Some uniqueness results for Bernoulli interior free boundary problem in convex domains

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Abstract : For Bernoulli interior free boundary problem in bounded convex domains, existence of an elliptic familly of convex solutions is established. It is also proved that there is a unique solution of Bernoulli problem associated with the so-called Bernoulli constant, and an estimate from above for this constant is given.

Résumé : Pour le problème de frontième libre dit "problème de Bernoulli intérieur" dans des ouverts convexes bornés, on établit l'existence d'une famille elliptique de solutions convexes. On montre aussi qu'il existe une unique solution au problème de Bernoulli associé à la constante de Bernoulli, pour laquelle on donne une estimation supérieure.

1 Introduction

The interior Bernoulli free boundary problem is the following: Let Ω be an open bounded subset of $\mathbb{R}^N$ and let $\lambda > 0$ be a fixed constant. We want to find a subdomain D of Ω such that the capacity potential u_D of D in Ω, i.e., the solution of

$$(1) \qquad \begin{cases} -\Delta u = 0 & \text{in } \Omega\backslash\overline{D} \\ u = 0 & \text{on } \partial\Omega \\ u = 1 & \text{on } \partial D \end{cases}$$

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satisfies the additional condition:

$$\forall x \in \partial D, \quad \frac{\partial u_D(x)}{\partial n_x} = -\lambda \ ,$$

where n_x is the outward normal to D at x. In the sequel, such a domain is called a solution of Bernoulli problem of level λ. Bernoulli problem has been extensively studied and we refer to the survey paper [14] for several motivations and references.

It is known that this problem does not have a solution for any positive level λ. For instance, when Ω is convex, it is proved in [19] that there is some positive constant λ_Ω such that Bernoulli problem has a solution of level λ if and only if $\lambda \geq \lambda_\Omega$. This constant λ_Ω is called Bernoulli constant.

It is also known that even if there are solutions to the problem for some λ, there is no uniqueness in general. For instance, if Ω is a ball, there are exactly two solutions to the problem for any $\lambda > \lambda_\Omega$, while for λ_Ω the solution is unique.

Let us now briefly describe the structure of the solutions when Ω is a ball. All the solutions are balls, with the same center as Ω (c.f. [22]). For any $\lambda > \lambda_\Omega$, let us denote by $\widetilde{D_\lambda}$ the largest solution of level λ and by $\widehat{D_\lambda}$ the smallest one. Then the family of largest solutions $(\widetilde{D_\lambda})_{\lambda > \lambda_\Omega}$ forms a continuous increasing family of balls, while the family of smallest solutions $(\widehat{D_\lambda})_{\lambda > \lambda_\Omega}$ forms a continuous decreasing family of balls. In Beurling terminology [5], the increasing family is called an *elliptic family of solutions* while the decreasing family is called a *hyperbolic family of solutions.* The unique solution corresponding to the Bernoulli constant λ_Ω is called *parabolic.* It is the limit of the $(\widetilde{D_\lambda})$ and of the $(\widehat{D_\lambda})$ when $\lambda \to \lambda_\Omega$. Finally, the limit of the elliptic family when $\lambda \to +\infty$ is equal to Ω, while the limit of the hyperbolic family when $\lambda \to +\infty$ is reduced to the center of the ball. In particular, the boundary of the solutions of Bernoulli problem completely cover Ω but the center.

A very interesting - and open - question is whether for general convex bounded sets Ω the structure of the solutions of Bernoulli problem enjoys similar features. In this paper we try to provide some positive evidence towards this conjecture.

Let us first recall that, for any bounded and convex domain Ω, and for any fixed volume $\sigma > 0$, there is at least one convex solution to Bernoulli free boundary problem of volume σ. This has already been proved in [2], Theorem 3 and in [9], Theorem 5.1. In the first part of this paper, we give a new - and we hope enlighting - proof of this result.

In [19], Henrot and Shahgholian proved the existence, for any $\lambda \geq \lambda_\Omega$, of a maximal convex solution $\widetilde{D_\lambda}$ of level λ. Moreover the family $(\widetilde{D_\lambda})_{\lambda \geq \lambda_\Omega}$ turns out to be increasing with respect to the inclusion. In order to prove that this family is an elliptic family of solutions, it remains to show that it is continuous. This has partially been established by Acker in [1], Theorem 6.5., where it is proved the existence of some constant λ_0 (with $\lambda_0 \geq \lambda_\Omega$) sufficently large such that the subfamily $(\widetilde{D_\lambda})_{\lambda \geq \lambda_0}$ is continuous ([1], Theorem 6.5 p. 1418). This implies the uniqueness of the solutions among the sets with a boundary close of the boundary of Ω.

One of the main contribution of this paper is the fact that the full family $(\widetilde{D_\lambda})_{\lambda \geq \lambda_\Omega}$ is continuous. More precisely we prove the following result:

Theorem 1.1 *The family $(\widetilde{D_\lambda})_{\lambda > \lambda_\Omega}$ is elliptic (i.e., increasing and continuous) and we have the following inclusion: For any λ_0 and λ_1 larger than λ_Ω, for any $s \in [0,1]$,*

$$(1-s)\widetilde{D_{\lambda_0}} + s\widetilde{D_{\lambda_1}} \subset \widetilde{D_{\gamma_s}} \,, \tag{2}$$

where $\gamma_s = 1/[(1-s)/\lambda_0 + s/\lambda_1]$.

Remark : Since $\gamma_s \leq (1-s)\lambda_0 + s\lambda_1$ and the family $(\widetilde{D_\lambda})_{\lambda \geq \lambda_\Omega}$ is increasing, Theorem 1.1 implies that

$$(1-s)\widetilde{D_{\lambda_0}} + s\widetilde{D_{\lambda_1}} \subset \widetilde{D}_{(1-s)\lambda_0 + s\lambda_1} \,.$$

This property can be viewed as a "concavity property" of the familly $(\widetilde{D_\lambda})_{\lambda \geq \lambda_\Omega}$.

