An Existence Result for a Class of Mean Field Games of Controls

Laurent Pfeiffer
Inria and Ecole Polytechnique, Institut Polytechnique de Paris

Joint work with J. Frédéric Bonnans, Justina Gianatti, and Saeed Hadikhanloo

Two-Days Online Workshop on Mean-Field Games,
June 18, 2020

Introduction

- **The Mean-Field Game (MFG) model:**
  - Coupling (à la Cournot) via endogenous price variable P
  - Price related to the distribution of (states,controls).

$\longrightarrow$ MFG of controls
$=$ extended MFG, strongly coupled MFG...

- **Topics:**
  - Existence (2nd order case)
  - Duality
  - Lagrangian approach (1st order case).

Cournot equilibria

Consider N producers, buy some raw material on a market.

- Quantity bought by producer i: v_i
- Benefit resulting from v_i: $-L_i(v_i)$
- Unitary price of raw material: $P = \Psi(\sum_{i=1}^{N} v_i)$.
- Nash equilibrium: a vector $\bar{v} \in \mathbb{R}^N$ such that

$$\bar{v}_i \in \underset{v_i \in \mathbb{R}}{\arg\min} \left\{ L_i(v_i) + \Psi\left(\sum_{j=1}^{N} \bar{v}_j\right) v_i \right\},$$

for $i = 1, \dots, N$.

Remark

The producers do not take into account their contribution to the equilibrium price P.

Potential formulation

Assumptions:

- $L_1,\ldots,L_N$ are strongly convex
- $\Psi = \nabla\Phi$, with Φ convex

Potential formulation:
Let $B\colon v \in \mathbb{R}^N \mapsto B(v) = \sum_{i=1}^N L_i(v_i) + \Phi\big(\sum_{i=1}^N v_i\big)$. Then,

$$\begin{aligned} &\bar{v} \in \mathbb{R}^N \text{ is a Nash equilibrium} \\ \Longleftrightarrow\ & \nabla L_i(\bar{v}_i) + \Psi\Big(\sum_{j=1}^N \bar{v}_j\Big) = 0 = \nabla_{v_i} B(\bar{v}), \quad \forall i = 1,\ldots,N \\ \Longleftrightarrow\ & \bar{v} \text{ minimizes } B. \end{aligned}$$

Implies **existence** and **uniqueness** (B is strongly convex).

Remark

MFG model: a dynamic version with infinitely many agents.

MFG model

Coupled system:

$$
\begin{cases}
(i) & -\partial_t u - \sigma \Delta u + H(\nabla u + P) = 0 & (x,t) \in Q, \\
(ii) & \partial_t m - \sigma \Delta m + \mathrm{div}(vm) = 0 & (x,t) \in Q, \\
(iii) & P(t) = \Psi \left(\int_{\mathbb{T}^d} v(x,t) m(x,t) \, \mathrm{d}x \right) & t \in [0,T], \\
(iv) & v = -\nabla H(\nabla u + P) & (x,t) \in Q, \\
(v) & m(x,0) = m_0(x), \quad u(x,T) = g(x) & x \in \mathbb{T}^d,
\end{cases}
\qquad \text{(MFGC)}
$$

Unknowns:

$u(x,t)$	value function	$v(x,t)$	feedback
$m(x,t)$	distribution	$P(t)$	price

Data:

H	Hamiltonian	Ψ	price function
m_0	initial distrib.	g	terminal cost

MFG of controls

- **Equation (i): Hamilton-Jacobi-Bellman (HJB) equation.**

  Associated stochastic optimal control problem:

$$u(x,t) = \begin{cases} \inf_{\alpha \in \mathbb{L}^2(t,T)} \mathbb{E}\Big[\int_t^T L(\alpha_s) + \langle P(s), \alpha_s \rangle \, \mathrm{d}s + g(X_T)\Big], \\ \text{s.t.: } \mathrm{d}X_s = \alpha_s \, \mathrm{d}s + \sqrt{2\sigma} \, \mathrm{d}W_s, \quad X_t = x. \end{cases}$$

X_s	stock at time s
α_s	bought/sold quantity
$P(s)$	unitary price.

