# A viability approach to first-order mean field games

Yurii Averboukh

Krasovskii Institute of Mathematics and Mechanics
ayv@imm.uran.ru

18 June, 2020

# Overview of the talk

- Object: deterministic mean field game examined within the probabilistic approach.
- Probabilistic approach consider the MFG as a symmetric Nash equilibrium in the infinite player differential game.
- Aim: study the dependence of the solution of MFG on the initial distribution of players.

# Related works

- Probabilistic approach:
  [Carmona, Delarue, 2015], [Carmona, Delarue, 2018], [Lacker, 2015], [A., 2015].
- Dependence of the solution of MFG on initial distribution of players was considered by means of the master equation: [Cardaliaguet, Delarue, Lasry, Lions, 2015/2019], [Carmona, Delarue, 2018]. The key ingridient of the master equation is the function $\varphi(t_0, x_0, m_0)$.

# State spaces and outcomes

| | state of the system | expected outcome |
|---|---|---|
| control problem | vector in $\mathbb{R}^d$ | real number |
| $N$ player game with weakly coupled dynamics | vector in $(\mathbb{R}^d)^N$ | vector in $\mathbb{R}^N$ |
| infinite player game | vector in $(\mathbb{R}^d)^{\mathfrak{c}}$ | vector in $\mathbb{R}^{\mathfrak{c}}$ |
| mean field game | probability on $\mathbb{R}^d$ | continuous function from $\mathbb{R}^d$ to $\mathbb{R}$ |

# Deterministic mean field game

Find a symmetric Nash equilibrium in the infinite-player game when

- the dynamics of each player is given by

$$\frac{d}{dt}x(t) = f(t, x(t), m(t), u(t));$$

- the reward of each player is equal to

$$\sigma(x(T), m(T)) + \int_{t_0}^{T} g(t, x(t), m(t), u(t))dt;$$

- the initial distribution of players is $m_0$.

Here

- $m(t)$ is a distribution of players at time $t$; $m(t)$ is a probability on the phase space;
- $u(t) \in U$ is a control; $U$ is a control space.

# Phase space

$$\mathbb{T}^d \triangleq \mathbb{R}^d/\mathbb{Z}^d.$$

- Elements of $\mathbb{T}^d$ are the sets $x = \{x' + n : n \in \mathbb{Z}^d\}$.
- Distance on $\mathbb{T}^d$: if $x, y \in \mathbb{T}^d$, then

$$\|x - y\| \triangleq \min\{\|x' - y'\| : x' \in x, y' \in y\}.$$

# Space of trajectories

- $\mathcal{C}_{s,r} \triangleq C([s,r], \mathbb{T}^d \times \mathbb{R})$.
- If $t \in [s,r]$, $w(\cdot) = (x(\cdot), z(\cdot)) \in \mathcal{C}_{s,r}$, then

$$e_t(w(\cdot)) \triangleq x(t),$$

$$\hat{e}_t(w(t)) \triangleq w(t).$$

# Assumptions

- $U$ is compact is a metric space;
- $f$, $g$, $\sigma$ are continuous;
- $f$ and $g$ are Lipschitz continuous w.r.t. to $x$ and $m$;
- $\sigma$ is Lipschitz continuous w.r.t. to $x$.

# Mean field game

Find a symmetric Nash equilibrium in the infinite-player game when

- the dynamics of each player is given by

$$\frac{d}{dt}x(t) = f(t, x(t), m(t), u(t));$$

- the reward of each player is equal to

$$\sigma(x(T), m(T)) + \int_{t_0}^{T} g(t, x(t), m(t), u(t))dt;$$

- the initial distribution of player is $m_0$.

Here $m(t)$ is a distribution of players at time $t$. For each $t$, $m(t)$ is a probability on $\mathbb{T}^d$.