The continuity of the family is a simple consequence of the "inequality of concavity" (2). This continuity is important to get uniqueness results. Indeed, by standard comparison principle, the ellipticity of the family implies that any classical solution of Bernoulli problem D containing $\widetilde{D_{\lambda_\Omega}}$ belongs to the family $(\widetilde{D_\lambda})_{\lambda \geq \lambda_\Omega}$. Namely there is some $\lambda \geq \lambda_\Omega$ with $D = D_\lambda$. This remark which goes back to Beurling, can be also found for instance in [1], Theorem 6.4.

Let us also point out that Theorem 1.1 (or, more precisely its proof) implies that $(\widetilde{D_\lambda})_{\lambda \geq \lambda_\Omega}$ is the unique elliptic family of convex solution (cf. Corollary 3.4).

Concerning the parabolic solution, our main contribution is the following result:

Theorem 1.2 *There is exactly one solution $\widetilde{D_{\lambda_\Omega}}$ of level λ_Ω.*

More precisely, we show in Theorem 4.1 below that $\widetilde{D_{\lambda_\Omega}}$ is the unique *subsolution* of level λ_Ω (the definition of a subsolution is given later). This result is much deeper than Theorem 1.1. Indeed, it cannot simply rely upon an "inequality of concavity" of the form (2). In fact the key point for the proof of Theorem 1.1 is an inequality due to Borell in [6], whereas for Theorem 1.2 it is necessary to investigate the cases of equality in Borell inequality. This later result has been obtained by the authors in [11].

Since the Bernoulli constant plays a crucial role in this study, we complete the paper by giving a new estimate from above for the Bernoulli constant. This inequality is optimal in the sense that it is exact for balls.

In conclusion, this paper gives a fairly complete picture for the elliptic solutions and for a parabolic solution of the interior Bernoulli free boundary problem. However the question of the (conjectured) uniqueness of the hyperbolic family of solutions remains open.

Let us just give a possible starting point in this direction: Following [14], it is known that, when the volume $\sigma \to 0^+$, the solutions of Bernoulli problem of volume σ become closer and closer to balls and concentrate at some points of Ω called the harmonic centers of Ω. It is known that the harmonic center of a convex bounded domain is unique (this is proved in [9] for $N = 2$, and in [11] for $N \geq 3$). Therefore the solutions of volume σ concentrate to this unique harmonic center when $\sigma \to 0^+$.

Let us finally explain how this paper is oganized. In section 2, we give a new proof for the existence, for any volume σ, of a solution of Bernoulli problem of volume σ. Sections 3 and 4 are respectively devoted to the proof of Theorems 1.1 and 1.2. In the last part of the paper we give some estimate for the Bernoulli constant.

2 Existence of convex solutions

The aim of this section is to give a new proof of the following result:

Theorem 2.1 ([9, 2]) *If Ω is an open bounded convex domain of $\mathbb{R}^N$, then, for any volume $\sigma \in (0, |\Omega|)$ there is at least one convex solution of Bernoulli problem of volume σ. More precisely, there is a solution which is a minimizer of the problem:*

$$\inf\{cap(D) \mid D \subset\subset \Omega \text{ is convex and } |D| = \sigma\} .$$

Remark : It would be interesting to know if, for any $\sigma \geq |\widetilde{D_{\lambda_\Omega}}|$, the set $\widetilde{D_\lambda}$ of volume σ is a minimizer of the problem.

For proving Theorem 2.1, let us introduce the following function: $\forall \sigma > 0$,

$$F(\sigma) = \inf\{cap(D) \mid D \subset\subset \Omega \text{ is convex and } |D| = \sigma\} , \tag{3}$$

where $|D|$ stands for the volume of D and $cap(D)$ is the capacity of D in Ω, i.e.,

$$cap(D) = \inf\{\int_\Omega |\nabla u|^2 \mid u \in H_0^1(\Omega),\ u \geq 1 \text{ in } D\} = \int_{\partial D} |\nabla u_D| . \tag{4}$$

Standard arguments show that the infimum in the problem defining F is attained (see for instance [3], [7]).

Lemma 2.2 *The function F is monotone increasing.*

Proof : Indeed, the capacity $D \to cap(D)$ is monotone increasing with respect to the inclusion.

□

Lemma 2.3 *Let σ be a point of derivability of F. Then any convex domain realizing the minimum in (3) is a solution to Bernoulli problem of level $\lambda = \sqrt{F'(\sigma)}$.*

Remark : The behaviour of the function F seems to be extremely relevant for describing the general behaviour of the solutions of the Bernoulli free boundary problem. In particular the question of the derivability of F, of its convexity (or its concavity) properties are of crucial importance. From Lemma 2.3 one could expect F to be convex on $[\bar{\sigma}, |\Omega|)$ and concave on $(0, \bar{\sigma}]$, where $\bar{\sigma}$ is the volume of the solution of level λ_Ω.