- **Equation (ii): Fokker-Planck equation.**

MFG of controls

- **Equation (iii): price relation.**

$$P(t) = \Psi\Big(\underbrace{\int_{\mathbb{T}^d} v(x,t) m(x,t) \, dx}_{\to \text{ Demand}} \Big)$$

- **Equation (iv): optimal feedback law.**

$$v(x,t) = -\nabla H(\nabla u(x,t) + P(t)).$$

Remark

- *Given m and u, the feedback v cannot be recovered in an explicit fashion $\to$ MFG of controls[1].*
- *Equilibrium problem for each time t (involving P and v).*

[1] Graber & Bensoussan '15, Gomes & Voskanyan '16, Cardaliaguet & Lehalle '18, Kobeissi '19, Graber, Ignazio & Neufeld '20,...

An example

An idealized model from **electrical engineering**[2]:

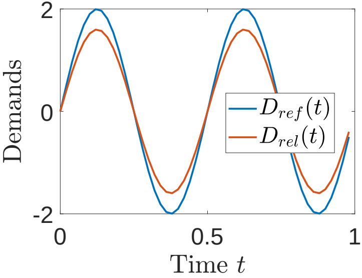
- Large population of **storage devices**
- State variable $x \in (0, 2)$: State-of-charge
- Control α: Relative loading speed
- Reference demand $D_{\text{ref}}(t)$
- Price function:

$$P(t) = \beta D_{\text{rel}}(t), \quad D_{\text{rel}}(t) = D_{\text{ref}}(t) + \int_0^2 v(x,t)m(x,t)\,\mathrm{d}x$$

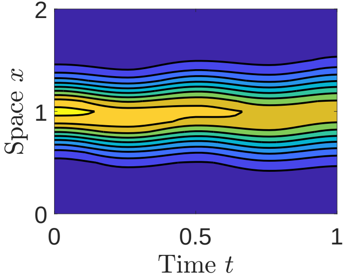
- Cost: $L(\alpha) = \frac{1}{2}\alpha^2$, $g(x) = -\beta D_{\text{ref}} x$
- Deterministic dynamics: $\sigma = 0$

[2]Couillet et al. '12, De Paola et al. '16, Alasseur, Ben Tahar & Matoussi '20, Gomes & Saúde '20

Results



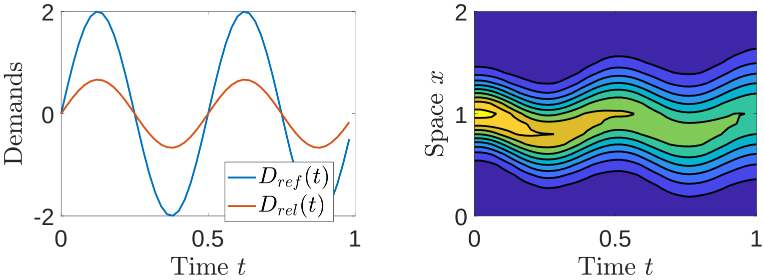
(a) Reference and relative demands



(b) Distribution

Figure: Equilibrium results $\beta = 0.25$ (small coupling)

Results



(a) Reference and relative demands (b) Distribution

Figure: Equilibrium results $\beta = 2$ (strong coupling)