# Relaxed problem

Each player tries to maximize

$$\sigma(x(T), m(T)) + z(T) - z(t_0)$$

subject to

$$(\dot{x}(t), \dot{z}(t)) \in F(t, x(t), m(t)),$$

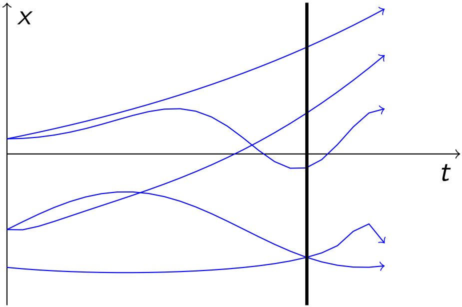
where

$$F(t, x, m) \triangleq \mathrm{co}\big\{(f(t, x, m, u), g(t, x, m, u)) : u \in U\big\}$$

# Distribution of trajectories

Let $\chi \in \mathcal{P}^1(\mathcal{C}_{t_0,T})$ be a distribution of curves in the extended phase space.

- $m(t) \triangleq e_t \sharp \chi \in \mathcal{P}^1(\mathbb{T}^d)$ is a distribution on the phase space;
- $\nu(t) \triangleq \hat{e}_t \sharp \chi \in \mathcal{P}^1(\mathbb{T}^d \times \mathbb{R})$ is a distribution on the extended phase space.



# Distribution of players' trajectories

$\chi \in \mathcal{P}^1(\mathcal{C}_{t_0,T})$ is the distribution of players' trajectories if,

- $\chi$ is concentrated on the set of absolutely continuous curves;
- $\chi$-a.e. $(x(\cdot), z(\cdot)) \in \mathcal{C}_{t_0,T}$ satisfy the differential inclusion

$$(\dot{x}(t), \dot{z}(t)) \in F(t, x(t), m(t)),$$

  where $m(t) \triangleq e_t \sharp \chi$ is a distribution on the phase space;
- $m(t_0) = e_{t_0} \sharp \chi = m_0$.

# Equilibrium distribution of trajectories

We say that $\chi \in \mathcal{P}^1(\mathcal{C}_{t_0,T})$ is an equilibrium distribution of players' trajectories if,

- $\chi$ is concentrated on the set of absolutely continuous curves;
- $\chi$-a.e. $(x(\cdot), z(\cdot)) \in \mathcal{C}_{t_0,T}$ satisfy the differential inclusion

$$(\dot{x}(t), \dot{z}(t)) \in F(t, x(t), m(t)),$$

  where $m(t) \triangleq e_t \sharp \chi$;
- $m(t_0) = e_{t_0} \sharp \chi = m_0$;
- for $\chi$-a.e. $(x(\cdot), z(\cdot)) \in \mathcal{C}_{t_0,T}$ and any $(x_1(\cdot), z_1(\cdot)) \in \mathcal{C}_{t_0,T}$ satisfying $(\dot{x}_1(t), \dot{z}_1(t)) \in F(t, x_1(t), m(t))$ and $x(t_0) = x_1(t_0)$,

$$\begin{aligned}\sigma(x(T), &m(T)) + z(T) - z(t_0)\\ &\geq \sigma(x_1(T), m(T)) + z_1(T) - z_1(t_0).\end{aligned}$$

# Solution of the mean field game

Let $\chi$ be an equilibrium distribution of players' trajectories.
Set

- $m(t) \triangleq e_t \sharp \chi$;
-

$$V(s,y) \triangleq \sup\Big\{ \sigma(x(T)) + z(T) - z(s) : \\ (x(\cdot), z(\cdot)) \text{ satisfying} \\ (\dot{x}(\cdot), \dot{z}(\cdot)) \in F(t, x(t), m(t)), \quad x(s) = y \Big\}$$

$(V, m(\cdot))$ is a solution of the mean field game.

# Theorem. Existence

There exists at least one solution of the mean filed game.

Proof is by fixed point arguments.

# Purpose

Study the dependence $m_0 \mapsto V(t_0, \cdot)$.

# Dependence on initial distribution

The mapping which assign to $m_0$ the reward of the representative player $V(t_0, \cdot)$, where $V$ is a value function of the mean field game, is multivalued!