Proof of Lemma 2.3 : We follow several arguments of Acker [1]. Let D realize the minimum in (3). From Poincaré's variational formula for the capacity (which can be applied since the boundary of D is Lipschitz, see for instance [14]), we have

$$\frac{d}{dh} cap(D - hB)_{|h=0} = -\int_{\partial D} |\nabla u_D|^2 \tag{5}$$

where we have set

$$D - hB = \{x \in D \mid d_{\partial D}(x) > h\} \ ,$$

($d_{\partial D}(x)$ denotes the distance of the point x to the set ∂D). Since, for h sufficiently small, we have

$$|D - hB| = |D| - h|\partial D| + o(h) = \sigma - h|\partial D| + o(h) \ ,$$

we can deduce that

$$F(\sigma - h|\partial D| + o(h)) \leq cap(D - hB) \ .$$

Then, using (5) and equality $F(\sigma) = cap(D)$, we get:

$$F'(\sigma)|\partial D| \geq \int_{\partial D} |\nabla u_D|^2 \ . \tag{6}$$

Let us now consider $D_h = \{u_D > 1 - h\}$. We know that

$$cap(D_h) = \frac{cap(D)}{1-h} \ .$$

Moreover,

$$|D_h| = |D| + h \int_{\partial D} \frac{1}{|\nabla u_D|} + o(h) \ .$$

Let us recall that $\nabla u \neq 0$ in $\Omega \backslash D$ (see [21]). Hence, since $|D| = \sigma$, this gives

$$F(\sigma + h \int_{\partial D} \frac{1}{|\nabla u_D|} + o(h)) \leq cap(D_h) = \frac{cap(D)}{1-h} \ .$$

Therefore, we have

$$F'(\sigma) \left(\int_{\partial D} \frac{1}{|\nabla u_D|} \right) \leq cap(D) = \int_{\partial D} |\nabla u_D| \ . \tag{7}$$

Putting (6) and (7) together gives:

$$\left(\int_{\partial D} \frac{1}{|\nabla u_D|} \right) \left(\int_{\partial D} |\nabla u_D|^2 \right) \leq |\partial D| \left(\int_{\partial D} |\nabla u_D| \right) = \left(\int_{\partial D} 1 \right) \left(\int_{\partial D} |\nabla u_D| \right)$$

(we have used here equality (4)). Let us set for simplicity $S = \partial D$ and $a(x) = |\nabla u_D(x)|$. Then the previous inequality can be rewritten as

$$\int_{S\times S} \left(\frac{a(y)^2}{a(x)} - a(x)\right) \;\le\; 0\;.$$

Since this expression is symmetric with respect to x and y, it implies

$$\int_{S\times S} \left(\frac{a(y)^2}{a(x)} + \frac{a(x)^2}{a(y)} - a(x) - a(y)\right) \;\le\; 0\;,$$

i.e.,

$$\int_{S\times S} \frac{a(y)^2}{a(x)}[1 + \frac{a(x)^3}{a(y)^3} - \frac{a(x)^2}{a(y)^2} - \frac{a(x)}{a(y)}] \;\le\; 0\;.$$

Since the polynomial $t \to 1 + t^3 - t^2 - t$ is positive for $t \ge 0$ unless $t = 1$, the previous inequality implies that

$$a(x) = a(y) \;\text{ for almost every } (x, y) \in S \times S\;.$$

Therefore $a = |\nabla u_D|$ is constant on $S = \partial D$. This means that D is a solution of Bernoulli problem. Using (6) and (7) shows easily that it is a solution of level $\lambda = \sqrt{F'(\sigma)}$.

□

Proof of Theorem 2.1: Let $\sigma > 0$ be fixed. Since F is almost everywhere derivable, there is a sequence (σ_n) converging to σ such that $F'(\sigma_n)$ exists. Let D_n be a minimizer for $F(\sigma_n)$. Then, since the D_n are convex and bounded, they converge, up to a subsequence again denoted (D_n) to some convex set D with $|D| = \sigma$. Moreover, from standard arguments in convex analysis, we also have that $|\partial D_n|$ converges to $|\partial D| > 0$. Let us now prove that the sequence $F'(\sigma_n)$ is bounded. Indeed, we have

$$cap(D_n) = \int_{\partial D_n} |\nabla u_{D_n}| = |\partial D_n| \sqrt{F'(\sigma_n)}\;,$$

because, from Lemma 2.3, D_n is a solution of Bernoulli problem of level $\sqrt{F'(\sigma_n)}$. Since $cap(D_n) = F(\sigma_n)$ and $1/|\partial D_n|$ are bounded, we have proved that $F'(\sigma_n)$ is bounded. Thus $(F'(\sigma_n))$ converges (up to a subsequence again denoted $(F'(\sigma_n)))$ to some $\lambda \ge 0$. Then standard arguments show that D, as a limit of convex solutions of Bernoulli problem of level $F'(\sigma_n)$, is also a convex solution of Bernoulli problem of level λ. Since $|D| = \sigma$, this completes the proof of Theorem 2.1.

□

3 The elliptic family of solutions

The aim of this section is to prove Theorem 1.1. Let us first recall the main results of [19] concerning the construction of the maximal solution $\widetilde{D_\lambda}$ of Bernoulli problem of level λ.

A *subsolution of Bernoulli problem of level* λ is a set $D \subset\subset \Omega$ such that

$$u_D \text{ is Lipschitz continuous and } \frac{\partial u_D}{\partial \nu_x} \geq -\lambda \text{ on } \partial D\ .$$

Let us introduce for any λ the family of subsolutions:

$$\mathcal{F}_\lambda = \{D \subset\subset \Omega \mid u_D \text{ is Lipschitz continuous and } \frac{\partial u_D}{\partial \nu_x} \geq -\lambda \text{ on } \partial D\}\ ,$$

where ν_x denotes the outward normal to D at x, and u_D is the capacity potential of D, i.e., the solution of (1). Let us point out that, if a domain D is a solution of Bernoulli problem of level λ, then D belongs to $\mathcal{F}_\lambda$. Let us set

$$\lambda_\Omega = \inf\{\lambda \mid \mathcal{F}_\lambda \neq \emptyset\}\ .$$

Then it is proved in [19] that $\lambda_\Omega > 0$ and that $\forall \lambda \geq \lambda_\Omega$, the set $\mathcal{F}_\lambda$ is not empty. The main result of [19] states that, for any $\lambda \geq \lambda_\Omega$, the set

$$\widetilde{D_\lambda} = \overline{\mathrm{Co}}\left(\bigcup_{D \in \mathcal{F}_\lambda} D\right)$$

is the maximal solution of Bernoulli problem of level λ, where $\overline{\mathrm{Co}}(A)$ denotes the closed convex hull of a set A.

