Control on Hilbert Spaces and Application to Mean Field Type Control Theory

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The Master Equation

Our project is to study

$$-\frac{\partial U}{\partial t}(x,m,t) + A_x U(x,m,t) + \int_{\mathbb{R}^n} A_\xi \frac{\mathrm{d}}{\mathrm{d}m} U(\xi,m,t)(x)\,\mathrm{d}m(\xi)$$
$$+ \frac{1}{2\lambda}|D_x U(x,m,t)|^2 + \frac{1}{\lambda}\int_{\mathbb{R}^n} D_\xi U(\xi,m,t)\cdot D_\xi \frac{\mathrm{d}}{\mathrm{d}m} U(\xi,m,t)(x)\,\mathrm{d}m(\xi)$$
$$= \frac{\mathrm{d}}{\mathrm{d}m} F(m)(x), \quad U(x,m,T) = \frac{\mathrm{d}}{\mathrm{d}m} F_T(m)(x)$$

where $x \in \mathbb{R}^n$, $m \in \mathcal{P}(\mathbb{R}^n)$, and $t \in [0,T]$. Here A_x is second order elliptic differential operator with usual assumptions:

$$A_x\varphi(x) = -\frac{1}{2}\mathrm{tr}\,\left(\sigma\sigma^* D^2\varphi(x)\right).$$

See

- P. L. Lions lectures in *Collège de France*
- Book by Cardaliaguet, Delarue, Lasry, Lions (Annals of Mathematics Studies)
- Two-volume set by Carmona, Delarue

Interpreting the Master Equation

First interpretation: mean field Nash equilibrium equation

- $\dfrac{\mathrm{d}}{\mathrm{d}m}F(m)$ and $\dfrac{\mathrm{d}}{\mathrm{d}m}F_T(m)$ give the mean field's contribution to individual cost (running and terminal, resp.)
- Game is *potential*

Second interpretation: $U(x, m, t)$ is the decoupling field for a mean field type control problem, which is formally

$$\inf\left\{ J_{m,t}(v) := \frac{\lambda}{2}\int_t^T \int_{\mathbb{R}^n} |v(\xi,s)|^2 \,\mathrm{d}m_s(\xi)\,\mathrm{d}s + \int_t^T F(m_s)\,\mathrm{d}s + F_T(m_T) \right| \partial_s m_s + A_x m_s + \nabla\cdot(v m_s) = 0,\ m_t = m \right\}.$$

- Euler-Lagrange equations formally give HJ-FP system typical of MFG

Bellman equation

For the mean field type control problem, the value function satisfies

$$-\frac{\partial V}{\partial t}(m,t) + \int_{\mathbb{R}^n} A_x \frac{\mathrm{d}}{\mathrm{d}m} V(m,t)(x)\,\mathrm{d}m(x) + \frac{1}{2\lambda}\int_{\mathbb{R}^n}\left|D\frac{\mathrm{d}}{\mathrm{d}m}V(m,t)(x)\right|^2 \mathrm{d}m(x) = F(m), \quad V(m,T) = F_T(m).$$

We can differentiate with respect to m to get the master equation.

- See Bensoussan, Frehse, Yam "Interpretation of the Master Equation..."
- See also Pham, Wei 2018 for viscosity solutions theory of Bellman equations on Wasserstein space

Our goal

Prove existence of classical solutions to Bellman and Master equations using only optimal control techniques.

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Lifting the problem

Main idea

"Lift" the mean field type control problem from the space of probability measures to a Hilbert space.

Motivation:

- Lions proposed lifting from measures to random variables as a way of defining derivatives on Wasserstein space
- L^2 random variables form a Hilbert space
- Classical optimal control works well on a Hilbert space
- Lifting the problem gives reasonable results in the first-order case: Bensoussan Yam 2018
- A previous preprint uses exactly this idea

New contribution of the present work

(i) Our version of "lifting" is radically different from that of Lions.
(ii) The results are not new; however, the method of proof is completely different.