Functional framework

Given $\alpha \in (0,1)$ and $X = [0,T]$, $X = \mathbb{T}^d$, or $X = Q$,

$$
\begin{aligned}
C^{j+\alpha}(X) :=& \big\{ u \in C^j(X) \,|\, \exists C > 0,\ \forall x, y \in X, \\
& \quad \|D^i u(y) - D^i u(x)\| \le C\|y - x\|_X^\alpha,\ \forall i \le j \big\} \\
C^{\alpha,\alpha/2}(Q) :=& \big\{ u \in C(Q) \,|\, \exists C > 0,\ \forall x, y \in X, \\
& \quad |u(x_2,t_2) - u(x_1,t_1)| \le C\big(\|x_2 - x_1\|^\alpha + |t_2 - t_1|^{\alpha/2}\big) \big\} \\
C^{2+\alpha,1+\alpha/2}(Q) :=& \big\{ u \in C^{\alpha,\alpha/2}(Q) \,|\, \partial_t u \in C^{\alpha,\alpha/2}(Q), \\
& \quad \nabla u \in C^{\alpha,\alpha/2}(Q),\ \nabla^2 u \in C^{\alpha,\alpha/2}(Q) \big\}.
\end{aligned}
$$

We fix $p > d + 2$ and define the Sobolev space

$$
W^{2,1,p}(Q) := L^p(0,T; W^{2,p}(Q)) \cap W^{1,p}(Q).
$$

Embedding: $\|u\|_{C^\alpha(Q)} + \|\nabla u\|_{C^\alpha(Q)} \le C\|u\|_{W^{2,1,p}(Q)}$.

Assumptions

Monotonicity assumptions:

- $\Psi = \nabla\Phi$, where Φ is convex
- L is strongly convex.

Growth assumptions:

- $L(v) \leq C(1 + \|v\|^2)$
- $\Psi(z) \leq C(1 + \|z\|)$.

Regularity assumptions:

- $H \in C^2(\mathbb{R}^d)$, H, ∇H, $\nabla^2 H$ are locally Hölder continuous
- Ψ is locally Hölder continuous
- $m_0 \in C^{2+\alpha}(\mathbb{T}^d)$, $g \in C^{2+\alpha}(\mathbb{T}^d)$
- $m_0 \in \mathcal{D}_1(\mathbb{T}^d) := \{h \in L^\infty(\mathbb{T}^d) \mid h \geq 0, \ \int_{\mathbb{T}^d} h(x)\,\mathrm{d}x = 1\}$.

Auxiliary mappings

We analyse (*iii*) and (*iv*) to eliminate v and P from (MFGC).

Lemma

For all $m \in \mathcal{D}_1(\mathbb{T}^d)$, *for all* $w \in L^\infty(\mathbb{T}^d, \mathbb{R}^d)$, *there exists a unique pair* $(v, P) = (\mathbf{v}(m,w), \mathbf{P}(m,w)) \in L^\infty(\mathbb{T}^d, \mathbb{R}^d) \times \mathbb{R}^d$ *such that*

$$\begin{cases} v(x) = & -\nabla H(w(x) + P), \quad \forall x \in \mathbb{T}^d, \\ P = & \Psi\left(\int_{\mathbb{T}^d} v(x) m(x)\, dx\right). \end{cases} \qquad (*)$$

Elements of proof. If $m > 0$, then (v, P) satisfies $(*)$ if and only if v minimizes the following convex functional:

$$J(v)\colon v \mapsto \Phi\left(\int_{\mathbb{T}^d} v(x) m(x)\, \mathrm{d}x\right) + \int_{\mathbb{T}^d} \big(L(v(x)) + \langle w(x), v(x)\rangle\big) m(x)\, \mathrm{d}x,$$

which possesses a unique minimizer.