To prove Theorem 1.1, we need some preliminary results about an inequality due to Borell [6] that we describe now. Let D_0 and D_1 be two convex, open subdomains of Ω. For $s \in [0,1]$, we denote by D_s the following set:

$$D_s = (1-s)D_0 + sD_1 = \{x \in I\!R^N,\ \exists x_0 \in D_0,\ \exists x_1 \in D_1 \text{ with } x = (1-s)x_0 + sx_1\}\ .$$

Let us recall that D_s is a convex, open subdomain of Ω. Following Borell, we denote by $\widetilde{u}_s$ the function

$$\forall x \in \Omega \backslash D_s,\quad \widetilde{u}_s(x) = \sup_{x_0, x_1} \min\{u_{D_0}(x_0), u_{D_1}(x_1)\} \tag{8}$$

where the supremum is taken over the $x_0 \in \Omega \backslash D_0$ and $x_1 \in \Omega \backslash D_1$ such that $x = (1-s)x_0 + sx_1$. Borell's inequality states that

$$u_{D_s}(\cdot) \geq \widetilde{u}_s(\cdot) \ \text{ in } \Omega \backslash \overline{D_s} \ . \tag{9}$$

Moreover, $\widetilde{u}_s$ is continuous on $\overline{\Omega} \backslash D_s$ and

$$\widetilde{u}_s = 0 \text{ on } \partial\Omega \qquad \text{and} \qquad \widetilde{u}_s = 1 \text{ on } \partial D \ .$$

In [11], we have refined Borell's inequality in establishing that the map $\widetilde{u}_s$ is in fact a subsolution of Laplace equation in the viscosity sense (for the definition of this notion, see [12]). Namely:

Lemma 3.1

$$-\Delta \widetilde{u}_s \leq 0 \quad \text{ in } \quad \Omega \backslash \overline{D_s} \quad \text{ in the viscosity sense.}$$

We use this fact in the next section together with the following sharp estimate of the case of equality in Borell's inequality, that we have also established in [11]:

Theorem 3.2 *Let us assume that for some $s \in (0,1)$ the function $\widetilde{u}_s$ is harmonic. Then $D_0 = D_1$.*

Remark : We shall mainly use this result combined with Lemma 3.1 and Borell's inequality (9) in the following way: If $D_0 \neq D_1$ and $s \in (0,1)$, then $u_{D_s} - \widetilde{u}_s$ is a non-negative, non-zero, superharmonic function.

The key point of the proof of Theorem 1.1 is the following

Lemma 3.3 *Let λ_0 and λ_1 be not smaller than λ_Ω. Then, for any convex sets D_0 and D_1, such that $D_0 \in \mathcal{F}_{\lambda_0}$ and $D_1 \in \mathcal{F}_{\lambda_1}$, for any $s \in [0,1]$, the set $D_s = (1-s)D_0 + sD_1$ belongs to $\mathcal{F}_{\gamma_s}$, where*

$$\gamma_s = \frac{1}{\frac{1-s}{\lambda_0} + \frac{s}{\lambda_1}} \ .$$

Remark: Let us notice that we have $\gamma_s \leq (1-s)\lambda_0 + s\lambda_1$ and thus

$$D_s = (1-s)D_0 + sD_1 \ \in \ \mathcal{F}_{(1-s)\lambda_0 + s\lambda_1} \ .$$

Proof of Lemma 3.3 : For simplicity, we set $u_0 = u_{D_0}$ and $u_1 = u_{D_1}$. Following Gabriel [15, 16, 17] and Lewis [21], the level sets of the functions

u_i are smooth and strictly convex, and $\nabla u_i \neq 0$ in $\Omega \backslash \overline{D_i}$. From Lemma 2.2 in [19], D_s belongs to $\mathcal{F}_{\gamma_s}$ if and only if

$$\frac{\partial u_{D_s}(x)}{\partial \nu_x} \geq -\gamma_s \text{ for almost all } x \in \partial D_s \ ,$$

where ν_x denotes the outward normal to D_s at x. The partial derivative has to be understood in the sense

$$\frac{\partial u_{D_s}(x)}{\partial \nu_x} = \lim_{h \to 0^+} \frac{u_{D_s}(x + h\nu_x) - 1}{h}$$

and exists almost everywhere on ∂D_s (see [13]). Let us now fix some $x \in \partial D_s$ point where the previous limit exists and where D_s has a unique unit normal ν_x. From Borell's inequality (9), we have

$$\lim_{h \to 0^+} \frac{u_{D_s}(x + h\nu_x) - 1}{h} \geq \limsup_{h \to 0^+} \frac{\widetilde{u}_s(x + h\nu_x) - 1}{h} \ .$$

Let us now recall some results of [6] (see also [11]): First it is proved that $\widetilde{u}_s$ is $\mathcal{C}^1$ in $\Omega \backslash \overline{D_s}$. Second, it is also proved that for any $x \in \Omega \backslash \overline{D_s}$, there exists a unique pair (x_0, x_1) belonging to $(\Omega \backslash \overline{D_0}) \times (\Omega \backslash \overline{D_1})$, such that

$$\widetilde{u}_s(x) = u_0(x_0) = u_1(x_1) \text{ and } x = (1-s)x_0 + sx_1 \ .$$

Moreover, $\nabla \widetilde{u}_s(x)$ is given by

$$\frac{\nabla \widetilde{u}_s(x)}{|\nabla \widetilde{u}_s(x)|} = \frac{\nabla u_0(x_0)}{|\nabla u_0(x_0)|} = \frac{\nabla u_1(x_1)}{|\nabla u_1(x_1)|}$$

and

$$\frac{1}{|\nabla \widetilde{u}_s(x)|} = \frac{1-s}{|\nabla u_0(x_0)|} + \frac{s}{|\nabla u_1(x_1)|} \ .$$