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Wasserstein space

We denote by $\mathcal{P}_2(\mathbb{R}^n)$ the Wasserstein space of Borel probability measures m on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} |x|^2 \, \mathrm{d}m(x) < \infty$, endowed with the metric

$$W_2(\mu, \nu) = \sqrt{\inf \left\{ \int |x - y|^2 \, \mathrm{d}\pi(x, y) : \pi \in \Pi(\mu, \nu) \right\}}. \tag{1}$$

Consider an atomless probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and on it the space $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)$ of square integrable random variables with values in $\mathbb{R}^n$. For $X \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)$ we denote by $\mathcal{L}_X$ the law of X, given by $\mathcal{L}_X(A) = \mathbb{P}(X \in A)$. To any m in $\mathcal{P}_2(\mathbb{R}^n)$, one can find a random variable X_m in $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)$ such that $\mathcal{L}_{X_m} = m$. We then have

$$W_2^2(m, m') = \inf_{\mathcal{L}_{X_m} = m, \ \mathcal{L}_{X_{m'}} = m'} \mathbb{E}[|X_m - X_{m'}|^2], \tag{2}$$

where the infimum is attained.

First derivative

We say F is *continuously differentiable* provided there exists a continuous function $\frac{\mathrm{d}F}{\mathrm{d}m} : \mathcal{P}_2 \times \mathbb{R}^n \to \mathbb{R}$ such that, for some $c : \mathcal{P}_2(\mathbb{R}^n) \to [0, \infty)$ that is bounded on bounded subsets, we have

$$\left| \frac{\mathrm{d}F}{\mathrm{d}m}(m, x) \right| \leq c(m)\left(1 + |x|^2\right) \tag{3}$$

and

$$\lim_{\epsilon \to 0} \frac{F(m + \epsilon(m' - m)) - F(m)}{\epsilon} = \int \frac{\mathrm{d}F}{\mathrm{d}m}(m, x)\, \mathrm{d}(m' - m)(x) \tag{4}$$

for any $m' \in \mathcal{P}_2$. Since $\frac{\mathrm{d}F}{\mathrm{d}m}$ is unique only up to a constant, we require the normalization condition

$$\int \frac{\mathrm{d}F}{\mathrm{d}m}(m, x)\, \mathrm{d}m(x) = 0, \tag{5}$$

which in particular ensures the functional derivative of a constant is 0.

We will often denote $\frac{\mathrm{d}F}{\mathrm{d}m}(m)(x) := \frac{\mathrm{d}F}{\mathrm{d}m}(m, x)$. Then $\frac{\mathrm{d}F}{\mathrm{d}m}(m) \in L^2_m(\mathbb{R}^n)$.

Second derivative I

We say F is *twice continuously differentiable* provided there exists a continuous function $\frac{\mathrm{d}^2 F}{\mathrm{d}m^2} : \mathcal{P}_2 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that, for some $c : \mathcal{P}_2(\mathbb{R}^n) \to [0, \infty)$ that is bounded on bounded subsets,

$$\left| \frac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m, x, \tilde{x}) \right| \leq c(m) \left(1 + |x|^2 + |\tilde{x}|^2 \right) \tag{6}$$

and

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int \left(\frac{\mathrm{d}F}{\mathrm{d}m}(m + \epsilon(\tilde{m}' - m), x) - \frac{\mathrm{d}F}{\mathrm{d}m}(m, x) \right) \mathrm{d}(m' - m)(x) = \iint \frac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m, x, \tilde{x}) \, \mathrm{d}(m' - m)(x) \, \mathrm{d}(\tilde{m}' - m)(\tilde{x}) \tag{7}$$

Second derivative II

for any $m', \tilde{m}' \in \mathcal{P}_2$. To ensure $\dfrac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m, x, \tilde{x})$ is uniquely defined, we will use the normalization convention

$$\int \frac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m, x, \tilde{x})\, \mathrm{d}m(\tilde{x}) = 0 \ \forall x, \quad \int \frac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m, x, \tilde{x})\, \mathrm{d}m(x) = 0 \ \forall \tilde{x}. \tag{8}$$

Again, we will write $\dfrac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m, x, \tilde{x}) = \dfrac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m)(x, \tilde{x})$, where we note that $\dfrac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m) \in L^2_{m \times m}$.

Standard arguments show that $\dfrac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m)$ is symmetric, i.e.

$$\frac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m)(x, \tilde{x}) = \frac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m)(\tilde{x}, x).$$

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Let $(\Omega, \mathcal{A}, \mathbb{P})$ be an atomless probability space. For $m \in \mathcal{P}_2$, let $\mathcal{H}_m := L^2(\Omega, \mathcal{A}, \mathbb{P}; L^2_m(\mathbb{R}^n; \mathbb{R}^n))$. On $\mathcal{H}_m$ we define the inner product

$$\langle X, Y \rangle_{\mathcal{H}_m} = \mathbb{E} \int X(x) \cdot Y(x) \, \mathrm{d}m(x) = \int_\Omega \int_{\mathbb{R}^n} X(\omega, x) \cdot Y(\omega, x) \, \mathrm{d}m(x) \, \mathrm{d}\mathbb{P}(\omega). \quad (9)$$

When it is sufficiently clear which inner product we mean, we will often drop the subscript $\mathcal{H}_m$.