# Designations

Let $\mathsf{p} : \mathbb{T}^d \times \mathbb{R} \to \mathbb{T}^d$ be defined by the rule: for $(x, z)$

$$\mathsf{p}(x, z) \triangleq x.$$

If $\phi \in C(\mathbb{T}^d)$, $\nu \in \mathcal{P}^1(\mathbb{T}^d \times \mathbb{R})$, then

$$[\phi, \nu] \triangleq \int_{\mathbb{T}^d \times \mathbb{R}} (\phi(x) + z)\nu(d(x, z)).$$

# Mean field type differential inclusion (MFDI)

$$\frac{d}{dt}\nu(t) \in \left\langle \widehat{F}(t, \cdot, \nu(t)), \nabla \right\rangle.$$

Here, for $t \in [0, T]$, $w = (x, z) \in \mathbb{T}^d \times \mathbb{R}$, $\nu \in \mathcal{P}^1(\mathbb{T}^d \times \mathbb{R})$,

$$\widehat{F}(t, w, \nu) \triangleq F(t, \mathrm{p}(w), \mathrm{p}\sharp\nu) = F(t, x, \mathrm{p}\sharp\nu).$$

# Mean field type differential inclusion

Let $s, r \in [0, T]$, $s < r$.
We say that $[s, r] \ni t \mapsto \nu(t) \in \mathcal{P}^1(\mathbb{T}^d)$ solves MFDI on $[s, r]$ if there exists $\chi \in \mathcal{P}^1(\mathcal{C}_{s,r})$ such that

- $\nu(t) = \hat{e}_t \sharp \chi$ i.e. for any measurable $E \subset \mathbb{T}^d \times \mathbb{R}$, $\nu(t, E) = \chi\{w(\cdot) : w(t) \in E\}$;
- $\chi$ is concentrated on the set of absolutely continuous curves;
- $\chi$-a.e. $(x(\cdot), z(\cdot))$ satisfy the differential inclusion

$$(\dot{x}(t), \dot{z}(t)) \in F(t, x(t), m(t)),$$

where $m(t) = \mathrm{p} \sharp \nu(t) = e_t \sharp \chi$.

# Bellman propagator

Let $[s,r] \ni t \mapsto m(t) \in \mathcal{P}^1(\mathbb{T}^d)$ be given. For $\psi \in C(\mathbb{T}^d)$,

$$(B^{s,r}_{m(\cdot)}\psi)(y) \triangleq \sup\Big\{\psi(x(r)) + z(r) - z(s) : \\ (x(\cdot), z(\cdot)) \text{ satisfying} \\ (\dot{x}(t), \dot{z}(t)) \in F(t, x(t), m(t)), \quad x(s) = y\Big\}$$

- $B^{s,r}_{m(\cdot)} : C(\mathbb{T}^d) \to C(\mathbb{T}^d)$;
- $B^{s,s}_{m(\cdot)} = \mathrm{Id}$;
- $B^{s,\theta}_{m(\cdot)} B^{\theta,r}_{m(\cdot)} = B^{s,r}_{m(\cdot)}$.

# Proposition. Equivalent definition of solution of the mean field game

A pair $(V, m(\cdot))$ is a solution to mean field game if and only if there exists a solution of MFDI on $[t_0, T]$ $\nu(\cdot)$ such that, for any $s \in [t_0, T]$,

1. $m(s) = \mathrm{p}\sharp\nu(s)$, $s \in [t_0, T]$;
2. $m(t_0) = m_0$;
3. $V(s, \cdot) = B^{s,T}_{m(\cdot)}\sigma(\cdot, m(T))$;
4. $[\sigma(\cdot, m(T)), \nu(T)] \geq [V(s, \cdot), \nu(s)]$.