Auxiliary mappings

Reduced coupled system:

$$\begin{cases} -\partial_t u - \sigma\Delta u + H(\nabla u + \mathbf{P}(m(\cdot,t), \nabla u(\cdot,t))) = & 0, \\ \partial_t m - \sigma\Delta m + \operatorname{div}(\mathbf{v}(m(\cdot,t), \nabla u(\cdot,t))m) = & 0, \\ u(x,T) = g(x), \quad m(x,0) = m_0(x). \end{cases} \qquad (MFGC')$$

Lemma (Stability lemma)

Let $R > 0$, let m_1 and $m_2 \in \mathcal{D}_1(\mathbb{T}^d)$, let w_1 and $w_2 \in L^\infty(\mathbb{T}^d, \mathbb{R}^d)$ with $\|w_i\|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \leq R$. There exists $C > 0$ and $\alpha \in (0,1)$, depending on R only such that

$$\begin{aligned} &\|\mathbf{P}(m_2, w_2) - \mathbf{P}(m_1, w_1)\| \\ &\quad \leq C\left(\|w_2 - w_1\|^\alpha_{L^\infty(\mathbb{T}^d)} + \|m_2 - m_1\|^\alpha_{L^1(\mathbb{T}^d)}\right). \end{aligned}$$

Idea of proof: stability analysis for convex optimization problems.

Potential formulation

Consider the cost function $B\colon W^{2,1,p}(Q) \times L^\infty(Q) \to \mathbb{R}$,

$$B(m,v) = \iint_Q L(v(x,t))m(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^d} g(x)m(x,T)\,\mathrm{d}x + \int_0^T \Phi\left(\int_{\mathbb{T}^d} v(x,t)m(x,t)\,\mathrm{d}x\right)\mathrm{d}t.$$

Lemma

Let $(\bar{u},\bar{m},\bar{v},\bar{P}) \in W^{2,1,p}(Q)^2 \times L^\infty(Q,\mathbb{R}^d) \times L^\infty(0,T;\mathbb{R}^k)$ be a solution to (MFGC). Then, $(\bar{m},\bar{v})$ is a **solution** *to:*

$$\min_{\substack{m\in W^{2,1,p}(Q)\\ v\in L^\infty(Q,\mathbb{R}^k)}} B(m,v) \quad s.t.: \begin{cases} \partial_t m - \sigma\Delta m + div(vm) = 0, \\ m(x,0) = m_0(x). \end{cases} \qquad (\mathcal{P})$$

Result and approach

Theorem

There exists a classical solution to (*MFGC*) *with*

$$u \in C^{2+\alpha,1+\alpha/2}(Q), \qquad m \in C^{2+\alpha,1+\alpha/2}(Q),$$
$$v \in C^{\alpha}(Q), D_x v \in C^{\alpha}(Q), \quad P \in C^{\alpha}(0,T).$$

Theorem (Leray-Schauder)

Let X *be a Banach space and let* $\mathcal{T} \colon X \times [0,1] \to X$ *satisfy:*

1. $\mathcal{T}$ *is a continuous and compact mapping,*
2. $\exists \tilde{x} \in X$, $\mathcal{T}(x,0) = \tilde{x}$ *for all* $x \in X$,
3. $\exists C > 0$, $\forall (x,\tau) \in X \times [0,1]$,
$$\mathcal{T}(x,\tau) = x \Longrightarrow \|x\|_X \leq C.$$

Then, there exists $x \in X$ *such that* $\mathcal{T}(x,1) = x$.

Construction of $\mathcal{T}$

Let $X = (W^{2,1,p}(Q))^2$. For $(u, m, \tau) \in X \times [0,1]$, $(\tilde{u}, \tilde{m}) = \mathcal{T}(u, m, \tau) \in W^{2,1,p}(Q)^2$ where:

- $\tilde{u}$ is the solution to

$$\begin{cases} -\partial_t \tilde{u} - \sigma \Delta \tilde{u} + \tau H(\nabla u + \mathbf{P}(\rho(m), \nabla u)) = & 0, \\ \tilde{u}(T, x) = & \tau g(x), \end{cases}$$

- $\tilde{m}$ is the solution

$$\begin{cases} \partial_t \tilde{m} - \sigma \Delta \tilde{m} + \tau \mathrm{div}(\mathbf{v}(\rho(m), \nabla u) m) = & 0, \\ \tilde{m}(x, 0) = & m_0(x), \end{cases}$$

Here $\rho \colon L^\infty(\mathbb{T}^d) \to \mathcal{D}_1(\mathbb{T}^d)$ is a kind of regular projection operator $(\rho(m) = m$ for $m \in \mathcal{D}_1)$.