Note $\mathcal{H}_m \cong L^2(\Omega \times \mathbb{R}^n, \mathcal{A} \otimes \mathcal{B}, \mathbb{P} \times m; \mathbb{R}^n)$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^n$.

Definition

Let $m \in \mathcal{P}_2, X \in \mathcal{H}_m$. We define $X \otimes m \in \mathcal{P}_2$ by duality: for all continuous functions $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $x \mapsto \frac{|\phi(x)|}{1+|x|^2}$ is bounded, we have

$$\int \phi(x) \, \mathrm{d}(X \otimes m)(x) = \mathbb{E} \left[\int \phi\left(X(x)\right) \mathrm{d}m(x) \right] = \int_\Omega \int_{\mathbb{R}^n} \phi\left(X(\omega, x)\right) \mathrm{d}m(x) \, \mathrm{d}\mathbb{P}(\omega).$$

Lemma

The map $X \mapsto X \otimes m$ is a contraction from $\mathcal{H}_m$ to $\mathcal{P}_2$, i.e. $W_2(X \otimes m, Y \otimes m) \leq \|X - Y\|_{\mathcal{H}_m}$.

If $X(\omega, x) = X(x)$ is deterministic, then $X \otimes m = X\sharp m$ where $X\sharp m(E) := m(X^{-1}(E))$.

Lemma

Let $X, Y \in \mathcal{H}_m$, and suppose $X \circ Y \in \mathcal{H}_m$. Then $(X \circ Y) \otimes m = X \otimes (Y \otimes m)$.

Examples

(i) If $X(x) = x$ is the identity map, then $X \otimes m = m$.
(ii) If $X(x) = a$ is a constant map, then $X \otimes m = \delta_a$, the Dirac delta mass concentrated at a.

If $X(\omega, x) = X(\omega)$ is just an L^2 random variable in $\mathbb{R}^n$, then $X \otimes m = \mathcal{L}_X$. Proof:

$$\int \phi(x)\, d(X \otimes m)(x) = \mathbb{E}\left[\int \phi(X)\, dm(x)\right] = \mathbb{E}\left[\phi(X)\right] \tag{10}$$

Lifting functionals on $\mathcal{P}_2$ I

Let $F : \mathcal{P}_2 \to \mathbb{R}$. For every $m \in \mathcal{P}_2$, the map $X \mapsto F(X \otimes m)$ is a functional on $\mathcal{H}_m$. We define the "partial derivative" of F with respect to $X \in \mathcal{H}_m$ as the unique element $D_X F(X \otimes m)$ of $\mathcal{H}_m$, if it exists, such that

$$\lim_{\epsilon \to 0} \frac{F\left((X + \epsilon Y) \otimes m\right) - F(X \otimes m)}{\epsilon} = \langle D_X F(X \otimes m), Y \rangle \ \forall Y \in \mathcal{H}_m. \qquad (11)$$

Lifting functionals on $\mathcal{P}_2$ II

Theorem (Bensoussan, PJG, Yam)

Let $F : \mathcal{P}_2 \to \mathbb{R}$ be continuously differentiable and assume $x \mapsto \frac{\mathrm{d}F}{\mathrm{d}m}(m,x)$ is continuously differentiable in $\mathbb{R}^n$. Assume that its derivative $D\frac{\mathrm{d}F}{\mathrm{d}m}(m)(x)$ is continuous in both m and x with

$$\left| D\,\frac{\mathrm{d}F}{\mathrm{d}m}(m)(x) \right| \leq c(m)\left(1+|x|\right) \tag{12}$$

for some constant $c(m)$ depending only on m. Then

$$D_X F(X \otimes m) = D\frac{\mathrm{d}F}{\mathrm{d}m}(X \otimes m)(X(\cdot)). \tag{13}$$

If $X(x) = x$, then $X \otimes m = m$, and thus (13) gives the **L-derivative**

$$D_X F(m) = D\frac{\mathrm{d}F}{\mathrm{d}m}(m)(\cdot). \tag{14}$$

Lifting functionals on $\mathcal{P}_2$ III

Proof.