Let us now consider $\xi_h \in [x, x + h\nu_x]$ such that

$$\frac{\widetilde{u}_s(x + h\nu_x) - 1}{h} = < \nabla \widetilde{u}_s(\xi_h), \nu_x > \ . \tag{10}$$

Let us notice that $\xi_h \to x$ when $h \to 0^+$. We now apply the results of [6] recalled above to the point ξ_h: There are $\xi_h^i \in \Omega \backslash \overline{D_i}$ such that

$$\widetilde{u}_s(\xi_h) = u_0(\xi_h^0) = u_1(\xi_h^1) \text{ and } \xi_h = (1-s)\xi_h^0 + s\xi_h^1$$

and

$$\frac{\nabla\widetilde{u}_s(\xi_h)}{|\nabla\widetilde{u}_s(\xi_h)|} = \frac{\nabla u_0(\xi_h^0)}{|\nabla u_0(\xi_h^0)|} = \frac{\nabla u_1(\xi_h^1)}{|\nabla u_1(\xi_h^1)|}$$

and, finally,

(11)
$$\frac{1}{|\nabla\widetilde{u}_s(\xi_h)|} = \frac{1-s}{|\nabla u_0(\xi_h^0)|} + \frac{s}{|\nabla u_1(\xi_h^1)|} .$$

Since D_i belongs to $\mathcal{F}_{\lambda_i}$, we have

$$|\nabla u_0(\xi_h^0)| \leq \lambda_0 \text{ and } |\nabla u_1(\xi_h^1)| \leq \lambda_1 .$$

Hence, from (11) and the previous inequalities, we obtain

(12)
$$|\nabla\widetilde{u}_s(\xi_h)| \leq 1/[(1-s)/\lambda_0 + s/\lambda_1] = \gamma_s .$$

Let us now consider a sequence $h_n \to 0^+$ such that

(13)
$$\limsup_{h\to 0^+} \frac{\widetilde{u}_s(x+h\nu_x)-1}{h} = \lim_n \frac{\widetilde{u}_s(x+h_n\nu_x)-1}{h_n} .$$

Let us set $\xi_n = \xi_{h_n}$. Since $a_n = -\nabla\widetilde{u}_s(\xi_n)/|\nabla\widetilde{u}_s(\xi_n)|$ is an outward normal to the convex set $\{\widetilde{u}_s \geq \widetilde{u}_s(\xi_n)\}$ at ξ_n, a standard passage to the limit shows that $a = \lim_n a_n$ is an outward normal to the set D_s at x because the ξ_n converge to x, x belongs to ∂D_s and the convex set $\{\widetilde{u}_s \geq \widetilde{u}_s(\xi_n)\}$ converges to the convex set D_s. Since, from our assumption, D_s has a unique outward normal at x, namely ν_x, we have $a = \nu_x$. Hence, from (10) and (13), we get

$$\limsup_{h\to 0^+} \frac{\widetilde{u}_s(x+h\nu_x)-1}{h} = \lim_n < \nabla\widetilde{u}_s(\xi_n), \nu_x >= -\lim_n |\nabla\widetilde{u}_s(\xi_n)| .$$

Using (12) finally proves the desired result:

$$\lim_{h\to 0^+} \frac{u_{D_s}(x+h\nu_x)-1}{h} \geq -\gamma_s .$$

□

Proof of Theorem 1.1 : We first prove (2). Let λ_0 and λ_1 be fixed. From [19], the convex sets $\widetilde{D_{\lambda_0}}$ and $\widetilde{D_{\lambda_1}}$ belong respectively to $\mathcal{F}_{\lambda_0}$ and to $\mathcal{F}_{\lambda_1}$. Lemma 3.3 then states that, for any $s \in [0,1]$, the convex set

$$D_s = (1-s)\widetilde{D_{\lambda_0}} + s\widetilde{D_{\lambda_1}}$$

belongs to $\mathcal{F}_{\gamma_s}$, where γ_s is defined as in Lemma 3.3. Accordingly, from the construction of the solution $\widetilde{D_\lambda}$, we have

$$D_s \subset \widetilde{D_{\gamma_s}}$$

This proves (2).

We now prove the continuity of the family $(\widetilde{D_\lambda})_{\lambda > \lambda_\Omega}$. Let $\lambda > \lambda_\Omega$ be fixed. From the construction of the solution $\widetilde{D_\lambda}$ and standard stability results, we have easily that

$$\widetilde{D_\lambda} = Int \left(\bigcap_{\lambda' > \lambda} \widetilde{D_{\lambda'}} \right) \ .$$

This proves the continuity on the right. For proving the continuity on the left, let us set

$$D = \bigcup_{\lambda' < \lambda} \widetilde{D_{\lambda'}} \ .$$

We already know that $D \subset \widetilde{D_\lambda}$ and we want to prove the equality. We argue by contradiction, by assuming that $D \neq \widetilde{D_\lambda}$.

Let us first notice that D is a convex solution of Bernoulli problem of level λ, as limit of convex solutions of Bernoulli problem of level λ', with $\lambda' \to \lambda$. Thus its boundary is smooth (see for instance [19]). Since $\widetilde{D_\lambda}$ is also a solution of the Bernoulli problem of level λ, the strong maximum principle implies that the boundary of the sets D and $\widetilde{D_\lambda}$ are disjoint. Hence $D \subset\subset \widetilde{D_\lambda}$.