Parabolic estimates

Consider the parabolic equation:

$$\begin{cases} \partial_t u - \sigma \Delta u + \langle b, \nabla u \rangle + cu = & h, \qquad (x,t) \in Q, \\ u(x,0) = & u_0(x), \quad x \in \mathbb{T}^d. \end{cases}$$

Assume that $u_0 \in C^{2+\alpha}(\mathbb{T}^d)$.

Theorem

1. *Assume that* $b \in L^p(Q)$, $c \in L^p(Q)$, *and* $h \in L^p(Q)$. *Then,* $u \in W^{2,1,p}(Q)$, $u \in C^\alpha(Q)$, *and* $\nabla u \in C^\alpha(Q)$.
2. *Assume that* $b \in C^{\beta,\beta/2}(Q)$, $c \in C^{\beta,\beta/2}(Q)$, *and* $h \in C^{\beta,\beta/2}(Q)$. *Then,* $u \in C^{2+\alpha,1+\alpha/2}(Q)$.

Regularity of $\mathcal{T}$

Lemma

1. *The mapping $\mathcal{T}$ is continuous.*
2. *For all $R > 0$, there exist $C > 0$ and $\alpha \in (0, 1]$ such that for all $(u, m) \in W^{2,1,p}(Q)$ and for all $\tau \in [0, 1]$,*

$$\|u\|_{W^{2,1,p}(Q)} + \|m\|_{W^{2,1,p}(Q)} \leq R \implies \|\tilde{u}\|_{C^{2+\alpha,1+\alpha/2}(Q)} + \|\tilde{m}\|_{C^{(2+\alpha,1+\alpha/2}(Q)} \leq C,$$

where $(\tilde{u}, \tilde{m}) = \mathcal{T}(u, m, \tau)$.

Consequence: $\mathcal{T}$ is compact, by the theorem of Arzelà-Ascoli.

Estimates for fixed points

Proposition

There exist $C > 0$ and $\alpha \in (0,1)$ such that for all $(u, m, \tau) \in X \times [0,1]$ satisfying $(u, m) = \mathcal{T}(u, m, \tau)$, we have

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(Q)} \leq C, \qquad \|m\|_{C^{2+\alpha,1+\alpha/2}(Q)} \leq C,$$
$$\|v\|_{C^{\alpha}(Q)} + \|D_x v\|_{C^{\alpha}(Q)} \leq C, \quad \|P\|_{C^{\alpha}(0,T)} \leq C,$$

where $P = \mathbf{P}(m, \nabla u)$ and $v = \mathbf{v}(m, \nabla u)$.

Proof. For $\tau = 1$. The pair (m, v) is a solution to $(\mathcal{P})$. Thus,

$$C \iint_Q \|v(x,t)\|^2 m(x,t)\,\mathrm{d}x\,\mathrm{d}t - C \leq B(m, v) \leq B(m_0, v_0 = 0) \leq C.$$

Thus,

$$\|P\|^2_{L^2(0,T)} \leq C\Big(1 + \int_0^T \|\textstyle\int_{\mathbb{T}^d} vm\,\mathrm{d}x\|^2 \mathrm{d}t\Big) \leq C\Big(1 + \iint_Q \|v\|^2 m\,\mathrm{d}x\,\mathrm{d}t\Big) \leq C.$$