Note that, by (12), $D\frac{\mathrm{d}F}{\mathrm{d}m}(X \otimes m, X(\cdot)) \in \mathcal{H}_m$ for any $X \in \mathcal{H}_m$. Let $Y \in \mathcal{H}_m$ be arbitrary. For $\epsilon \neq 0$, let $\mu = (X + \epsilon Y) \otimes m, \nu = X \otimes m$, and for $t \in [0,1]$ set $\nu_t = \nu + t(\mu - \nu)$. Then we have

$$\begin{aligned}
&\frac{1}{\epsilon}\Big(F\left((X+\epsilon Y)\otimes m\right) - F(X \otimes m)\Big) = \frac{1}{\epsilon}\int_0^1 \int_{\mathbb{R}^n} \frac{\mathrm{d}F}{\mathrm{d}m}(\nu_t, x)\,\mathrm{d}(\mu-\nu)(x)\,\mathrm{d}t \\
&= \frac{1}{\epsilon}\mathbb{E}\int_0^1 \int_{\mathbb{R}^n}\left(\frac{\mathrm{d}F}{\mathrm{d}m}(\nu_t, X(x)+\epsilon Y(x)) - \frac{\mathrm{d}F}{\mathrm{d}m}(\nu_t, X(x))\right)\mathrm{d}m(x)\,\mathrm{d}t \\
&\to \mathbb{E}\int_{\mathbb{R}^n} D\frac{\mathrm{d}F}{\mathrm{d}m}(X \otimes m, X(x)) \cdot Y(x)\,\mathrm{d}m(x) \\
&= \langle D\frac{\mathrm{d}F}{\mathrm{d}m}(X \otimes m, X(\cdot)), Y\rangle_{\mathcal{H}_m}
\end{aligned}$$

using the continuity of $D\frac{\mathrm{d}F}{\mathrm{d}m}$. □

Partial derivatives wrt m I

Let $F : \mathcal{P}_2 \to \mathbb{R}$ and let $X \in \cap_{m \in \mathcal{P}_2} \mathcal{H}_m$. We define the partial derivative of $F(X \otimes m)$ with respect to m, denoted $\frac{\partial F}{\partial m}(X \otimes m)(x)$, to be the derivative of $m \mapsto F(X \otimes m)$ in the sense given before.

Lemma

Let $X : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a $(\mathcal{A} \otimes \mathcal{B}, \mathcal{B})$ measurable vector field (where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^n$) such that

$$\mathbb{E}\left|X(x)\right|^2 \leq c(X)\left(1 + |x|^2\right) \ \forall x \in \mathbb{R}^n, \tag{15}$$

where $c(X)$ is a constant depending only on X. Then $X \in \cap_{m \in \mathcal{P}_2} \mathcal{H}_m$.

Partial derivatives wrt m II

Proposition

Let $F : \mathcal{P}_2 \to \mathbb{R}$ be continuously differentiable and let $X \in \cap_{m\in\mathcal{P}_2}\mathcal{H}_m$. Then

$$\frac{\partial F}{\partial m}(X \otimes m)(x) = \mathbb{E}\frac{\mathrm{d}F}{\mathrm{d}m}(X \otimes m)(X(x)). \tag{16}$$

Partial derivatives wrt m III

Proof.

For $\epsilon \neq 0$ let $\mu = X \otimes \left(m + \epsilon(m' - m)\right), \nu = X \otimes m$, and for $t \in [0,1]$ set $\nu_t = \nu + t(\mu - \nu)$. We have, as $\epsilon \to 0$,

$$\frac{1}{\epsilon}\left(F\left(X \otimes \left(m + \epsilon(m' - m)\right)\right) - F\left(X \otimes m\right)\right) = \frac{1}{\epsilon}\int_0^1 \int_{\mathbb{R}^n} \frac{\mathrm{d}F}{\mathrm{d}m}(\nu_t, x)\,\mathrm{d}(\mu - \nu)(x)$$

$$= \mathbb{E}\int_0^1 \int_{\mathbb{R}^n} \frac{\mathrm{d}F}{\mathrm{d}m}\left(\nu_t, X(x)\right)\mathrm{d}(m' - m)(x)$$

