona-Delarue '15]: convergence rate, using strong convexity assumptions.

Outline

1. N-agent optimization
   - Value function V^N

Outline

1. N-agent optimization
   - Value function V^N

2. Mean field control problem
   - Value function V is viscosity solution of HJB equation

Outline

1. N-agent optimization
   - Value function V^N
2. Mean field control problem
   - Value function V is viscosity solution of HJB equation
3. Convergence of V^N to V with convergece rate, under general assumption, using viscosity solution property

Outline

1. N-agent optimization
   - Value function V^N
2. Mean field control problem
   - Value function V is viscosity solution of HJB equation
3. Convergence of V^N to V with convergece rate, under general assumption, using viscosity solution property
4. Under convexity assumptions
   - Unique solution to MFCP
   - Regularity of V
   - Convergence of optimal trajectories, with convergence rate

Agents dynamics

N identical agents $X^1, \dots, X^N$ with $X^k_t \in [\![d]\!]$,
Evolve in continuous time, finite horizon T.
denote $\boldsymbol{x} = (x_1, \dots, x_N), \boldsymbol{X}_t = (X^1_t, \dots, X^N_t) \in [\![d]\!]^N$.

Player k chooses its transition rate $\beta^k_j(t, \boldsymbol{x}) \geq 0$ in Markovian feedback form:

$$\mathbb{P}\left[X^k_{t+h} = j | \boldsymbol{X}_t = \boldsymbol{x}\right] = \beta^k_j(t, \boldsymbol{x})h + o(h)$$

in order to minimize the cost

$$J^k(\beta^1, \dots, \beta^N) = \mathbb{E}\left[\int_0^T \ell(X^k_t, \beta^k(t, \boldsymbol{X}_t)) + f(X^k_t, \mu^N_t)dt + g(X^k_T, \mu^N_T)\right]$$

Given $\boldsymbol{x}$ denote the empirical measure $m^N_{\boldsymbol{x}} = \frac{1}{N}\sum_{k=1}^N \delta_{x_k}$
$N \cdot m^N_{i,t} = \sum_k \mathbb{1}_{\{X^k_t = i\}}$ is the number of players in state i.

Optimization

Consider for simplicity

$$\ell^i(\beta^k(\boldsymbol{x})) = \ell(i, \beta^k(\boldsymbol{x})) = \frac{1}{2}\sum_{j\neq i} |\beta^k_j(t, i, \boldsymbol{x}^{-k})|^2$$

Agents are cooperative: common reward to minimize

$$J_N(\boldsymbol{\beta}) = \frac{1}{N}\sum_{k=1}^{N} J^k(\boldsymbol{\beta})$$

strategy vector $\boldsymbol{\beta} = (\beta^1, \dots, \beta^N)$ not necessarily exchangeable.

Optimization

Consider for simplicity

$$\ell^i(\beta^k(\boldsymbol{x})) = \ell(i, \beta^k(\boldsymbol{x})) = \frac{1}{2}\sum_{j\neq i} |\beta^k_j(t, i, \boldsymbol{x}^{-k})|^2$$

Agents are cooperative: common reward to minimize

$$J_N(\boldsymbol{\beta}) = \frac{1}{N}\sum_{k=1}^{N} J^k(\boldsymbol{\beta})$$

strategy vector $\boldsymbol{\beta} = (\beta^1, \dots, \beta^N)$ not necessarily exchangeable.

Single optimization problem for the process $\boldsymbol{X} = (X^1, \dots, X^N)$ defined by the generator

$$\mathcal{L}^N_t \phi(\boldsymbol{x}) = \sum_{k=1}^{N}\sum_{j\neq i} \beta^k_j(x_k, \boldsymbol{x}^{-k})[\phi(j, \boldsymbol{x}^{-k}) - \phi(\boldsymbol{x})]$$

Value function

Value function $v^N(t, \boldsymbol{x})$

- HJB equation is ODE, indexed by $\boldsymbol{x} \in [\![d]\!]^N$;
- Well-posedness of HJB;
- Existence and uniqueness of optimal startegy $\boldsymbol{\beta}$.
- Possible to consider also open-loop controls and non-convex ℓ: multiple optimizers, non exchangeable.