Estimates for fixed points

$u, \nabla u \in L^\infty(Q)$	u value function of opt. control pb.
$P \in L^\infty(0,T;\mathbb{R}^k)$	Stability lemma
$H(\nabla u + P) \in L^\infty(Q)$ $u \in W^{2,1,p}(Q)$	Regularity of H HJB: parabolic eq. with L^p coeff.
$v \in L^\infty(Q,\mathbb{R}^d)$ $D_x v \in L^p(Q,\mathbb{R}^{d\times d})$	Stability lemma $D_x v = -\nabla^2 H(\nabla u + P)\nabla^2 u$
$m \in W^{2,1,p}(Q)$	FP: parabolic eq. with L^p coeff.
$P \in C^\alpha(Q)$	Stability lemma
$H(\nabla u + P) \in C^\alpha(Q)$ $u \in C^{2+\alpha,1+\alpha/2}(Q)$	Regularity of H HJB: parabolic eq. with Hölder coeff.
$v, D_x v \in C^\alpha(Q)$	Stability lemma
$m \in C^\alpha(Q)$	FP: parabolic eq. with Hölder coeff.

Convexity of the potential problem

Reformulate the potential problem:

- Change of variable $(m, v) \to (m, w) := (m, mv)$.
- New variable A and new constraint $A(t) = \int_{\mathbb{T}^d} v(x,t)m(x,t)\mathrm{d}x = \int_{\mathbb{T}^d} w(x,t)\mathrm{d}x$.

Yields an equivalent problem:

$$\min_{(m,w,A)} \iint_Q L\left(\frac{w}{m}\right) m \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^d} gm(\cdot, T)\mathrm{d}x + \int_0^T \Phi(A(t))\,\mathrm{d}t$$

$$\text{s.t.:} \quad \begin{cases} \partial_t m - \sigma\Delta m + \mathrm{div}(w) = 0, \\ m(x,0) = m_0(x), \\ A(t) = \int_{\mathbb{T}^d} w(x,t)\,\mathrm{d}x. \end{cases}$$

with **convex cost function** and **affine constraints**.

Duality

Consider the following criterion:

$$D(u,P) = -\int_{\mathbb{T}^d} u(x,0)m_0(x)\,\mathrm{d}x - \int_0^T \Phi^*(P(t))\,\mathrm{d}t,$$

for $u \in W^{2,1,p}(Q)$ and $P \in L^\infty(0,T)$, and the **dual** problem:

$$\sup_{\substack{u\in W^{2,1,p}(Q)\\ P\in L^\infty(0,T)}} D(u,P), \quad \text{s.t.:} \begin{cases} -\partial u_t - \sigma\Delta u + H(\nabla u + P) = & 0, \\ u(x,T) = & g(x). \end{cases}$$

Lemma

For all solutions $(\bar{u}, \bar{m}, \bar{v}, \bar{P})$ to (MFGC), the pair $(\bar{u}, \bar{P})$ is a **solution** *to the dual problem.*

Duality

Proof. Let (u, P) be feasible.

$$
\begin{aligned}
&\int_{\mathbb{T}^d} u(x,0)m_0(x)\,\mathrm{d}x - \int_{\mathbb{T}^d} u(x,T)\bar{m}(x,T)\,\mathrm{d}x \\
&\quad = -\iint_Q \partial_t u \bar{m}\,\mathrm{d}x\,\mathrm{d}t - \iint_Q u\partial_t \bar{m}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \iint_Q (\sigma\Delta u - H(\nabla u + P))\bar{m}\,\mathrm{d}x\,\mathrm{d}t - \iint_Q u(\sigma\Delta\bar{m} - \mathrm{div}(\bar{v}\bar{m}))\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq \iint_Q \big(L(\bar{v}) + \langle \nabla u + P, \bar{v}\rangle\big)\bar{m}\,\mathrm{d}x\,\mathrm{d}t - \iint_Q \langle \nabla u, \bar{v}\rangle \bar{m}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \iint_Q \big(L(\bar{v}) + \langle P, \bar{v}\rangle\big)\bar{m}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Therefore,

$$
\int_{\mathbb{T}^d} u(x,0)m_0(x)\,\mathrm{d}x \leq \int_{\mathbb{T}^d} g(x)\bar{m}(x,T)\,\mathrm{d}x + \iint_Q \big(L(\bar{v}) + \langle P, \bar{v}\rangle\big)\bar{m}\,\mathrm{d}x\,\mathrm{d}t.
$$