$$\to \mathbb{E}\int_{\mathbb{R}^n} \frac{\mathrm{d}F}{\mathrm{d}m}\left(X \otimes m, X(x)\right)\mathrm{d}(m' - m)(x),$$

using the continuity of $\frac{\mathrm{d}F}{\mathrm{d}m}$. The claim follows. □

A formula for second derivatives

Let $F : \mathcal{P}_2 \to \mathbb{R}$ be twice continuously differentiable and let $X, Z \in \mathcal{H}_m$. Then $D_X^2 F(X \otimes m)$ exists in a Gâteaux sense and

$$D_X^2 F(X \otimes m)(Z)(x) = D^2 \frac{\mathrm{d}F}{\mathrm{d}m}(X \otimes m)(X(x))Z(x) + \tilde{\mathbb{E}} \int_{\mathbb{R}^n} D_1 D_2 \frac{\mathrm{d}^2 F}{\mathrm{d}m^2}(X \otimes m)(\tilde{X}(\tilde{x}), X(x))\tilde{Z}(\tilde{x})\,\mathrm{d}m(\tilde{x}) \quad (17)$$

in which $\tilde{X}(\tilde{x}), \tilde{Z}(\tilde{x})$ are independent copies of $X(x), Z(x)$, and in which the expectation $\tilde{\mathbb{E}}$ is independent of $X(x)$.

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Recall that $(\Omega, \mathcal{A}, \mathbb{P})$ is an atomless probability space, and for $m \in \mathcal{P}_2$, $\mathcal{H}_m := L^2(\Omega, \mathcal{A}, \mathbb{P}; L^2_m(\mathbb{R}^n; \mathbb{R}^n))$. Now assume

- $(\Omega, \mathcal{A}, \mathbb{P})$ is sufficiently large to contain a standard Wiener process in $\mathbb{R}^n$, denoted $w(t)$, with filtration $\mathcal{W}_t = \{\mathcal{W}_t^s\}_{s \geq t}$ where $\mathcal{W}_t^s = \sigma\left((w(\tau) - w(t)) : t \leq \tau \leq s\right)$.
- $(\Omega, \mathcal{A}, \mathbb{P})$ is rich enough to support random variables that are independent of the entire Wiener process.

Denote by $\mathcal{H}_{m,t}$ the space of all $X = X_t \in \mathcal{H}_m$ such that X is independent of $\mathcal{W}_t$.

For $X \in \mathcal{H}_{m,t}$ we define σ-algebras $\mathcal{W}_{Xt}^s = \sigma(X) \vee \mathcal{W}_t^s$, and the filtration generated by these will be denoted $\mathcal{W}_{Xt}$.

$L^2_{\mathcal{W}_{Xt}}(t, T; \mathcal{H}_m)$ will be the set of all processes in $L^2(t, T; \mathcal{H}_m)$ that are adapted to $\mathcal{W}_{Xt}$, $L^2_{\mathcal{W}_t}(t, T; \mathcal{H}_{X \otimes m})$ the set of all processes in $L^2(t, T; \mathcal{H}_{X \otimes m})$ that are adapted to $\mathcal{W}_t$.

Lemma

There is a linear isometry $L^2_{\mathcal{W}_t}(t, T; \mathcal{H}_{X \otimes m}) \to L^2_{\mathcal{W}_{Xt}}(t, T; \mathcal{H}_m)$ given by $v(s)(x) \mapsto v(s)(X(x))$.

Control problem: first formulation

Define the cost functional $J_{X\otimes m,t} : L^2_{\mathcal{W}_t}(t,T;\mathcal{H}_{X\otimes m}) \to \mathbb{R}$ by

$$J_{X\otimes m,t}(v_{\cdot t}(\cdot)) = \frac{\lambda}{2}\int_t^T \int_{\mathbb{R}^n} \mathbb{E}|v_{\xi t}(s)|^2 \,\mathrm{d}(X\otimes m)(\xi)\,\mathrm{d}s + \int_t^T F(X_{\cdot t}(s;v_{\cdot t}(\cdot))\otimes(X_{\cdot t}\otimes m))\,\mathrm{d}s + F_T(X_{\cdot t}(T;v_{\cdot t}(\cdot))\otimes(X_{\cdot t}\otimes m)) \quad (18)$$

where $X_{\cdot t}(s;v_{\cdot t}(\cdot))$ is defined by

$$X_{xt}(s;v_{\cdot t}(\cdot)) = x + \int_t^s v_{xt}(\tau)\,\mathrm{d}\tau + \sigma(w(s)-w(t)). \quad (19)$$