Value function

Value function $v^N(t, \boldsymbol{x})$

- HJB equation is ODE, indexed by $\boldsymbol{x} \in [\![d]\!]^N$;
- Well-posedness of HJB;
- Existence and uniqueness of optimal startegy $\boldsymbol{\beta}$.
- Possible to consider also open-loop controls and non-convex ℓ: multiple optimizers, non exchangeable.

Mean-field assumption: There exists $\alpha^N : [0, T] \times S_d^N \to [0, +\infty)^{d \times d}$ such that

$$\beta_j^k(t, x_k, \boldsymbol{x}^{-k}) = \alpha^N_{x_k, j}(t, \mu^N_{\boldsymbol{x}})$$

$S_d = \{m \in \mathbb{R}^d : m_i \geq 0, \quad \sum_{i=1}^d m_i = 1\}$, $S_d^N = S_d \cap \frac{1}{N}\mathbb{Z}^d$

Value function

Value function $v^N(t, \boldsymbol{x})$

- HJB equation is ODE, indexed by $\boldsymbol{x} \in [\![d]\!]^N$;
- Well-posedness of HJB;
- Existence and uniqueness of optimal startegy $\boldsymbol{\beta}$.
- Possible to consider also open-loop controls and non-convex ℓ: multiple optimizers, non exchangeable.

Mean-field assumption: There exists $\alpha^N : [0, T] \times S_d^N \to [0, +\infty)^{d \times d}$ such that

$$\beta_j^k(t, x_k, \boldsymbol{x}^{-k}) = \alpha^N_{x_k, j}(t, \mu^N_{\boldsymbol{x}})$$

$S_d = \{m \in \mathbb{R}^d : m_i \geq 0, \quad \sum_{i=1}^d m_i = 1\}$, $S_d^N = S_d \cap \frac{1}{N}\mathbb{Z}^d$

Assume $\alpha_{i,j} \in [0, M]$ for any N.

Mean field N-agent problem

The cost becomes

$$J_N(\alpha^N) = \frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\left[\sum_{i=1}^{d}\mathbb{1}_{\{X_T^k=i\}}g^i(\mu_T^N)\right.$$

$$\left.+\int_0^T\sum_{i=1}^{d}\mathbb{1}_{\{X_t^k=i\}}\big(\ell(i,\alpha^N(t,i,\mu_t^N))+f(i,\mu_t^N)\big)\,dt\right]$$

$$=\mathbb{E}\left[\int_0^T\sum_{i=1}^{d}\mu_{i,t}^N\big(\ell^i(\alpha^N(t,i,\mu_t^N))+f^i(\mu_t^N)\big)\,dt+\sum_{i=1}^{d}\mu_{i,T}^N g^i(\mu_T^N)\right]$$

Mean field N-agent problem

The cost becomes

$$J_N(\alpha^N) = \frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\left[\sum_{i=1}^{d}\mathbb{1}_{\{X_T^k=i\}}g^i(\mu_T^N)\right.$$
$$\left.+\int_0^T\sum_{i=1}^{d}\mathbb{1}_{\{X_t^k=i\}}\left(\ell(i,\alpha^N(t,i,\mu_t^N))+f(i,\mu_t^N)\right)dt\right]$$
$$=\mathbb{E}\left[\int_0^T\sum_{i=1}^{d}\mu_{i,t}^N\left(\ell^i(\alpha^N(t,i,\mu_t^N))+f^i(\mu_t^N)\right)dt+\sum_{i=1}^{d}\mu_{i,T}^N g^i(\mu_T^N)\right]$$

Optimal control problem for Markov chain $\mu_t^N \in S_d^N$

$$\mathbb{P}\left(\mu_{t+h}^N = m + \frac{1}{N}(e_j - e_i)\,\middle|\, \mu_t^N = m\right) = N m_i \alpha_{i,j}^N(t,m) h + o(h)$$