# Value multifunction

A multifunction $\mathcal{V} : [0, T] \times \mathcal{P}^1(\mathbb{T}^d) \rightrightarrows C(\mathbb{T}^d)$ is said to be a value multifunction of the mean field game if, for any $t_0 \in [0, T]$, $m_0 \in \mathcal{P}^1(\mathbb{T}^d)$, and $\phi \in \mathcal{V}(t_0, m_0)$, there exists a solution to the mean field game such that

$$V(t_0, \cdot) = \phi(\cdot), \quad m(t_0) = m_0.$$

# Properties of value multifunctions

- If $\mathcal{V}_\alpha$, $\alpha \in \mathcal{A}$ are value multifunctions, then
$$\mathcal{V} \triangleq \mathrm{cl}\left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{V}_\alpha\right)$$
is also a value multifunction.
- There exists a maximal value multifunction.

# Mean field game dynamics

The multifunction $\Psi^{r,s} : \mathcal{P}^1(\mathbb{T}^d) \times C(\mathbb{T}^d) \rightrightarrows \mathcal{P}^1(\mathbb{T}^d) \times C(\mathbb{T}^d)$ is defined by the rule: $(\mu, \psi) \in \Psi^{r,s}(m, \phi)$ if and only if there exists a solution of the MFDI on $[s, r]$ $\nu(\cdot)$ satisfying the following properties, for $m(t) = \mathrm{p}\sharp\nu(t)$:

($\Psi$1) $m(s) = m$, $m(r) = \mu$;

($\Psi$2) $\phi = B^{s,r}_{m(\cdot)}\psi$;

($\Psi$3) $[\psi, \nu(r)] \geq [\phi, \nu(s)]$.

# Semigroup property

For any $s_0, s_1, s_2 \in [0, T]$, $s < r < \theta$,

$$\Psi^{s_2,s_0} = \Psi^{s_2,s_1} \circ \Psi^{s_1,s_0}.$$

Here, given multivalued mappings $\Phi_1, \Phi_2 : X \rightrightarrows X$,

$$(\Phi_2 \circ \Phi_1)(x) \triangleq \bigcup_{y \in \Phi_1(x)} \Phi_2(y).$$

# Viability w.r.t. to mean field dynamics

We say that a upper semicontinuous multifunction $\mathcal{V} : [0, T] \times \mathcal{P}^1(\mathbb{T}^d) \rightrightarrows C(\mathbb{T}^d)$ is viable with respect to the mean field game dynamics if, for any $s, r \in [0, T]$, $s \leq r$, $m \in \mathcal{P}^1(\mathbb{T}^d)$, $\phi \in \mathcal{V}(s, m)$, there exist $\mu \in \mathcal{P}^1(\mathbb{T}^d)$ and $\psi \in C(\mathbb{T}^d)$ such that

- $(\mu, \psi) \in \Psi^{r,s}(m, \phi)$;
- $\psi \in \mathcal{V}(r, \mu)$.

# Property

The maximal value function is viable with respect to the mean field game dynamics.

# Theorem. Viability and MFG

Assume that a upper semicontinuous multifunction $\mathcal{V} : [0, T] \times \mathcal{P}^1(\mathbb{T}^d) \rightrightarrows C(\mathbb{T}^d)$ is viable with respect to the mean field game dynamics and $\mathcal{V}(T, m) = \{\sigma(\cdot, m)\}$. Then $\mathcal{V}$ is a value multifunction.

Proof is by the discretization of the time interval $[t_0, T]$.

# Probabilities on the tangent space

Let $m \in \mathcal{P}^1(\mathbb{T}^d)$ and let $c > 0$.

Denote by $\mathcal{L}^c(m)$ the set of probabilities $\beta \in \mathcal{P}^1(\mathbb{T}^d \times \mathbb{R}^{d+1})$ such that

- the marginal distribution of $\beta$ on $\mathbb{T}^d$ is $m$ i.e., for any measurable $E \in \mathbb{T}^d$, $\beta(E \times \mathbb{R}^{d+1}) = m(E)$;
- $\mathrm{supp}\beta \subset \mathbb{T}^d \times \mathbb{B}_c \times [-c, c]$, where $\mathbb{B}_c$ stands for the ball of radius $c$ centered in the origin.