We now choose some $\lambda_0 \in (\lambda_\Omega, \lambda)$. Let s be the largest real in $(0,1)$ such that

$$(1-s)\widetilde{D_{\lambda_0}} + s\widetilde{D_\lambda} \subset D \ .$$

Let us notice that s belongs to $(0,1)$ and that, if we set $D_s = (1-s)\widetilde{D_{\lambda_0}} + s\widetilde{D_\lambda}$, the boundaries of D_s and D have a non empty intersection. Let x belong to this intersection. From Lemma 3.3, we know that D_s belongs to $\mathcal{F}_{\gamma_s}$, where $\gamma_s = 1/[(1-s)/\lambda_0 + s/\lambda]$. Let us notice that γ_s is smaller than λ. Since $D_s \subset D$, we have $u_{D_s} \leq u_D$. Let ν_x be the normal to D_s and D at x. We have

$$\lambda = \lim_{h \to 0^+} \frac{1 - u_D(x + h\nu_x)}{h} \leq \lim_{h \to 0^+} \frac{1 - u_{D_s}(x + h\nu_x)}{h} \leq \gamma_s \ .$$

Hence there is a contradiction, since we have in fact $\gamma_s < \lambda$. This completes the proof.

□

The same proof shows that the family $(\widetilde{D_\lambda})_{\lambda>\lambda_\Omega}$ is the unique elliptic family of convex solutions. Namely:

Corollary 3.4 *Let D_0 and D_1 be two convex solutions of Bernoulli problem respectively of level λ_0 and λ_1. Assume that*

$$D_0 \subset\subset D_1 \qquad \text{and} \qquad \lambda_0 < \lambda_1 \ .$$

Then $D_1 = \widetilde{D_{\lambda_1}}$.

Proof : Replace in the proof of Theorem 1.1 the set D_{λ_Ω} by D_0, D by D_1 and $\widehat{D_\lambda}$ by $\widehat{D_{\lambda_1}}$.

□

4 Uniqueness of the parabolic solution

We finally investigate the special case of the parabolic solution, i.e., the solution of level λ_Ω. Our aim is to establish the uniqueness of the solution. We are in fact going to prove a stronger result. Namely:

Theorem 4.1 *With the notations of the previous section, we have*

$$\mathcal{F}_{\lambda_\Omega} = \{\widetilde{D_{\lambda_\Omega}}\} \ .$$

Let us notice that Theorem 4.1 implies Theorem 1.2, since a solution D of the Bernoulli problem of level λ_Ω always belongs to $\mathcal{F}_{\lambda_\Omega}$.

Proof of Theorem 4.1 : We argue by contradiction. Let us assume that there is an open set D belonging to $\mathcal{F}_{\lambda_\Omega}$, with $D \neq \widetilde{D_{\lambda_\Omega}}$. The definition of $\widetilde{D_{\lambda_\Omega}}$ implies that $D \subset \widetilde{D_{\lambda_\Omega}}$. From Lemma 2.4 of [19], we know that the convex hull of D, denoted by D_0 also belongs to $\mathcal{F}_{\lambda_\Omega}$. We claim that $D_0 \subset\subset \widetilde{D_{\lambda_\Omega}}$. Indeed, otherwise, there should exist some x belonging to the intersection of the boundary of D and the boundary of $\widetilde{D_{\lambda_\Omega}}$. Since $D \subset \widetilde{D_{\lambda_\Omega}}$ and $D \neq \widetilde{D_{\lambda_\Omega}}$, Hopf maximum principle then would imply that $|\nabla u_D(x)| > |\nabla u_{\widetilde{D_{\lambda_\Omega}}}(x)|$ at this point x. But this is impossible since $|\nabla u_D(x)| \leq \lambda_\Omega$ because D is a sub-solution and $|\nabla u_{\widetilde{D_{\lambda_\Omega}}}(x)| = \lambda_\Omega$ because $\widetilde{D_{\lambda_\Omega}}$ is a solution of Bernoulli problem of level λ_Ω. Hence we have proved that $D_0 \subset\subset \widetilde{D_{\lambda_\Omega}}$.

We now consider the convex combination $D_s = (1-s)D_0 + s\widetilde{D_{\lambda_\Omega}}$ for some $s \in (0,1)$. To achieve the proof of our Theorem, it suffices to prove that D_s belongs in fact to $\mathcal{F}_{\lambda_\Omega - \epsilon}$, for some $\epsilon > 0$. Indeed this leads to a contradiction because $\mathcal{F}_{\lambda_\Omega-\epsilon}$ is empty from the definition of λ_Ω.

We now prove that D_s belongs to $\mathcal{F}_{\lambda_\Omega-\epsilon}$ for some $\epsilon > 0$. At this step, we have to underline that Borell's inequality is no longer enough for proving that D_s belongs to some $\mathcal{F}_{\lambda_\Omega-\epsilon}$. Indeed, Borell's inequality only gives that D_s belongs to $\mathcal{F}_{\lambda_\Omega}$ (see Lemma 3.3). Therefore we have to use a stronger argument: This argument is Theorem 3.2, which states that, since $D_0 \neq \widetilde{D_{\lambda_\Omega}}$, the map $\widetilde{u}_s$ defined by (8) *cannot* be a solution of Laplace equation. Since $\widetilde{u}_s$ is a subsolution of this equation (cf. Lemma 3.1), this shows that the map $u_{D_s} - \widetilde{u}_s$ is a *non-negative* viscosity supersolution of Laplace equation, vanishing at the boundary $\partial\Omega \cup \partial D_s$. Using the fact that D_s is convex and bounded, Hopf maximum principle states that there is a neighborhood U of ∂D_s and some positive constant ϵ such that

$$\forall x \in U\backslash D_s,\ u_{D_s}(x) - \widetilde{u}_s(x) \geq \epsilon d_{D_s}(x)\ ,$$

where $d_{D_s}(x)$ denote the distance of the point x to the set D_s. We now argue as in the proof of Lemma 3.3: We have, for almost every $x \in \partial D_s$, where there is a unique outward normal ν_x to D_s at x,

$$\lim_{h\to 0^+} \frac{u_{D_s}(x+h\nu_x) - 1}{h} \geq \limsup_{h\to 0^+} \frac{\widetilde{u}_s(x+h\nu_x) - 1}{h} + \epsilon$$

since $d_{D_s}(x+h\nu_x) = h$. We can estimate the term in lim sup as in the proof of Lemma 3.3 (with now $\lambda_0 = \lambda_1 = \lambda_\Omega$): This gives

$$\limsup_{h\to 0^+} \frac{\widetilde{u}_s(x+h\nu_x) - 1}{h} \geq -\lambda_\Omega\ .$$