HJB-N equation

$V^N(t,m)$ value function, $m \in S_d^N$, HJB equation is ODE

$$
\begin{cases}
-\frac{d}{dt} V^N + \sum_{i \in [\![d]\!]} m_i H(D^{N,i} V^N(t,m)) = \sum_{i \in [\![d]\!]} m_i f^i(m) \\
V^N(T,m) = \sum_{i \in [\![d]\!]} m_i g^i(m),
\end{cases}
\tag{HJB-N}
$$

where $[D^{N,i} V^N(t,m)]_j := N\left[V^N\left(m + \frac{1}{N}(e_j - e_i)\right) - V^N(m)\right]$,

$$
H^i(z) = \sum_{j \neq i} \{-a^*(-z_j) z_j - \tfrac{1}{2} |a^*(-z_j)|^2\}, \; a^*(r) = \begin{cases} 0 & r \leq 0 \\ r & 0 \leq r \leq M \\ M & r \geq M \end{cases}
$$

HJB-N equation

$V^N(t,m)$ value function, $m \in S_d^N$, HJB equation is ODE

$$
\begin{aligned}
&-\frac{d}{dt}V^N + \sum_{i\in[\![d]\!]} m_i H(D^{N,i}V^N(t,m)) = \sum_{i\in[\![d]\!]} m_i f^i(m) \\
&V^N(T,m) = \sum_{i\in[\![d]\!]} m_i g^i(m),
\end{aligned}
\qquad \text{(HJB-N)}
$$

where $[D^{N,i}V^N(t,m)]_j := N\big[V^N(m+\frac{1}{N}(e_j-e_i)) - V^N(m)\big]$,

$$
H^i(z) = \textstyle\sum_{j\neq i}\{-a^*(-z_j)z_j - \frac{1}{2}|a^*(-z_j)|^2\}, \; a^*(r) = \begin{cases} 0 & r \le 0 \\ r & 0 \le r \le M \\ M & r \ge M \end{cases}
$$

- HJB well-posed, unique optimal control

$$
\alpha^N_{i,j}(t,m) = a^*\Big(-N\Big[V^N(m+\frac{1}{N}(e_j-e_i)) - V^N(m)\Big]\Big)
$$

- Result: $\min_{\beta} J_N(\beta) = \min_{\alpha_N} J_N(\alpha_N)$ and $v^N(t,\boldsymbol{x}) = V^N(t,\mu^N_{\boldsymbol{x}})$

Mean field control problem

$N \to \infty$: $X^1, X^2, \dots$ i.i.d., $\mu^N \to \mu = \mathcal{L}(X)$, $\lim_N V^N$??

One reference player X chooses its transition rate $\alpha = (\alpha_{i,j})_{i,j=1}^d$, $\alpha_{i,j}(t) \in [0, M]$ deterministic (in feedback form)

$$\mathbb{P}(X_{t+h} = j | X_t = i) = \alpha_{i,j}(t) h + o(h) \qquad j \neq i$$

in order to minimize, $\mathcal{L}(X_t) = \mathbb{P} \circ X_t^{-1} \in S_d$,

$$J(\alpha) = \mathbb{E}\left[\int_0^T \frac{1}{2} \sum_{j \neq X_t} |\alpha_{X_t,j}(t)|^2 + f(X_t, \mathcal{L}(X_t)) dt + g(X_T, \mathcal{L}(X_T)) \right]$$

Mean field control problem

$N \to \infty$: $X^1, X^2, \dots$ i.i.d., $\mu^N \to \mu = \mathcal{L}(X)$, $\lim_N V^N$??