Duality

We also have:

$$-\int_0^T \Phi^*(P(t))\,\mathrm{d}t \le -\int_0^T \left\langle P(t), \textstyle\int_{\mathbb{T}^d} \bar{v}\bar{m}\,\mathrm{d}x \right\rangle + \int_0^T \Phi\big(\textstyle\int_{\mathbb{T}^d} \bar{v}\bar{m}\big)\,\mathrm{d}t.$$

Therefore,

$$\begin{aligned} D(u,P) \le & \iint_Q L(\bar{v})\bar{m}\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^d} g(x)\bar{m}(x,T)\,\mathrm{d}x + \int_0^T \Phi\big(\textstyle\int_{\mathbb{T}^d} \bar{v}\bar{m}\big)\,\mathrm{d}t \\ = & B(\bar{v},\bar{m}). \end{aligned}$$

Equality is reached for $(u,P) = (\bar{u},\bar{P})$. □

Remark

Allows use of dual methods for **numerical resolution**[3].

[3] Benamou, Carlier, Santambrogio '17.

Deterministic optimal control

- Let $\Gamma = H^1(0,T;\mathbb{R}^n)$. Given $x \in \mathbb{R}^d$ and $P \in L^\infty(0,T;\mathbb{R}^n)$, the representative agent solves:

$$\min_{\gamma\in\Gamma,\,\gamma(0)=x} \int_0^T L(\gamma(t),\dot{\gamma}(t)) + \langle P(t),\dot{\gamma}(t)\rangle \,\mathrm{d}t + g(\gamma(T)).$$

Set of optimal trajectories: $\Gamma_{\text{opt}}^P[x]$.

- **Lagrangian formulation**[4]: description of equilibrium via a probability measure η on Γ. Consider

$$\mathcal{P}_{m_0} = \left\{ \eta \in \mathcal{P}(\Gamma) \,\middle|\, \int_\Gamma \|\gamma\|_\Gamma \,\mathrm{d}\eta(\gamma) < \infty,\; e_0\sharp\eta = m_0 \right\},$$

where $e_0 \colon \gamma \in \Gamma \mapsto e_0(\gamma) = \gamma(0)$.

[4] Benamou, Carlier & Santambrogio '17, Cannarsa & Capuani '18.

MFG equilibrium

- Given $\eta \in \mathcal{P}_{m_0}$, define the price

$$P^\eta(t) = \Psi(A(t)), \quad \text{for a.e. } t \in (0,T), \quad \text{where:}$$
$$A = \int_\Gamma \dot\gamma \, \mathrm{d}\eta(\gamma) \in L^2(0,T;\mathbb{R}^n).$$

Definition

We call **MFG equilibrium** any $\eta \in \mathcal{P}_{m_0}$ such that

$$\eta \in E(\eta) := \left\{ \hat\eta \in \mathcal{P}_{m_0} \;\middle|\; \mathrm{supp}(\hat\eta) \subseteq \cup_{x \in \mathbb{R}^n} \Gamma_{\mathrm{opt}}^{P^\eta}[x] \right\}.$$

Difficulties:

- Here $P^\eta \in L^\infty$, thus optimal trajectories have low regularity.
- Kakutani's fixed point theorem does not apply to $\eta \in E(\eta)$ $\rightarrow$ lack of compactness.