# Shift operator

For $t \geq 0$, let the operator $\Theta^\tau : \mathbb{T}^d \times \mathbb{R}^{d+1} \to \mathbb{T}^d \times \mathbb{R}$ be given by

$$\Theta^\tau(x, a, b) \triangleq (x + \tau a, \tau b).$$

If $m$ is a probability on $\mathbb{T}^d$, $\beta \in \mathcal{L}^c(m)$ is a probability on the tangent space, $\tau \geq 0$, then

$$\nu \triangleq \Theta^\tau \sharp \beta \in \mathcal{P}^1(\mathbb{T}^d \times \mathbb{R})$$

is a shift of the probability $m$ in the direction given by $\beta$.

# 'Frozen' Bellman propagator

If $m \in \mathcal{P}^1(\mathbb{T}^d)$, $s, r \in [0, T]$, $s \leq r$, then

$$(A_m^{s,r}\phi)(x) \triangleq \sup\Big\{\phi(x + (r-s)a) + (r - s)b : (a, b) \in F(s, x, m)\Big\}.$$

$A_m^{s,r} : C(\mathbb{T}^d) \to C(\mathbb{T}^d)$.

# Set-valued derivative of the multifunction

Let $\mathcal{V} : [0, T] \times \mathcal{P}^1(\mathbb{T}^d) \rightrightarrows C(\mathbb{T}^d)$ be upper semicontinuous, $t \in [0, T]$, $m \in \mathcal{P}^1(\mathbb{T}^d)$, $\phi \in C(\mathbb{T}^d)$.

Recall that, for $\phi \in C(\mathbb{T}^d)$, $\nu \in \mathcal{P}^1(\mathbb{T}^d \times \mathbb{R})$,

$$[\phi, \nu] \triangleq \int_{\mathbb{T}^d \times \mathbb{R}} (\phi(x) + z)\nu(dxdz).$$

If $m \in \mathcal{P}^1(\mathbb{T}^d)$, $\widehat{m} \in \mathcal{P}^1(\mathbb{T}^d \times \mathbb{R})$ is defined by the rule: for $\varphi \in C(\mathbb{T}^d \times \mathbb{R})$,

$$\int_{\mathbb{T}^d \times \mathbb{R}} \varphi(x, z)\widehat{m}(d(x, z)) \triangleq \int_{\mathbb{T}^d} \varphi(x, 0)m(dx).$$

# Set-valued derivative of the multifunction

A probability $\beta \in \mathcal{L}^c(m)$ belongs to $\mathcal{D}_F^c\mathcal{V}(t,m,\phi)$ if, for some $\{\tau_n\}_{n=1}^\infty \subset (0,+\infty)$, $\{\beta_n\}_{n=1}^\infty \subset \mathcal{L}^c(m)$, $\{\phi_n\}_{n=1}^\infty \subset C(\mathbb{T}^d)$ and $\nu_n \triangleq \Theta^{\tau_n}\sharp\beta_n$, $m_n \triangleq \mathrm{p}\sharp\nu_n$,

1. $\tau_n, W_1(\beta,\beta_n) \to 0$ as $n \to \infty$;
2. $\phi_n \in \mathcal{V}(t+\tau_n, m_n)$;
3. $$\lim_{n\to\infty} \frac{\|A_m^{t,t+\tau_n}\phi_n - \phi\|}{\tau_n} = 0;$$
4. $$\lim_{n\to\infty} \frac{[\phi_n,\nu_n] - [\phi,\widehat{m}]}{\tau_n} \geq 0;$$
5. $$\int_{\mathbb{T}^d\times\mathbb{R}^{d+1}} \mathrm{dist}(v;F(t,x,m))\beta(d(x,v)) = 0.$$

# Bounded and Lipschitz continuous functions

For $M, C > 0$, let $\mathrm{BL}_{M,C}$ denote the set of functions $\phi$ such that

- $\|\phi\| \leq M$;
- $\phi$ is $C$-Lipschitz continuous.