One reference player X chooses its transition rate $\alpha = (\alpha_{i,j})_{i,j=1}^d$, $\alpha_{i,j}(t) \in [0, M]$ deterministic (in feedback form)

$$\mathbb{P}(X_{t+h} = j | X_t = i) = \alpha_{i,j}(t) h + o(h) \qquad j \neq i$$

in order to minimize, $\mathcal{L}(X_t) = \mathbb{P} \circ X_t^{-1} \in S_d$,

$$J(\alpha) = \mathbb{E}\left[\int_0^T \frac{1}{2} \sum_{j \neq X_t} |\alpha_{X_t,j}(t)|^2 + f(X_t, \mathcal{L}(X_t)) dt + g(X_T, \mathcal{L}(X_T)) \right]$$

Rewrite in the form, $\mu_t^i = \mathbb{P}(X_t = i)$, $\mu_t = \mathcal{L}(X_t)$

$$\dot{\mu}_t^i = \sum_{j \neq i} \left(\mu_t^j \alpha_{j,i}(t) - \mu_t^i \alpha_{i,j}(t) \right)$$

$$J(\alpha) = \int_0^T \sum_{i=1}^d \mu_t^i \left(\frac{1}{2} \sum_{j \neq i} |\alpha_{i,j}(t)|^2 + f^i(\mu_t) \right) dt + \sum_{i=1}^d \mu_T^i g^i(\mu_T)$$

HJB equation

Single deterministic optimal control problem

HJB equation for value function $V(t, m)$:

$$
\begin{aligned}
&-\partial_t V + \sum_{i \in [\![d]\!]} m_i H\Big(\{(\partial_{m_j} - \partial_{m_i})V\}_{j=1}^{d} \Big) = \sum_{i \in [\![d]\!]} m_i f^i(m) \\
&V(T, m) = \sum_{i \in [\![d]\!]} m_i g^i(m),
\end{aligned}
\tag{HJB}
$$

HJB equation

Single deterministic optimal control problem

HJB equation for value function $V(t,m)$:

$$
\begin{aligned}
& -\partial_t V + \sum_{i\in[\![d]\!]} m_i H\Big(\{(\partial_{m_j} - \partial_{m_i})V\}_{j=1}^{d} \Big) = \sum_{i\in[\![d]\!]} m_i f^i(m) \\
& V(T,m) = \sum_{i\in[\![d]\!]} m_i g^i(m),
\end{aligned}
\qquad \text{(HJB)}
$$

- First order PDE in $[0,T]\times S_d$, no boundary conditions;
- $N\big[V^N(m+\frac{1}{N}(e_j-e_i)) - V^N(m)\big] \to (\partial_{m_j}-\partial_{m_i})V(m)$;
- no classical solutions;
- existence of optimal controls, non-uniqueness;
- Potential mean field game: $f^i(m) = F(m)$.

Viscosity solution

- Value function V is the unique viscosity solution of (HJB) in $[0, T) \times S_d$.
- Test functions in $\mathcal{C}^1([0, T) \times S_d)$.
- V is Lipschitz-continuous.

Viscosity solution

- Value function V is the unique viscosity solution of (HJB) in $[0, T) \times S_d$.
- Test functions in $\mathcal{C}^1([0, T) \times S_d)$.
- V is Lipschitz-continuous.

- $\mu_t \in \mathrm{Int}(S_d)$ if $\mu_0 \in \mathrm{Int}(S_d)$.
- gives property on $D_z H$.

Theorem

Value function V is the unique viscosity solution of (HJB) in $[0, T) \times \mathrm{Int}(S_d)$.

Viscosity solution

- Value function V is the unique viscosity solution of (HJB) in $[0, T) \times S_d$.
- Test functions in $\mathcal{C}^1([0, T) \times S_d)$.
- V is Lipschitz-continuous.

- $\mu_t \in \mathrm{Int}(S_d)$ if $\mu_0 \in \mathrm{Int}(S_d)$.
- gives property on $D_z H$.

Theorem

Value function V is the unique viscosity solution of (HJB) in $[0, T) \times \mathrm{Int}(S_d)$.

- Analogous results for continuous state space: [Pham-Wei '18], [Wu-Zhang '19].

Previous results

Aim: $V^N \to V$ with convergence rate.

Previous results

Aim: $V^N \to V$ with convergence rate.