Extended equilibria

- **Key idea:** consider the **adjoints** of the representative agents[5]. Given x and P, for any $\gamma \in \Gamma^P_{\text{opt}}[x]$, let p be the solution to

$$-\dot{p}(t) = \nabla L_x(\gamma(t), \dot{\gamma}(t)), \quad p(T) = \nabla g(\gamma(T)). \qquad (*)$$

  We define

$$\tilde{\Gamma}^P_{\text{opt}}[x] = \left\{ (\gamma, p) \in \Gamma^P_{\text{opt}}[x] \times \Gamma \,\middle|\, (\gamma, p) \text{ satisfies } (*) \right\}$$

- Consider

$$\tilde{\mathcal{P}}_{m_0} = \left\{ \kappa \in \mathcal{P}(\Gamma \times \Gamma) \,\middle|\, \int_\Gamma \|\gamma\|_\Gamma \, \mathrm{d}\kappa(\gamma, p) < \infty, \ \tilde{e}_0 \sharp \eta = m_0 \right\},$$

  where $\tilde{e}_0 \colon (\gamma, p) \in \Gamma \times \Gamma \mapsto e_0(\gamma, p) = \gamma(0)$.

[5] Gomes & Voskanyan, '16

Extended equilibria

- Appropriate definition of the price via an **auxiliary mapping**:

$$\tilde{P}_t^\kappa = \mathbf{P}(t, \mu_t^\kappa), \quad \mu_t^\kappa := \hat{e}_t \sharp \kappa,$$

where $\hat{e}_t \colon (\gamma, p) \mapsto (\gamma(t), p(t))$. Now $\tilde{P}^\kappa$ is Hölder!

Definition

We call **extended MFG equilibrium** any $\kappa \in \tilde{\mathcal{P}}_{m_0}$ such that

$$\kappa \in \tilde{E}(\kappa) := \left\{ \hat{\kappa} \in \mathcal{P}_{m_0} \,\middle|\, \text{supp}(\hat{\kappa}) \subseteq \cup_{x \in \mathbb{R}^n} \tilde{\Gamma}_{\text{opt}}^{\tilde{P}^\kappa}[x] \right\}.$$

Theorem

There exists an extended MFG equilibrium κ. Moreover, its first marginal is an MFG equilibrium.

References

- C. Alasseur, I. Ben Tahar, A. Matoussi. An extended Mean Field Game for storage in smart grids. *J. Optim. Theory & Applications*, 2020.
- J.D. Benamou, G. Carlier, F. Santambrogio, Variational Mean Field Games, *Active Particles*, Vol. 1, 2017.
- J.F. Bonnans, S. Hadikhanloo, L. Pfeiffer. Schauder estimates for a class of potential Mean Field Games of Controls, *Appl. Math. Optim.*, 2019.
- P. Cannarsa, R. Capuani. Existence and Uniqueness for Mean Field Games with State Constraints, *PDE Models for multi-agents phenomena*, 2018.
- P. Cardaliaguet, C.-A. Lehalle. Mean Field Game of Controls and an application to trade crowding. *Math. Financ. Econ.*, 2018.
- R. Couillet, S. Perlaza, H. Tembine, M. Debbah. Electrical vehicles in the smart grid: a mean field game analysis. *IEEE J. Sel. Areas Commun.*, 2012.

References

- A. De Paola, D. Angeli, G. Strbac. Distributed control of micro-storage devices with mean field games. *IEEE Trans. Smart Grid*, 2016.
- D.A. Gomes, V. Voskanyan. Extended deterministic Mean Field Games. *SICON*, 2016.
- D. Gomes, J. Saude. A Mean-Field Game Approach to Price Formation. *Dyn. Games and Applications*, 2020.
- P.J. Graber, A. Bensoussan. Existence and uniqueness of solutions for Bertrand and Cournot Mean Field Games. *Applied Math. Optim.*, 2015.
- J. Graber, V. Ignazio, A. Neufeld. Nonlocal Bertrand and Cournot Mean Field Games with general nonlinear demand schedule, Preprint, 2020.
- Z. Kobeissi. On classical solutions of the Mean Field Game system of Controls. Preprint, 2019.

Thank you for your attention!