Hence we have

$$\lim_{h\to 0^+} \frac{u_{D_s}(x+h\nu_x) - 1}{h} \geq -\lambda_\Omega + \epsilon\ .$$

Using Lemma 2.2 of [19], this proves that D_s belongs to $\mathcal{F}_{\lambda_\Omega-\epsilon}$ and gives the desired contradiction.

$\square$

5 Some estimate of the Bernoulli constant

Our aim is to estimate from above the Bernoulli constant λ_Ω. For an estimate from below, let us recall that the following question is still open (see [14]): Let Ω be an open, bounded, convex subset of $I\!\!R^N$ and let $\tilde{\Omega}$ be the ball centered at 0 with $|\Omega| = |\tilde{\Omega}|$. Do we always have

$$\lambda_{\tilde{\Omega}} \leq \lambda_\Omega \ ?$$

In order to explain our result, we have to introduce some definitions and notations: Let Ω be an open, bounded convex subset of $I\!\!R^N$. Let us denote by $F = F(|\cdot|)$ the fondamental solution of Laplace equation in $I\!\!R^n$ and, for any $x \in \Omega$, let $H_x(\cdot)$ be the regular part of the Green function of Laplace equation with Dirichlet boundary condition, i.e., the solution of

$$\begin{cases} -\Delta H_x(\cdot) = 0 & \text{in } \Omega \\ H_x(\cdot) = F(|\cdot - x|) & \text{on } \partial\Omega \end{cases}$$

The Robin function $t : \Omega \to I\!\!R$ and the harmonic radius $r : \Omega \to I\!\!R$ are respectively defined by

$$\forall x \in \Omega,\ t(x) = H_x(x) \text{ and } t(x) = F^{-1}(t(x)) \ .$$

We also denote by $\bar{r}_\Omega$ the maximum of the harmonic radius in Ω (which exists since Ω is convex and bounded) and by $\bar{x}_\Omega$ the harmonic center of Ω (i.e., the point of maximum of the strictly concave function $r(\cdot)$, see [11] for instance). Let us recall that in dimension $N = 2$, the maximum of the harmonic radius is usually called the conformal radius.

In [19], the following estimate from above of the Bernoulli constant is given: If Ω is an open convex bounded subset of $I\!\!R^2$, we have

$$\lambda_\Omega \leq \frac{6.252}{\bar{r}_\Omega} \ .$$

We improve this result in the following:

Theorem 5.1 *We have, in any dimension $N \geq 2$,*

$$\lambda_\Omega \leq \lambda_{B_{\bar{r}_\Omega}(0)} = \begin{cases} \dfrac{N-2}{|(N-1)^{-\frac{N-1}{N-2}} - (N-1)^{-\frac{1}{N-2}}|} \dfrac{1}{\bar{r}_\Omega} & \text{if } N \geq 3 \\ \\ e/\bar{r}_\Omega & \text{if } N = 2 \end{cases}$$

Remarks :

1. Let us recall that the following inequality holds true for open, bounded and convex sets:

   (14) $$\text{If } \Omega_1 \subset \Omega_2,\ \lambda_{\Omega_1} \geq \lambda_{\Omega_2}\ .$$

   This is a straightforward consequence of the construction of [19]. This inequality gives an easy estimate from below of the Bernoulli constant

   $$\lambda_\Omega \geq \lambda_{B_R(0)}$$

   where R is the radius of the smallest ball containing Ω.

2. Let us point out that the estimate from above given in the Theorem does not derive from inequality (14) because the ball $B(\bar{x}_\Omega, \bar{r}_\Omega)$ is not contained in Ω, unless Ω is a ball.

3. Let us finally notice that the estimate of the Theorem is optimal for balls, because in this case the maximum of the harmonic radius $\bar{r}_\Omega$ is equal to the radius of the ball.

Proof : Let us set $B = B_{\bar{r}_\Omega}(0)$ and $\lambda = \lambda_B$ the Bernoulli constant of the ball B. There is a unique radius $r \in (0, \bar{r}_\Omega)$ such that $D = B_r(0)$ is the solution of Bernoulli problem of level λ in B (indeed, in the case of balls, it is known that any solution is radial, cf. [22]). Let $\bar{G}_x(\cdot)$ and $G_x(\cdot)$ be the Green functions of the sets B and Ω respectively, for the Dirichlet problem, i.e., the solutions of

$$\begin{cases} -\Delta \bar{G}_x(\cdot) = \delta_x & \text{in } B \\ \bar{G}_x(\cdot) = 0 & \text{on } \partial B \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta G_x(\cdot) = \delta_x & \text{in } \Omega \\ G_x(\cdot) = 0 & \text{on } \partial\Omega \end{cases}$$

where δ_x is the Dirac measure at x. Then, for $x = 0$, the solution $\bar{G}_0(\cdot)$ is radial and we denote by $\bar{t}$ its value on ∂D. Let us set

$$\forall s \in [0, +\infty),\ \phi(s) = \begin{cases} s/\bar{t} & \text{if } s \leq \bar{t} \\ 1 & \text{otherwise} \end{cases}$$

Then $\bar{u}(\cdot) = \phi \circ \bar{G}_0(\cdot)$ is nothing but the capacity potential of D in B.