# Theorem

Assume that a upper semicontinuous multifunction $\mathcal{V} : [0, T] \times \mathcal{P}^1(\mathbb{T}^d) \rightrightarrows C(\mathbb{T}^d)$ has nonempty values and there exist constants $M$ and $C$ such that, for any $t \in [0, T]$, $m \in \mathcal{P}^1(\mathbb{T}^d)$,

$$\mathcal{V}(t, m) \subset \mathrm{BL}_{M,C}(\mathbb{T}^d).$$

Then, $\mathcal{V}$ is viable with respect to the mean field game dynamics if and only if, there exists a constant $c > 0$ such that, for any $t \in [0, T]$, $m \in \mathcal{P}^1(\mathbb{T}^d)$, $\phi \in \mathcal{V}(t, m)$,

$$\mathcal{D}_F^c \mathcal{V}(t, m, \phi) \neq \varnothing.$$

# Proof of the viability theorem

The proof is analogous to the proof of Nagumo-type viability theorems. It involves the properties of ‘frozen’ dynamics in the space $\mathcal{P}^1(\mathbb{T}^d) \times C(\mathbb{T}^d)$.

# Corollary. Infinitesimal condition on the value multifunction

Let $\mathcal{V} : [0, T] \times \mathcal{P}^1(\mathbb{T}^d) \rightrightarrows C(\mathbb{T}^d)$ be upper semicontinuous.
Assume that, for any $t \in [0, T]$, $m \in \mathcal{P}^1(\mathbb{T}^d)$, $\phi \in \mathcal{V}(t, m)$,

- $\mathcal{V}(t, m) \neq \varnothing$;
- $\mathcal{V}(t, m) \subset \mathrm{BL}_{M,C}(\mathbb{T}^d)$ where the constants $M$ and $C$ do not dependent on $t$ and $m$;
- $\mathcal{V}(T, m) = \{\sigma(\cdot, m)\}$;
- $\mathcal{D}^c_F \mathcal{V}(t, m, \phi) \neq \varnothing$, where the constant $c$ does not depend on $t$, $m$ and $\phi$.

Then $\mathcal{V}$ is a value multifunction of the mean field game.

# Master equation

$$\frac{\partial \varphi}{\partial t} + H(t, x, m, \nabla_x \varphi) + \int_{\mathbb{T}^d} \frac{\partial H(t, x, m, \nabla_x \varphi)}{\partial p} \partial_m \varphi(t, x, m)(y) m(dy) = 0.$$

Here

- $\varphi$ is a function from $[0, T] \times \mathbb{T}^d \times \mathcal{P}^1(\mathbb{T}^d)$ with values in $\mathbb{R}$,
- $\partial \varphi / \partial t$, $\nabla_x \varphi$, $\nabla_m \varphi$ stand for its derivatives w.r.t to time, state and measure variable.
- $H$ is Hamiltonian

$$H(t, x, m, p) \triangleq \max_{u \in U} [\langle p, f(t, x, m, u) \rangle + g(t, x, m, u)].$$

# Master equation and value function [Carmona, Delarue, 2018]

- Let $\Gamma(t_0, x_0, m_0)$ be an expected reward of the representative player who starts at the time $t_0$, at the state $x_0$ under the condition that the distribution of all players is $m_0$.
- Under certain regularity conditions the classical solution to the master equation is equal to $\Gamma$.
- If the coefficients of the master equation are continuous, the value function $\Gamma$ is well-defined, unique and continuous with its derivative w.r.t. $x$, then it solves master equation in the viscosity sense.

# Value function and value multifunction

- If $\Gamma$ is a value function, then the multifunction $\mathcal{V}$ defined by the rule
$$\mathcal{V}(t_0, m_0) \triangleq \{\Gamma(t_0, \cdot, m_0)\}$$
is a value multifunction.
- If $\mathcal{V}$ is single-valued i.e. $\mathcal{V}(t_0, m_0) = \{\phi_{t_0,m_0}\}$, then the function $\Gamma(t_0, x_0, m_0) \triangleq \phi_{t_0,m_0}(x_0)$ is a value function.

# Future problems

- Extend the results to the case when the dynamics of each player is driven by SDE.
- Find the link between the sub- and superdifferentials of the function of probability and the set-valued derivative.

Thank you for your attention!