Results for continuous state space:

- [Lacker '17], [Djete-Possamai-Tan '20] and [Fornasier-Lisini -Orrieri -Savaré '19]: convergence of optimal controls via compactness arguments (common noise, deterministic, ...);
- [Carmona-Delarue '15]: convergence rate, using convexity of f and g in (x, m).

Previous results

Aim: $V^N \to V$ with convergence rate.

Results for continuous state space:

- [Lacker '17], [Djete-Possamai-Tan '20] and [Fornasier-Lisini -Orrieri -Savaré '19]: convergence of optimal controls via compactness arguments (common noise, deterministic, ...);
- [Carmona-Delarue '15]: convergence rate, using convexity of f and g in (x, m).

Prove convergence using viscosity solution characterization of V:

- (HJB-N) is finite difference scheme for (HJB).
- results on approximation scheme for viscosity solutions in
  - [Capuzzo Dolcetta-Ishii '84]: discounted control problem;
  - [Souganidis '85]: general HJ equation.

Main convergence result

Theorem

Assume $m \to \sum_{i \in [\![d]\!]} m_i f^i(m)$ and $m \to \sum_{i \in [\![d]\!]} m_i g^i(m)$ Lipschitz.

$$\sup_{t \in [0,T], m \in S_d^N} |V^N(t,m) - V(t,m)| \le \frac{C}{\sqrt{N}}$$

- Use Lipschitz-continuity of V and V^N, uniformly in N.

Main convergence result

Theorem

Assume $m \to \sum_{i\in[\![d]\!]} m_i f^i(m)$ and $m \to \sum_{i\in[\![d]\!]} m_i g^i(m)$ Lipschitz.

$$\sup_{t\in[0,T], m\in S_d^N} |V^N(t,m) - V(t,m)| \le \frac{C}{\sqrt{N}}$$

- Use Lipschitz-continuity of V and V^N, uniformly in N.
- Valid also for cost $c(t,x,\alpha,m)$ not convex in α and transition rate $Q_{i,j}(t,\alpha,\mu)$: in this case, non-uniqueness of N-optimal control and non-existence of limiting optimal control.

Approximation

Corollary

Let α be an optimal control for mean field control problem. Then α (not depending on m) is quasi-optimal for N-agent optimization:

$$J^N(\alpha) \leq \inf_{\alpha^N} J^N(\alpha^N) + \frac{C}{\sqrt{N}}$$

Approximation

Corollary

Let α be an optimal control for mean field control problem. Then α (not depending on m) is quasi-optimal for N-agent optimization:

$$J^N(\alpha) \leq \inf_{\alpha^N} J^N(\alpha^N) + \frac{C}{\sqrt{N}}$$

- Apply standard arguments in propagation of chaos to estimate

$$|J^N(\alpha) - J(\alpha)| \leq \frac{C}{\sqrt{N}}, \qquad \sup_{t\in[0,T]} \mathbb{E}|\mu_t^N(\alpha) - \mu(\alpha)| \leq \frac{C}{\sqrt{N}}$$

- Assume initial conditions X_0^k, $k = 1, \dots, N$, i.i.d.

Approximation

Corollary

Let α be an optimal control for mean field control problem. Then α (not depending on m) is quasi-optimal for N-agent optimization:

$$J^N(\alpha) \leq \inf_{\alpha^N} J^N(\alpha^N) + \frac{C}{\sqrt{N}}$$

- Apply standard arguments in propagation of chaos to estimate

$$|J^N(\alpha) - J(\alpha)| \leq \frac{C}{\sqrt{N}}, \qquad \sup_{t\in[0,T]} \mathbb{E}|\mu^N_t(\alpha) - \mu(\alpha)| \leq \frac{C}{\sqrt{N}}$$

- Assume initial conditions X_0^k, $k = 1, \ldots, N$, i.i.d.

Results for continuous state space:

- [Lacker '17], [Djete-Possamai-Tan '20]: no convergence rate
- [Carmona-Delarue '15]: rate, requires convexity of f and g.