Let us now consider the harmonic transplantation $u_0 = \phi \circ G_{\bar{x}}$ (see [20], [4]). Then u_0 is clearly the capacity potential of the set $D_0 = \{G_{\bar{x}} > \bar{t}\}$. Following [4], Theorem 18, we have

$$cap_B(D) = \int_B |\nabla \bar{u}|^2 = \int_\Omega |\nabla u_0|^2 = cap_\Omega(D_0)$$

and

$$|D| \leq |D_0|$$

(from Theorem 18 of [4], part 2, with $f(t) = 1$ if $t \geq 1$, $f(t) = 0$ otherwise). Accordingly,

$$cap_B(D) = cap_\Omega(D_0) \geq \inf\{cap_\Omega(C) \mid C \subset\subset \Omega,\ |C| = |D|,\ C \text{ convex}\} = F(|D|)\ ,$$

where F is defined in section 2, because the capacity is non decreasing with respect to the inclusion. From Theorem 2.1, we know that there is a open convex set C of volume $|D|$, which is solution of Bernoulli problem for some level $\bar{\lambda} > 0$. Moreover, from the proof of Theorem 2.1 (see section 2), we can choose C as a minimizer of $F(|D|)$. Accordingly, we have

$$cap_B(D) = \lambda|\partial D| \geq cap_\Omega(C) = \bar{\lambda}|\partial C|\ .$$

Since $|D| = |C|$, the isoperimetric inequality states that $|\partial C| \geq |\partial D|$. Hence we have proved that

$$\lambda = \lambda_B \geq \bar{\lambda} \geq \lambda_\Omega\ .$$

Since $B = B_{\bar{r}_\Omega}(0)$, the proof of Theorem 5.1 is complete.

$\square$

References

[1] Acker, A. *Uniqueness and monotonicity of solutions for the interior Bernoulli free boundary problem in the convex, n-dimensional case.* Nonlinear Anal., Theory Methods Appl. 13, No.12, 1409-1425 (1989).

[2] Acker, Andrew *Area-preserving domain perturbation operators which increase torsional rigidity or decrease capacity, with applications to free boundary problems.* Z. Angew. Math. Phys. 32, 435-449 (1981).

[3] Aguilera, N., Alt, H.W. and Caffarelli, L.A. *An optimization problem with volume constraint.* SIAM J. Control Optimization 24, 191-198 (1986).

[4] Bandle, C. and Flucher, M. *Harmonic radius and concentration of energy; hyperbolic radius and Liouville's equations* $\Delta U = e^U$ *and* $\Delta U = U^{\frac{n+2}{n-2}}$. SIAM Rev. 38, No.2, 191-238 (1996).

[5] Beurling, A. *On free-boundary problems for the Laplace equations.* Sem. analytic functions 1, 248-263 (1958).

[6] Borell, C. *Capacitary inequalities of the Brunn-Minkowski type.* Math. Ann. 263, 179-184 (1983).

[7] Buttazzo, G. and Dal Maso, G. *An existence result for a class of shape optimization problems.* Arch. Ration. Mech. Anal. 122, No.2, 183-195 (1993).

[8] Cabre, X. and Caffarelli, L. A. FULLY NONLINEAR ELLIPTIC EQUATIONS. Colloquium Publications. American Mathematical Society. 43. Providence, RI (1995).

[9] Caffarelli, L. A. and Friedman, A. *Convexity of solutions of semilinear elliptic equations.* Duke Math. J. 52, 431-456 (1985).

[10] Caffarelli, L. and Spruck, J. *Convexity properties of solutions to some classical variational problems.* Commun. Partial Differ. Equations 7, 1337-1379 (1982).

[11] Cardaliaguet, P. and Tahraoui, R *On the strict concavity of the harmonic radius in dimension* $N \geq 3$, J.Math. Pures Appl.., 81, 223-240 (2002).

[12] Crandall, M. G., Ishii, H. and Lions, P.-L. *User's guide to viscosity solutions of second order partial differential equations.* Bull. Am. Math. Soc., New Ser. 27, No.1, 1-67 (1992).

[13] Dahlberg, B. E.J. *Estimates of harmonic measure.* Arch. Ration. Mech. Anal. 65, 275-288 (1977)

[14] Flucher, M. and Rumpf, M. *Bernoulli's free-boundary problem, qualitative theory and numerical approximation.* J. Reine Angew. Math. 486, 165-205 (1997).

[15] Gabriel, R.M. *An extended principle of the maximum for harmonic functions in 3- dimensions.* J. London Math. Soc. 30, 388-401 (1955).

[16] Gabriel, R.M. *A result concerning convex level surfaces of 3-dimensional harmonic functions.* J. London math. Soc. 32, 286-294 (1957).

[17] Gabriel, R.M. *Further results concerning the level surface of the Green's function for a 3-dimensional convex domain. I, II.* J. London math. Soc. 32, 295-302, 303-306 (1957).

[18] Henrot, A. and Shahgholian, H. *Convexity of free boundaries with Bernoulli type boundary condition.* Nonlinear Anal., Theory Methods Appl. 28, No.5, 815-823 (1997).

[19] Henrot, A. and Shahgholian, H. *Existence of classical solutions to a free boundary problem for the* $p-$*Laplace operator: (II) the interior case.* Indiana Univ. Math. J., 49, N. 1 (2000) , pp. 311-323.

[20] Hersch, J. *Transplantation harmonique, transplantation par module et théorèmes isopérimétriques*, Comment. Math. Helv., 44 (1969), pp. 354-366.

[21] Lewis, J. *Capacitary functions in convex rings.* Arch. Ration. Mech. Anal. 66, 201-224 (1977).

[22] Reichel, W. *Radial symmetry by moving planes for semilinear elliptic boundary value problems on annulli and other non-convex domains.* Elliptic abd Parabolic problems: Pont-à-Mousson 1994, Pitman Res. Notes 325 (1995), 164-182.