This is just a classical SDE for each $x\in\mathbb{R}^n$.
The value function is defined as

$$V(X\otimes m,t) := \inf_v J_{X\otimes m,t}(v_{\cdot t}(\cdot)). \quad (20)$$

Control problem: second formulation

By the isometry $L^2_{\mathcal{W}_t}(t,T;\mathcal{H}_{X\otimes m}) \to L^2_{\mathcal{W}_{Xt}}(t,T;\mathcal{H}_m)$ given by $v_{xt}(s) \mapsto v_{Xt}(s)$, the first formulation is equivalent to a second:

Let $J_{Xt} : L^2_{\mathcal{W}_{Xt}}(t,T;\mathcal{H}_m) \to \mathbb{R}$ be given by

$$J_{Xt}(v_{Xt}(\cdot)) = \frac{\lambda}{2}\int_t^T \left\| v_{Xt}(s)\right\|^2_{\mathcal{H}_m} \mathrm{d}s + \int_t^T F(X_{Xt}(s;v_{Xt}(\cdot))\otimes m)\,\mathrm{d}s + F_T(X_{Xt}(T;v_{Xt}(\cdot))\otimes m) \quad (21)$$

where

$$X_{Xt}(s;v_{Xt}(\cdot)) = X + \int_t^s v_{Xt}(\tau)\,\mathrm{d}\tau + \sigma(w(s)-w(t)). \quad (22)$$

This is an SDE on the infinite dimensional space $\mathcal{H}_m$.

Remark

(i) $X_{Xt}(s;v_{Xt}(\cdot)) \in \mathcal{H}_{m,s}$ for all $s \geq t$.
(ii) $X_{Xt}(s;v_{Xt}(\cdot)) \otimes m$ is an abuse of notation. It actually means $X_{\cdot t}(s;v_{\cdot}(\cdot)) \otimes (X_{\cdot t}\otimes m)$.

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Assumptions I

F and F_T are twice continuously differentiable.
Essential estimates:

$$\begin{aligned} &\left| D^2 \frac{\mathrm{d}F}{\mathrm{d}m}(m)(x) \right| \leq c, \left| DD_2 \frac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m)(x,\tilde{x}) \right| \leq c, \\ &\left| D \frac{dF_T}{dm}(m)(x) \right| \leq c_T, \left| D_2 D_1 \frac{d^2 F_T}{dm^2}(m)(x,\tilde{x}) \right| \leq c_T, \forall x, \tilde{x} \in \mathbb{R}^n. \end{aligned} \tag{23}$$

Semi-convexity conditions:

$$\begin{aligned} D^2 \frac{\mathrm{d}F}{\mathrm{d}m}(m)(x)\xi \cdot \xi + D_2 D_1 \frac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m)(x,\tilde{x})\xi \cdot \tilde{\xi} \geq -c'|\xi|(|\xi| + |\tilde{\xi}|), \\ D^2 \frac{\mathrm{d}F}{\mathrm{d}m}(m)(x)\xi \cdot \xi + D_2 D_1 \frac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m)(x,\tilde{x})\xi \cdot \tilde{\xi} \geq -c'|\xi|(|\xi| + |\tilde{\xi}|), \forall x, \tilde{x}, \xi, \tilde{\xi}. \end{aligned} \tag{24}$$

Assumptions II

Continuity condition:

$$(m,x) \mapsto D^2 \frac{\mathrm{d}F}{\mathrm{d}m}(m)(x), (m,x,\tilde{x}) \mapsto D_2 D_1 \frac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m)(x,\tilde{x}) \text{ are continuous from } \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \text{and } \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n;\mathbb{R}^n), \text{ respectively,} \tag{25}$$

and likewise for F_T.
Condition to guarantee objective is convex:

$$\lambda - T(c_T' + \frac{c' T}{2}) > 0. \tag{26}$$

Smallness condition:

$$\lambda - T(c_T + c\frac{T}{2}) > 0 \tag{27}$$

Lifted Bellman Equation

The Bellman equation for $V(X \otimes m, t)$ is

$$\frac{\partial V}{\partial t}(X \otimes m, t) + \frac{1}{2}\langle D_X^2 V(X \otimes m, t)(\sigma N), \sigma N\rangle - \frac{1}{2\lambda}||D_X V(X \otimes m, t)||^2 + F(X \otimes m) = 0, \quad V(X \otimes m, T) = F_T(X \otimes m). \tag{28}$$

Our definition of **classical solution** includes these essential features:

- $V, D_X, D_X^2 V$ are continuous (V and $D_X V$ are Hölder in time)
- V is right-differentiable in time

Theorem (Bensoussan, PJG, Yam)

Under the above assumptions, $V(X \otimes m, t)$ is the unique classical solution to the Bellman equation.