Convexity

- Uniqueness of optimal control of MFCP ?
- Existence of classical solution to (HJB)?
- Convergence of optimal control and optimal trajectories?

Convexity

- Uniqueness of optimal control of MFCP ?
- Existence of classical solution to (HJB)?
- Convergence of optimal control and optimal trajectories?

Assume that the functions

$$S_d \ni m \to \sum_{i \in [\![d]\!]} m_i f^i(m), \qquad S_d \ni m \to \sum_{i \in [\![d]\!]} m_i g^i(m).$$

are convex and in $\mathcal{C}^{1,1}(S_d)$.

Classical solution

Theorem

Assume convexity. Then the value function $V \in \mathcal{C}^{1,1}([0,T] \times S_d)$ *and is a classical solution to* (HJB).
Unique optimal control, in feedback form,

$$\alpha_{i,j}(t,m) = a^*\Big((\partial_{m_i} - \partial_{m_j})V(t,m)\Big)$$

Classical solution

Theorem

Assume convexity. Then the value function $V \in \mathcal{C}^{1,1}([0,T] \times S_d)$ *and is a classical solution to* (HJB).
Unique optimal control, in feedback form,

$$\alpha_{i,j}(t,m) = a^*\Big((\partial_{m_i} - \partial_{m_j})V(t,m)\Big)$$

- Prove that V is semiconcave and semiconvex.
- Equivalent problem, set $w_{i,j} = \mu^i \alpha_{i,j}$,

$$\dot{\mu}^i_t = \sum_{j \neq i} (w_{j,i}(t) - w_{i,j}(t))$$

$$J(\alpha) = \int_0^T \sum_{i=1}^d \left(\frac{1}{\mu^i_t} \sum_{j \neq i} \frac{|w_{i,j}(t)|^2}{2} + \mu^i_t f^i(\mu_t) \right) dt + \sum_{i=1}^d \mu^i_T g^i(\mu_T)$$

Convergence

Convergence of optimal (μ^N, α^N) to (μ, α)?

Rely on regularity of V. Approximate

$$N\Big[V\Big(m + \frac{1}{N}(e_j - e_i)\Big) - V(m)\Big] = (\partial_{m_j} - \partial_{m_i})V(t,m) + \mathcal{O}\Big(\frac{1}{N}\Big)$$

Convergence

Convergence of optimal (μ^N, α^N) to (μ, α)?

Rely on regularity of V. Approximate

$$N\Big[V\Big(m + \frac{1}{N}(e_j - e_i)\Big) - V(m)\Big] = (\partial_{m_j} - \partial_{m_i})V(t,m) + \mathcal{O}\Big(\frac{1}{N}\Big)$$

Analogous convergence results for MFG:

- [Cardaliaguet-Delarue-Lasry-Lions '19]: continuous state space.
- [Bayraktar-Cohen '18] and [Cecchin-Pelino]: finite state space.

Convergence of optimal trajectories

μ^N empirical measure using optimal N-control $\alpha^N(t,m)$ given by V^N
ρ^N empirical measure using limiting control $\alpha(t,m)$, given by V
μ unique optimal trajectory of MFCP, optimal control $\alpha(t,m)$

Theorem

Assume $V \in \mathcal{C}^{1,1}([0,T] \times S_d)$.

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\mu_t^N - \rho_t^N|\right] \le \frac{C}{N^{1/4}},$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\mu_t^N - \mu_t|\right] \le \frac{C}{N^{1/9}}.$$

Conclusion and perspectives

We obtained

1. Convergence of V^N to V with convergence rate, via viscosity solutions, under general assumptions.
2. Convergence of optimal trajectories μ^N to μ if V is smooth, e.g. under convexity assumptions.

For future work:

- Analogous results for continuous state space.

Conclusion and perspectives

We obtained

1. Convergence of V^N to V with convergence rate, via viscosity solutions, under general assumptions.
2. Convergence of optimal trajectories μ^N to μ if V is smooth, e.g. under convexity assumptions.

For future work:

- Analogous results for continuous state space.

THANK YOU FOR YOUR ATTENTION