Back down to original Bellman equation

A simple corollary of our theorem is that $V(m,t)$ solves the original Bellman equation:

$$-\frac{\partial V}{\partial t}(m,t) + \int_{\mathbb{R}^n} A_x \frac{\mathrm{d}}{\mathrm{d}m} V(m,t)(x)\,\mathrm{d}m(x) + \frac{1}{2\lambda}\int_{\mathbb{R}^n}\left|D\frac{\mathrm{d}}{\mathrm{d}m} V(m,t)(x)\right|^2 \mathrm{d}m(x) = F(m), \quad V(m,T) = F_T(m).$$

To derive this, just plug in $X(x) = x$.

Solving the Master Equation I

Let $U(x, m, t) = \dfrac{\mathrm{d}}{\mathrm{d}m} V(m, t)(x)$. Then formally U satisfies the Master Equation:

$$
\begin{aligned}
&-\frac{\partial U}{\partial t}(x, m, t) + A_x U(x, m, t) + \int_{\mathbb{R}^n} A_\xi \frac{\mathrm{d}}{\mathrm{d}m} U(\xi, m, t)(x) \, \mathrm{d}m(\xi) \\
&\quad + \frac{1}{2\lambda}|D_x U(x, m, t)|^2 + \frac{1}{\lambda} \int_{\mathbb{R}^n} D_\xi U(\xi, m, t) \cdot D_\xi \frac{\mathrm{d}}{\mathrm{d}m} U(\xi, m, t)(x) \, \mathrm{d}m(\xi) \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}m} F(m)(x), \qquad U(x, m, T) = \frac{\mathrm{d}}{\mathrm{d}m} F_T(m)(x).
\end{aligned}
$$

Solving the Master Equation II

To justify this, we need more regularity:

$$\left|D\frac{\mathrm{d}}{\mathrm{d}m}F(m)(x)\right| \leq c(1+|x|),\quad \left|D^2\frac{\mathrm{d}}{\mathrm{d}m}F(m)(x)\right| \leq c,\quad \left|D^3\frac{\mathrm{d}}{\mathrm{d}m}F(m)(x)\right| \leq c,$$
$$\left|D_1\frac{\mathrm{d}^2}{\mathrm{d}m^2}F(m)(x,\widetilde{x})\right| \leq c(1+|\widetilde{x}|),\quad \left|D_2D_1\frac{\mathrm{d}^2}{\mathrm{d}m^2}F(m)(x,\widetilde{x})\right| \leq c,$$
$$\left|D_1^2\frac{\mathrm{d}^2}{\mathrm{d}m^2}F(m)(x,\widetilde{x})\right| \leq c(1+|\widetilde{x}|),\quad \left|D_1^2D_2\frac{\mathrm{d}^2}{\mathrm{d}m^2}F(m)(x,\widetilde{x})\right| \leq c. \tag{29}$$

Theorem (Bensoussan, PJG, Yam)

In addition to all the previous assumptions, take $\lambda \geq \lambda_T$ for λ_T sufficiently large depending on c, c_T, and T. Then U satisfies the Master Equation in a pointwise sense.

We do not fully treat uniqueness; see previous references.

Main idea behind all the proofs

The optimal trajectory for the control problem can be derived from a forward-backward system of SDEs:

$$Y_{Xt}(s) = X - \frac{1}{\lambda}\int_t^s Z_{Xt}(\tau)\,\mathrm{d}\tau + \sigma(w(s) - w(t)), \tag{30}$$

$$Z_{Xt}(s) = \mathbb{E}\left[\int_s^T D_X F(Y_{Xt}(\tau)\otimes m)\,\mathrm{d}\tau + D_X F_T(Y_{Xt}(T)\otimes m)\middle| \mathcal{W}_{Xt}^s\right]. \tag{31}$$

We look for a priori estimates on the pair $(Y_{Xt}(s), Z_{Xt}(s))$, including a sensitivity analysis wrt m. All of our estimates on V arise as a corollary.

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